Undergraduate Lecture Notes in Physics

Anders Malthe-Sørenssen

Elementary Mechanics Using Python

A Modern Course Combining Analytical and Numerical Techniques

Springer

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To Mina, Aurora and Olav.

Preface

This book was developed as a textbook for use in the course "Introduction to Mechanics" in the Department of Physics at the University of Oslo starting 2007. In this course we aimed at providing a seamless integration of analytical and numerical methods when solving physics problems, thereby allowing us to solve more advanced and applied problems in mechanics, and providing examples that are perceived as more relevant for the students. We could address not only the very special cases that have analytical solutions, but could instead focus on choosing problems that would initiate discussions and provide the students with physical insights.

Through the processes of introducing and developing advanced problems, it also became clear that this approach brought the students closer to the way physics is discovered and applied. In addition, it introduced the students to a more exploratory way of understanding phenomena and of developing their physical concepts. Well-developed examples that also include elements of numerical computations gave the students a feeling of discovering physical processes while also understanding how they are results of the underlying simple physical laws. In many cases, the advanced examples and exercises spawned interesting and rewarding discussions about the underlying physical processes, and also forced the students to understand the various forms of representation used to illustrate physical processes, such as motion diagrams and energy diagrams, and use these diagrams to reason about physical processes.

As the course, examples, and exercises were developed it also became clear that the introduction of numerical methods in an introductory course in physics also helped build the notion that numerical methods are no different from analytical methods—they are part of the theoretical toolbox that any physicist is supposed to master. Our aim became to make it as natural for our students to solve their problems by developing a small program and discussing the results, as it was to use a calculator.

It has been particularly rewarding to observe the way that many of the examples and exercises trigger discussions when students discover unexpected results, in the form of unexpected resonances in a simple model for friction or in the case of

Greenwood gaps in the distribution of asteroids in the solar system. The insight that the simple laws of mechanics that they learned actually had observable consequences and explanatory power was often an important insight as well as an important reinforcer for the students. We also believe that this helps the student build a more realistic image of how science actually is done.

In order to get most of the numerical parts of this text it is advantageous for the students to have some prior knowledge of scientific programming, preferably with a scripting type language such as Matlab or Python, but this is not absolutely necessary. We encourage readers who are not familiar with scripting type programming first to study Chap. 2. However, in our experience students who read the book, study the examples, and do the exercises will already be developing programmers by the end of the course.

This book grew out of a larger, collaborative effort at the University of Oslo. I would like to thank Morten Hjorth-Jensen and Arnt Inge Vistnes for including me in the physics part of the Computers in Science Education program. I also thank Hans Petter Langtangen and Knut Mørken at the Department of Informatics for their dedication, support, and inspiration for introducing numerical approaches in the basic curriculum. I thank the Faculty for Mathematics and Natural Sciences for their support used to develop exercises and examples used in this text. I would also like to thank Arnt Inge Vistnes, Jonas van den Brinck, and Sigve Bøe Skattum for developing some of the exercises that have been included in this book as examples or exercises. Sigve Bøe Skattum has also provided many of the illustrations.

Oslo Anders Malthe-Sørenssen
March 2015

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Chapter 1
Introduction

In this book we introduce the fundamental concepts in our understanding of nature and learn to use them to deepen your understanding of nature. This is a bold and sweeping goal—it is indeed the goal of physics. The tools and concepts from mechanics have a central role in how a physicist thinks about nature. And an important part of learning mechanics is to learn to think like a physicist. Unfortunately there are no short-cuts to acquiring the experience of an expert. The only way to learn physics, and mechanics, is through diligent application of the theory to example and exercises. We will help you by providing hints on how to structure your approach, by introducing well-tested problem solving techniques, and through worked examples, but in the end it is only the amount of work *you* spend on exercises that will determine your success. The examples also provide you with inspirations for what you can do when you master the basic principles of mechanics, and we hope this will indeed show you the power that lies in our knowledge of physics, and the exiting adventure it is to discover how nature works and apply that knowledge to develop technologies for the best of mankind.

1.1 Physics

Physics has several aspects: Physics as a science represents the quest to understand the basic laws of nature. Physics provides the tools to understand the processes occurring in nature on all time and length scales. Physics also provides the conceptual and theoretical background for developing new technologies. The fashionable directions in technological and scientific development change, but they all depend on a solid foundation in physics. Physics as a scientific venture is an interplay between the development of theory and experimental investigations.

How physics is used to understand nature is clearly expressed in the physics of biological processes. If you are interested in how a protein folds—and how it folds is important to understand its functions and interactions—we must understand the

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fundamental physics in the interactions between atoms, between the molecular parts of the protein, and between the protein and the surrounding fluid. Physics provides the tools to develop such an understanding.

Physics provides us with the tools to develop new, better technologies. Technologies that can help solve environmental- or energy-related problems. And physics tempts us with possibilities to develop completely new technologies, based on so-far unknown principles, that may lead to improvements larger than we could have imagined.

There are still unsolved, fundamental problems that are within the reach of physics. But in order to address these problems you must master the tools of the trade, you must develop an ability to understand and address the physics of problems, you must develop knowledge about the laws of physics, since we use this knowledge to guide our intuition when we think of physics, and you must develop your knowledge of mathematical tools so that you can solve real problems. This starts with learning mechanics.

You will learn beautiful laws in physics. Much of the theory you learn will be formulated in nice, mathematical equations, beautiful symmetries. It is elegant, concise, and beautiful. And this is indeed something we want to show you. Nature could have been in so many ways. But, look—it is so simple, and so beautiful.

But try not to be blinded by the beauty. The most beautiful and elegant mathematical formulations are found in the parts of physics that are finished. There is not really anything left to do but to find new decimals in the physical constants. When a field is under development it is often messy, unfinished, unready. It is uncharted territory waiting for someone to make sense of it. There may be many exiting discoveries waiting in the messiness. Such is often the nature of discovery.

1.2 Mechanics

Mechanics is the part of physics that addresses the motion of objects. However, in order to predict motion, we need quantitative tools to describe motion. Our main tool is calculus and associated analytical and numerical methods. The study of motion is traditionally called kinematics, which is in many ways closer to mathematics than to physics. When we approach a problem in physics we first use our physical insight to simplify the problem. We strive to make the problem so simple that we can use simple physical laws to formulate mathematical equations that describe the motion. The first part of this process, finding a good physical model and translating the model into a mathematical problem is what we typical refer to as the "physics of the problem".

When we have formulated a mathematical description of the problem, we find the motion and solve the problem using methods from our mathematical toolbox, which contains both analytical and numerical methods. In practice, there is a significant interplay between finding the right physical formulation and solving the mathematical problem, because our insight in physics, and, in particular, in more general concepts such as conservation laws, often allows us to find short-cuts that lead to an analytical solution. Although, for many problems, and arguably for almost all applied problems, there will never be a simple, analytical solution, and we must depend on our ability to address problems using robust, numerical techniques.

In this book we will take you through this procedure many times. So many times that it becomes deeply rooted in you. And as soon as you have grasped the simplicity of the method, we hope you will keep it a secret—physics is supposed to be difficult, and you are expected to uphold that tradition.

1.3 Integrating Numerical Methods

The most unusual part of this textbook is the integration of numerical and analytical methods into the exposition of theory, examples, and exercises. What do we mean by analytical and numerical methods? Analytical methods are the classical mathematical methods you have learned to use in calculus, giving you an exact analytical solution through derivation, integration, or the solution of differential equations. Numerical methods are a similar set of tools that you may have learned to use to solve the same types of problems on a computer: numerical derivation, numerical integration and numerical solution of differential equations. We have developed this integrated approach because we know that the use of computational methods are going to be important for you—probably more important than the use of analytical techniques; because it allows us to present you with more realistic and inspiring examples and applications; and because it also provides you with a deeper understanding of the underlying mathematics.

The use of computations to solve problems in mathematics and physics is not new. For example, when the famous physicist Richard Feynman introduced planetary motion in his classic lectures at CalTech in 1961, he used a simple numerical scheme to calculate the motion of the planets. However, with the advent of the computer we now have the possibility to do billions of computations per second with ease, and this completely changes the game. We can now solve very complicated problems on any computer—if we only know how. The use of computational methods is becoming increasingly important in most areas in science and engineering, in academia and in industry. Since the ambition of any education is to prepare you for a 40 year working life, we know that you need to master the use of computational methods just as well, if not even better, than you master classical analytical methods—since this is what you will be using to solve problems.

This text is based on the principle that you learn best what you do every day. That is why we have integrated the use of numerical methods into every part of the text—it is part of how we explain the theory, it is part of the examples, and it is part of the exercises you do. However, such an integration requires you to learn a particular programming language. This text comes in two versions, one version based on Python and one version based on Matlab. The text is identical, it is only the parts describing specifics of the programming languages that are different in the two cases. It is an advantage to know some basic scientific programming before reading this text, but it is not necessary—many students have become proficient at programming simply by reading this text, solving the exercises, and discussing with students and tutors.

1.4 Problems and Exercises

This book consists of several types of problems and exercises that have various functions:

Discussion questions: A classical type of problems in physics are called "Fermi" problems named after Enrico Fermi. They are mainly estimation problems of complex questions with many unknowns. The main point of such a problem is not to identify the correct answer—there may be none known—the point is the process of reasoning to find an order of magnitude estimation of the answer. Such problems are well suited for a group discussion. Similar questions have recently become very popular as part of job interviews—since they test how the applicant think and apply her knowledge and reasoning power to address an unknown problem.

Closed, structured problems: This is the classical physics problem. We call the problem "closed" if all the necessary data is given in the problem, and "structured" if the steps to go from the initial problem to the solution are given as subexercises. These problems are popular because the teach problem solving by example and practice by following a structured approach. The idea is that you will learn to do this automatically for yourself if you have done it a sufficient number of times in the exercises.

Open, unstructured problems: When you have practice in solving structured and closed problems, you should be ready for open and unstructured problems. In "open" problems, not all necessary details are given—you have to figure out or decide several key facts yourself. This is the type of problems you will meet in your professional life. Students may initially find these problems frustrating, in particular since they have to introduce many approximations themselves and evaluate whether they are appropriate. However, such problems may also be inspiring, since they allow for more creativity and for more discussions.

Examples of open, closed, structured, and unstructured problems: An open, unstructured problem could be: "A tank is filled with water from a faucet. How long does it take to fill the tank?". The corresponding closed, unstructured problem would be: "A cylindrical tank of diameter 10 cm and height 20 cm is filled with water at a rate of $0.1\,\mathrm{dm}^3/\mathrm{s}$. How long does it take to fill the tank?". The corresponding closed, structured problem would be: "A cylindrical tank of diameter 10 cm and height 20 cm is filled with water at a rate of $0.1\,\mathrm{dm}^3/\mathrm{s}$. (a) What is the area of the base of the tank? (b) What is the volume of the tank? (c) How long time does it take to fill this volume?".

Projects: We provide long, structured problems that require the application of both analytical and numerical methods. In addition, we focus on discussion and evaluation of the results, and evaluation of the approaches and approximation. The idea is that you will learn the work-flow used in actual research through these project. Solving the projects is considered a major objective of this book. Indeed, the text is meant to give you the background to solve these problems.

1.5 How to Learn Physics

We actually know quite a lot about how to learn and how to teach physics. The research area known as Physics Education Research (PER) is well developed, and provide teaching institutions good insights into what methods work and how they work. This knowledge is important for you, since it gives you a research based insight into how you can optimize your time and learn physics as efficiently as possible.

Almost all pedagogical research (on how to learn physics) can be boiled down to a single result: Students learn best when they are deeply involved in doing physics (reading, discussing, solving problems) with material that is adapted to the student, receiving immediate, individualized feedback.

Learning physics is an active, mental process where *you* have to construct the knowledge in your own mind. Reflecting on how you learn is therefore important for your own learning process. One aim of your education in physics is to be able to think like an expert. Experts think about physics using a few general principles that are organized in a hierarchy so that they cover all of physics. Novices tend to think of physics as many, independent results—one for each situation, one for each problem—which makes learning physics an hopeless endeavor in memorizing formulas.

The most effective way to learn physics is to have a private, competent tutor who can adapt the material, provide feedback, individualize explanations, give you problems that are at the appropriate level of difficulty, monitor your thinking by discussions, and help you correct your thinking to learn fruitful mental models. This is actually the way physics is taught at the graduate level.

The second best way is to learn physics in a social setting where you immerse yourself in physics discussions and thinking throughout the day. Discussions with your co-students as well as your teachers will guide you, give you feedback on your thinking, and provide you with direction on how to work. But you will have to provide important parts of the individualization yourself. You have to choose how much of the textbook you should read, you may have to select problems to solve and use solution manuals in useful ways, and you are responsible for your own progress. This is the environment we typically try to create at a college or university.

1.5.1 Advice for How to Succeed

- Everybody can learn physics. Do not believe otherwise. Experiments with one-to-one mentoring shows that large improvements can be gained by attending a proper learning regimen. You can do it—but it will require effort. Take responsibility for your own learning.
- You learn efficiently by trying to frame your understanding of a problem in words—you learn by discussing physics with other students and your teachers. It turns out that it is the one who does the explaining who learns the most. Asking another student about something is therefore an important part of learning, but it is actually the student you ask who benefits the most—even if it does not feel this way. You should therefore seek environments where you participate actively in discussing physics problems: Attend workshops and teaching groups, find a good group of students to work with, use web-forums actively to discuss and formulate your understanding. If you participate actively in making a good learning environment, you will benefit from this yourself.
- You learn from getting feedback on your work. You should therefore grab all chances of getting feedback constructively: Hand in written exercises, demand relevant feedback on the exercises, and use this feedback to guide your own teaching process: Read based on the feedback and choose problems based on the feedback. Try again if you fail.
- Act as your own coach. Set your own targets and measure how well you are doing in reaching them. Prepare for the teaching you participate in. Reading even just 10 min about a subject before attending class will greatly improve your learning outcome. Think about how you learn: What parts of the teaching activities do you find most useful for your learning? What environments help you learn better? How do you learn physics?
- Multitasking does not work—it is a myth. Your mind cannot pay attention to more than one thing at a time. Trying to do more things at once decreases your overall productivity. You should therefore try to work uninterrupted when as you learn new stuff.
- Learning new things "hurts"—you should be able to feel the mental strain as you push yourself to the limits. This text and many others provide solution manuals. Do not use them recklessly! You only fool yourself. Even just glancing at a sketch of a problem can significantly reduce the learning outcome from this problem. You will not have access to this on the exam, and you will definitely not have access to solutions when you solve real world problems. However, you will have access to other examples. Learn to use examples in a constructive manner—initially as templates for how to think and solve a problem, eventually as an automated problem-solving approach that you can apply to any problem.
- A textbook is a perfect adaptive learning tool, since you can choose for yourself what parts to read, when to read it, and how to read it. You can also choose whether you want to start by reading, start from the example, or start from the problems. Use this flexibility wisely, and reflect on how you use it.

1.6 How to Use This Book

This textbook is meant to be used as a stand-alone textbook in physics or as a supplementary text on numerical methods in introductory physics. The book is intended to be read linearly: first you read the text, then your read the examples, and then you solve the problems. However, I realize and even encourage you to choose your own learning strategies. The most important part is what you do in the form of tutorials and exercises, and not what you read. The text can therefore be seen as supporting material for the projects: In order to be able to do the projects, you need to read the text and study the examples. Still, the book has a certain organization, which is based on the knowledge that you may take several paths through the text:

Background: This text requires a course in calculus. It does not require a course in numerical methods and programming, but experience shows that programming requires maturity to master — and an additional course in programming is therefore useful to ensure that the methods learned here become integrated into the toolbox of each student.

Numerical methods: The main exposition of the material, the theoretical explanation, includes both numerical and analytical methods where they are apt. We do not separate them, since they are equally important. However, in addition to the use of numerical methods in the main text, we have added addition material on numerical methods, which is meant to provide a more solid mathematical foundation for the use of numerical methods.

Problem-solving strategies: We introduce a few, robust problem-solving strategies. These are meant to be general templates that you should become so used to, that you use them without thinking about it. Initially, we therefore suggest that you follow these strategies as closely as possible, but as you get more experienced you may take short cuts or automate larger parts of the solution strategies. However, if you are baffled by a problem, you always have the problem-solving strategies to fall back on.

Proofs: Most of the more advanced material and many of the longer derivations and proofs are left out of the text flow. However, you can find relevant derivations and associated mathematical theory at the end of the book or online.

Examples: Each concept can be explained by key worked examples. These are the central examples that may serve as templates for how to address a particular class of physical systems. They are often extended and provide the best background for solving the projects.

Chapter 2
Getting Started with Programming

In this text we integrate the use programming techniques and tools in our study of physics. You will therefore need to know a few programming basics in order to profit from this approach. However, if you do not have a relevant background in introductory scientific programming, do not despair. Experience shows that you can learn to program through your first physics course—many students have done this successfully and with good results. In order to prepare you for the main text, this chapter provides an introduction to programming.

2.1 A Python Calculator

We are in this text using the editor Spyder to work with Python. (Alternatively you can use iPython—type `ipython` in a terminal window to start). When you start Spyder, you get a window where you can type commands to be executed. Click on the `Console window` in the lower right corner and type:

```
>> 9*4
36
```

Notice the difference between the text *you* type, which is preceded by », and the results generated by the program, which are typeset without indentation.

Python can be used as an advanced calculator by typing expressions on the command line:

```
>> 3*2**3+4
28
```

Standard operators are plus (`+`), minus (`-`), multiplication (`*`), division (`/`), and power (`**`). Powers of ten are input using `e`:

```
>> 4.5e4
45000
>> 2.5e-10
2.5e-10
```

which also shows how Python displays numbers.

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_2

Python has most mathematical functions and constants built in, such as `pi`, `cos`, `sin`, and `exp`: To use them we need to load the `pylab` module first:

```
>> from pylab import *
```

After this we are ready to use the mathematical functions:

```
>> 4*pi
12.566370614359172
```

Python uses radians for the trigonometric functions:

```
>> sin(pi/6)
0.49999999999999994
```

As you can see, Python does not always round off as you may expect, even though the answer is close to the exact answer ($sin(\pi/6) = 0.5$). You can find a list of useful syntax, functions and expressions in the summary.

Python becomes more useful when you have a formula you want to use. For example, you may want to use the formula:

$$T_F = \frac{9}{5} T_C + 32 \ , \tag{2.1}$$

to find the temperature, T_F, in Fahrenheit, given the temperature T_C in centigrade. We may type this formula directly into Python

```
>> TF = 9/5*TC + 32
Traceback (most recent call last): File "<stdin >"...
NameError: name "TC" is not defined
```

Ooops. That did not work, because Python does not yet know the value of T_C. We give T_C a value and retype the formula:

```
>> TC = 40.0
>> TF = 9.0/5.0*TC + 32.0
```

To see the answer, type

```
print TF
104.0
```

Note that we are typing `.0` after each number. This is because Python differs between integers $(1, 2, 3, 4, \ldots)$ and real numbers, $(1.0, 2.3, 4.9, \ldots)$. Typing `.0` or only `.` (a dot) after each number ensures that the numbers are real. Also notice that integer division is not the same as real division. With integer division $9/5 = 1$, while with real numbers $9.0/5.0 = 1.8$.

Instead of retyping the formula, you can use the up arrow to find your previously typed commands and execute them again. We have now defined a variable, `TC`. You can see the value given to `TC` by typing:

```
>> print TC
40
```

Notice that we assigned a value to the variable `TF` through a calculation. We have not introduced a function for `TF`. What does this mean? It means that if you change the value of `TC`, the value of `TF` will not change automatically unless you retype the formula for TF. You can check this by assigning TC a new value, and then ask Python for the value of TF:

```
>> TC = 50
>> print TF
104.0
```

This is an important aspect of a programming language such as Python a variable does not change value unless you assign a new value to it!

2.2 Scripts and Functions

However, we do not want to type in the whole formula each time we want to calculate a new value for T_F. Instead we can make a *script*, a group of several statements, or a *function*, similar to an internal function such as `sin`.

Scripts

We can group several statements into a *script*, which we can reuse. You do this by opening a new file in the `File` menu in Spyder: `File` → `New file...`. This opens a new window with an editor. Here you can now type (or copy) the commands we already used:

```
TC = 40.0
TF = 9.0/5.0*TC + 32.0
print TF
```

Now, we need to save the script. In the editor window you do: 'File'→'Save'. You must give the script a name and choose where to place it. This will generate a file with an extension `.py` - we call such a file a py-file, because it shows that the file contains a Python script/program. You run the program from the editor window by typing the `F5` key. A dialog box will show up. Select `Execute in current ...`, leave all other options as they are and click the `Run` button. As a result the commands in the script are executed as if they were typed into the Python window, and the resulting output is shown in the Python window:

```
104.0
```

You can now change the value of TC in the script and rerun the script to redo the calculation for another temperature. Notice that we wrote the script so that the temperature TC is assigned inside the script. This means that if you change the value of TC on the command line, for example by typing:

```
>> TC = 45.0
```

and then run the script—the script will not use this new value of TC, but instead use the value from inside the script.

Writing scripts to solve simple problems will be our standard operating procedure throughout this text. This is an efficient way to develop a simple program, change the parameters (such as changing TC), and rerun the program with new parameters. While this is practical for developing short programs and solving simple problems, it is not good programming practice. In general, we encourage the development of good programming practices, but in this text we will prioritize making the code as simple as possible.

Functions

From a programming perspective, it is better to introduce a *function* to calculate the temperature. A user defined function acts just like a predefined mathematical function such as `sin` or `exp`. We define a function by opening a new py-file: Push `File` → `New File ...`. We define a function by typing the following into the editor:

```
def convertF(TC):
    # Converts from centigrade to Fahrenheit
    TF = 9.0/5.0*TC + 32.0
    return TF
```

and save it with the name `convertF.py`.

What do these statements do? We define a function by the command `def` followed by the name of the function, the input arguments. The line must end with a colon, `:`. Here the only argument is the temperature in centigrades, `TC`. Inside the function we must calculate the value of `TF`, because this is the value the function is supposed to calculate. Finally, we specify that the function will return the value of the variable `TF`.

We call our new function by typing:

```
>> convertF(45.0)
113.0
```

Notice that Python requires each such used-defined function to either be in the same file as the script using the function, of the function must be in a separate file, and that the file must be in the search path for Python. This means that the file `convertF.py` must be in the current directory or in the standard Python directory for this to work. I suggest that you always save the functions you need in the same directory as you save the scripts you are currently working on, and that you make new directories for each problem.

A particular feature of functions is that the internal calculations and variables used inside the function are lost as soon as the function is finished. For example, Python may use several calculation steps if we call the `sin` function, but this is hidden from us. Outside the function, we only see the result of the function. To illustrate this, we could break our short function into several steps:

```
def convertF2(TC):
    # Converts from centigrade to Fahrenheit
    ratio = 9.0/5.0
    constant = 32.0
    TF = factor*TC + constant
    return TF
```

Here, we have introduced two internal variables, `ratio` and `constant`, that are forgotten as soon as the function is finished. For example, if we type:

```
>> convertF2(45.0)
113
>> print ratio
Traceback (most recent call last): File "<stdin >", ...
NameError: name "ratio" is not defined
```

we see that Python does not know the value of `ratio` after the function has done its work.

Functions are powerful and necessary tools of more advanced programming techniques and will be gradually introduced throughout the text. But initially we try to make the programs as simple as possible, and we will therefore use simple scripting as our main tool.

2.3 Plotting Data-Sets

Python not only works as a numerical calculator, it also has advanced data visualization capabilities. For example, as part of a laboratory exercise you may have measured the volume and mass of a set of steel spheres. You number the measurements using the index, i, and record masses m_i and volumes V_i in Table 2.1, where we have used that 1 litre = 1 l = 1 dm^3.

Such a sequence of numbers are stored in an *array* (or a *vector*) in Python. We define the sequence of masses and volumes in Python using

```
>> m = array([1.0,2.0,4.0,6.0,9.0,11.0])
>> V = array([0.13,0.26,0.50,0.77,1.15,1.36])
```

We can find an individual mass value by:

```
>> print m[0]
1.0
>> m[3]
6.0
```

Table 2.1 Measurement of masses m_i as a function of volumes V_i

i	1	2	3	4	5	6
m_i (kg)	1	2	4	6	9	11
V_i	0.13 l	0.26 l	0.50 l	0.77 l	1.15 l	1.36 l

There are now 6 values for the masses, numbered `m[0]` to `m[5]`. Notice that Python starts enumeration at 0, so that 0 is the first element and 5 is the last element. We call the array `m` a 6×1 array or a vector of length 6. The volumes are stored the same way:

```
>> V[0]
0.1300
>> V[3]
0.7700
```

The enumeration of the two arrays is identical: element `m[3]` of the masses corresponds to element `V[3]` of the volumes.

The relation between m and V is illustrated by plotting V as a function of m. This is done by the `plot` command:

```
>> plot(m,V,'o')
```

where the string `'o'` ensures that a small circle is plotted at each data-point. The plot command makes a "scatter" plot—it contains a point for each of the data-points $m(i)$, $V(i)$ in the two arrays. The two arrays must therefore be the same length—they must have the same number of elements. We annotate the axes by:

```
>> xlabel('m (kg)')
>> ylabel('V (l)')
```

where the `xlabel` refers to the first array `m` in `plot(m,V,'o')`, and the `ylabel` refers to the second array—the `V` array. The resulting plot is shown in Fig. 2.1.

Where did the units (kg and liters) go when we defined the mass `m` and the volume `V`? We cannot use units when we introduce digital representations of the numbers. We can only input numbers into Python and we have to keep track of the units. This is why we specified the units along the axes in the `xlabel` and `ylabel` commands.

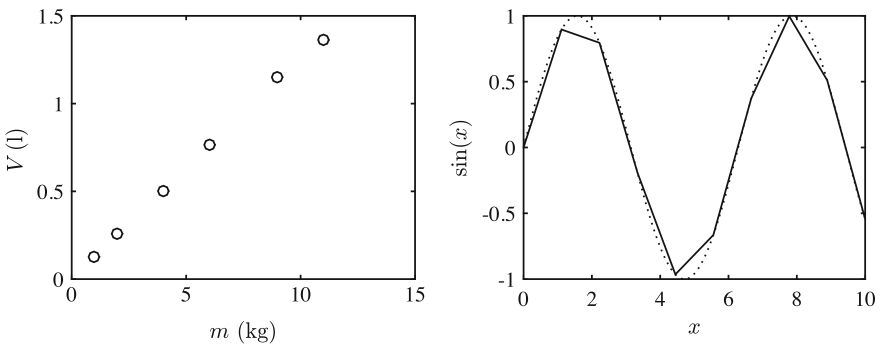

Fig. 2.1 The plot of V as a function of m (*Left*) and with the plot of $\sin(x)$ as a function of x for 10 points (*solid line*) and for 1000 points (*dotted line*) (*Right*)

Table 2.2 Sequence of i, x_i and $\sin(x_i)$

i	1	2	3	4	5	...	n
x_i	0.0	0.1	0.2	0.3	0.4	...	10.0
$\sin(x_i)$	sin(0.0)	sin(0.1)	sin(0.2)	sin(0.3)	sin(0.4)	...	sin(10.0)

2.4 Plotting a Function

Python cannot plot a function such as $\sin(x)$ directly. We must first generate two sequences of numbers, one sequence for the x'es and one sequence for the corresponding values of $\sin(x)$, and then plot the two sequences against each other. While this may sound complicated, Python has functions that ensure that you can almost directly write the mathematical expression into Python.

Loops

We want to make a sequence of x'es, such as 0.0, 0.1, 0.2, 0.3, ... etc., and then for each x_i we want to calculate the corresponding value for $\sin(x_i)$ in Table 2.2, where we generate x_i from 0.0 to 10.0 in steps of 0.1.

How do we generate such an array in Python? First, we have to generate the array.[1] How many elements do we need? Going from 0.0 to 10.0 in steps of 0.1 we need:

$$n = \frac{10.0 - 0.0}{0.1} + 1 \;, \tag{2.2}$$

steps, where we have added one in order to include the last step (otherwise we would stop at 9.9 instead of at 10.0. We define an array of this length by:

```
>> n = int(ceil((10.0-0.0)/0.1)+1)
>> x = zeros((n,1),float)
```

Here, the function `ceil()` rounds up. Notice the use of `int` in order to ensure that we return an integer and not a real. The function `zeros((n,1),float)` generates and returns an array of size `n` by `1` which is filled with zeros. Now we need to fill the array:

```
>> x[0] = 0.0
>> x[1] = 0.1
>> x[2] = 0.2
>> x[3] = 0.3
>> x[4] = 0.4
...
```

[1] In Python it is not necessary to define the size of the array before it is filled. We could just fill it as we go along, but this is not good coding practice, it will lead to very slow codes for large arrays, and may cause surprising errors in your programs. We will therefore always predefine the size of arrays.

Fortunately, there is a more efficient way of doing this—by using a `for`-loop. A `for`-loop allows us to loop through a list of values $0, 1, 2, \ldots, n-1$ for the variable `i`, and then execute a set of commands at each step—exactly what we need. We can replace the long list of `x[0] = 0.0` etc. by the loop:

```
>> for i in range(n):
...    x[i] = i*0.1
```

If you type this in, nothing will execute until you press `Enter` twice. Note also that you need to indent the first line after the colon in the line with the `for`-statement. It is only the set of commands that are indented that are part of the loop and that are run several times. Indenting the line means that you add four spaces to the beginning of the line (compared with `for` in the `for`-statement). Indentation is the way Python recognizes a block of commands.

You can check the generated values of `x` by:

```
>> print x
[[ 0. ]
 [ 0.1]
 [ 0.2]
 [ 0.3]
 [ 0.4]
 [ 0.5]
...
```

Notice how we specify the range of the loop, by specifying a sequence of numbers by use of the `range(n)` function. Typing `range(n)` at the command prompt gives you exactly the list of values for `i`. In Python it is important that `n` is an integer. If we had not been careful to define `n` as an integer above, we would have need to do it here by typing `range(int(n))`.

Now, let us put this into a small script. And let us also calculate the value for the function $\sin(x)$:

```
from pylab import *
x0 = 0.0, x1 = 10.0, dx = 0.1
n = int(ceil((x1-x0)/dx) + 1)
x = zeros((n,1),float)
y = zeros((n,1),float)
for i in range(n):
    x[i] = x0 + i*dx
    y[i] = sin(x[i])
plot(x,y)
show()
```

Which both generates and plots the function $\sin(x)$. The variables `x0`, `x1`, and `dx` provide the start, stop and step of the x-values used. You can now change them and rerun the script to generate plots for other ranges or with other resolutions.

Notice that `y[i] = sin(x[i])` must appear inside the loop—that is before the end. Otherwise it would only be executed once, using the value `i` had at the end of the loop. Putting commands outside a loop that should be inside a loop is a common mistake—sometimes also done by experienced programmers.

The `while`-Loop

The `for`-loop is probably the loop-structure you will use the most, but there are also other tools for making a loop, such as the `while`-loop. In the `while`-loop the commands inside the loop are executed until the expression in the `while` command is true. It does not automatically update a counter either. For example, we can change the script to calculate $\sin(x)$ to use a `while`-loop instead by the following changes:

```
...
y = zeros((n,1),float)
i = 0
while i<=n:
    x[i] = x0 + i*dx
    y[i] = sin(x[i])
    i = i + 1
...
```

Notice that we must assign `i=0` before the loop and `i=i+1` inside the loop, since we now need to update the counter "manually" inside the loop. We also introduce an "expression" `i<=n` which may be false (having the value 0) or true (having the value 1). The loop continues until the expression becomes false. Notice that a common source of error is to generate `while` loops that continue forever, for example because you have forgotten to update the counter inside the loop. You will notice this because your program never ends: Python will never stop or plot your results.

You may wonder what the point of the `while`-loop is, since it looks more cumbersome than the `for`-loop. We will use the `while` loop when we want to continue a calculation for an unknown number of steps. For example, you may want to find the motion of a falling ball up to the point where it hits the ground. However, you may not know beforehand how many steps are needed before it hits the ground. If the position of the ball is $x(t) = 1000 - 4.9\,t^2$, we can calculate the position as a function of time in steps of dt as long as x is positive using the following program:

```
from pylab import *
t0 = 0.0, t1 = 10.0, dt = 0.01
n = int(ceil((t1-t0)/dt) + 1)
t = zeros((n,1),float)
y = zeros((n,1),float)
t[0] = t0
y[0] = 100.0-4.9*t[0]**2
i = 0
while y[i]>0.0:
    i = i + 1
    t[i] = t0 + i*dt
    y[i] = 100.0-4.9*t[i]**2
#stop1
plot(t[0:i],y[0:i])
xlabel('t [s]'); ylabel('y [m]');
```

This script needs some explanation. First, we notice we update both now is `t` and `y` and not only `x` as before. In addition, we see that we calculate the value of `y[0]` before the loop starts, otherwise the loop would never start. This is also a common mistake. Therefore, ensure that you understand why and what is done before the `while`-loop starts in this script.

Vectorization

While loops are generally powerful and useful methods, there is a much simpler way to generate sequences of numbers and plot functions in Python—and the method also allows your code to follow the mathematical formulas more closely. This method is called *vectorization*.

We can make a sequence of x'es in several ways using functions that are built into Python instead of using a loop. For example, the function `linspace` generates a sequence of equidistant numbers from `0.0` to `10.0`:

```
>> x = linspace(0,10,10)
>> print x

[ 0.         1.11111111 2.22222222 3.33333333 4.44444444
  5.55555556 6.66666667 7.77777778 8.88888889 10.       ]
```

In this case we generated 10 numbers, but you are free to choose your own resolution. An alternative to specifying the number of points you want, as we do with `linspace`, is to specify the step size. The expression `r_[0.0:10.0:0.3]` returns an array starting at the value 0 and ending at 10.0 in steps of 0.3[2]:

```
>> x = arange(0.0, 10.0, 0.3)
array([ 0. , 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3. ,
        3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1, 5.4, 5.7, 6. , 6.3,
        6.6, 6.9, 7.2, 7.5, 7.8, 8.1, 8.4, 8.7, 9. , 9.3, 9.6,
        9.9])
```

Ok—so that was simply an easier way of generating the array `x`. Why all the fuzz? Because of a powerful feature of Python called vectorization: we can apply the function $\sin(x)$ to the whole array `x`. Python will then apply the function to each of the elements in `x` and return a new array with the same number of elements as `x`. The three lines:

```
>> x = linspace(0,10,10)
>> y = sin(x)
>> plot(x,y), show()
```

are equivalent to the program:

```
from pylab import *
x0 = 0.0
x1 = 10.0
dx = 1.0
n = int(ceil((x1-x0)/dx) + 1)
x = zeros((n,1),float)
y = zeros((n,1),float)
for i in range(n):
    x[i] = x0 + (i-1)*dx
    y(i) = sin(x[i])
plot(x,y)
show()
```

[2] Notice the small difference between the two methods: using `linspace` ensures that the first and the last numbers are included in the list, but when you use `arange(0.0, 10.0, 0.3)` the last number is 9.9 and not 10.0!.

Notice how simple the vectorized code is—it is almost identical to the mathematical formula. We only have to define the range of `x`-values before we call the `sin(x)` function. Beautiful and powerful.

The program above generates the solid line in Fig. 2.1. However, this plot has too sharp corners, because we have too few data-points. Let us generate 1000 x-values and plot $\sin(x)$ with this resolution in the same plot:

```
>> x = linspace(0,10,1000)
>> y = sin(x)
>> hold('on')
>> plot(x,y,':');
>> hold('off')
>> show()
```

The result is shown in Fig. 2.1 with a dotted line. Here, we have used a few more tricks. We use the command `hold('on')` to ensure that Python does not generate a new plot, which would remove the previous one, but instead plots the data in the same plot as we have already used. Typing `hold('off')` stops this behavior—otherwise all subsequent plots will be part of the same plot. We have also used the string `':'` to tell Python that we want a dotted line. You can find more plotting methods in the summary at the end of the chapter.

The vectorization technique is very general, and usually allows you to translate a mathematical formula to Python almost formula by formula. For example, we plot the function

$$f(x) = x^2 e^{-ax} \sin(\pi x) , \tag{2.3}$$

from $x = 0$ to $x = 10$ by typing (when $a = 1$):

```
>> x = linspace(0,10,1000)
>> a = 1.0
>> f = x**2*exp(-a*x)*sin(pi*x)
>> plot(x,f), show()
```

As soon as you have learned to transcribe mathematical expressions from the mathematical notation to Python you are ready to calculate and plot any function.

The technique of vectorization is a powerful and efficient technique. Python is usually very fast at calculating vectorized commands. And we can write very elegant programs using such techniques, ensuring that the Python code follows the mathematical formulation closely, which makes the code easy to understand.

2.5 Random Numbers

Sometimes you need randomness to enter your physical simulation. For example, you may want to model the motion of a tiny dust of grain in the air bouncing about due to random hits by the air molecules, so called Brownian motion. We model this by having the grain move a *random* distance during a given time interval. How do we create random numbers on the computer? Unfortunately, we cannot generate really random numbers, but most programs have decent pseudo-random number generators, which generate a sequence of numbers that appear to be random. In Python we can simulate the throw of a dice using

```
>> randint(6) + 1
3
```

where `randint(n)` generates a random integer between 0 and $n - 1$, where each outcome has the same probability. If you type the command several times, you will get a new answer each time. Python includes several functions that return random numbers: It can generate random real numbers between 0 and 1 using the `rand` function and normal-distributed numbers (with average 0 and standard derivation 1) using the function `randn`.

2.6 Conditions

Now, if we return to discuss the motion of the grain of dust, we want to model its motion according to a simple rule: I throw a dice. If I get between 1 and 3, the grain moves a step forward, otherwise it moves a step backward. How can we handle such conditions? We need a set of conditional statements, so that we can perform a given set of commands when a particular condition is fulfilled. We need an `if`-statement:

```
if (expr):
    <statement a1>
    <statement a2>
    ..
else:
    <statement b1>
    <statement b2>
    ..
```

Here the expression `(expr)` is an expression such as `randint(6)+1>3` which may be true or false. If the expression is true, statements `a1`, `a2`, ... are executed, otherwise the statements `b1`, `b2`, ... are executed.

Let us use this to find the motion of the grain. Every time we throw the dice, the grain moves a distance $dx = \pm 1$. If the grain is at position x_i at step i, the grain will be at a position

$$x_{i+1} = x_i + dx \ . \tag{2.4}$$

at step $i + 1$. We can use this rule and an `if`-statement to write the script to find the position at subsequent steps $i = 0, 1, 2, \ldots$:

```
from pylab import *
n = 1000
x = zeros(n,float)
for i in range(n):
    if (randint(6)+1<=3):
        dx = -1
    else:
        dx =  1
    x[i+1] = x[i] + dx
```

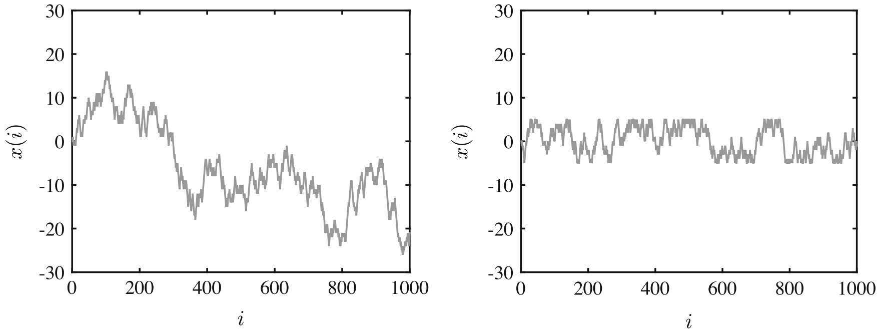

Fig. 2.2 Plot of the position $x(i)$ of a random walker (a bouncing grain of dust) as a function of the number of steps i done (*left*), and when the walker is constrained to the zone $-5 \le x \le +5$ (*right*)

The resulting motion is shown in Fig. 2.2.

We will use `if`-statements throughout the text, often to enforce particular conditions on the motion. For example, we could add a level of complexity to the motion of the grain by requiring that the grain moved inside a narrow channel of width 10: The grain cannot move outside a region spanning from −5 to +5:

```
#start1
from pylab import *
n = 1000
x = zeros(n,float)
for i in range(1,n):
    if (randint(6)+1<=3):
        dx = -1
    else:
        dx = +1
    x[i] = x[i-1] + dx
    if (x[i]> 5):
        x[i] =  5
    if (x[i]<-5):
        x[i] = -5
#end1
plot(x),
xlabel('i'), ylabel('x(i)')
show()
```

The resulting motion x_i as a function of i is shown in Fig. 2.2.

For the interested reader, we include a particularly compact formulation of the random walk, which you may have fun trying to understand.

```
>> x = cumsum(2*(randint(1,7,1000)<=3)-1)
>> plot(x), show()
```

You can research this expression by using the `help`-function in Python

```
>> help(cumsum)
```

2.7 Reading Real Data

When you work with physics you need to handle real data: NASA publishes data for the motion of most stellar objects; your mobile phone has an accelerometer; and a GPS that measures thousands of data-points in a few seconds. You do not want to type in these numbers by hand. Therefore you need to be able to read files containing numbers. For example, the motion of a sprinter running 100 m is given in the file 100m.d.[3] The file looks like this if you open it in a text editor (such as `emacs`):

```
0.0000000e+000 -2.1155775e-001
1.0000000e-002 -1.7485406e-001
2.0000000e-002 -1.3798607e-001
3.0000000e-002 -1.0095306e-001
4.0000000e-002 -6.3754256e-002
5.0000000e-002 -2.6388915e-002
...
```

A total of 972 lines of data. The first column gives the time, measured in seconds, and the second column gives the position of the runner, measured in meters. Fortunately, it is very simple to read such as file into Python. It is done by a single command:

```
>> run100m = loadtxt("run100m.d")
```

We split the data into two arrays `t` and `x` by:

```
>> t = run100m[:,0]
>> x = run100m[:,1]
```

and plot the data using

```
>> plot(t,x), show()
```

If you experience a problem where Python cannot find the file, getting an error message like:

```
>> loadtxt("run100m.d")
... IOError: [Errno 2] No such file or directory: "run100m.d"
```

It means that the file `run100m.d` is not in your current working directory.

2.7.1 Example: Plot of Function and Derivative

Problem: Plot the function

$$f(x) = e^{-x^2} , \tag{2.5}$$

and its derivative by using the formula:

$$f'(x) \simeq \frac{f(x+h) - f(x-h)}{2h} , \tag{2.6}$$

[3] http://folk.uio.no/malthe/mechbook/100m.d.

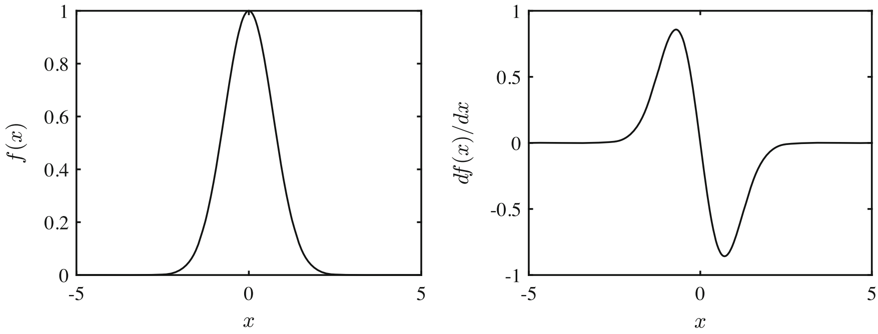

Fig. 2.3 Plot of $f(x)$ as a function of x and its derivative df/dx as a function of x calculated using a numerical method

as an approximation for the derivative on the interval $-5 \leq x \leq 5$. You may use the value $h = 0.001$ for h.

Solution: The function can be plotted directly by a vectorized approach:

```
>> from pylab import *
>> x = linspace(-5,5,1000)
>> f = exp(-x**2)
>> plot(x,f), show()
```

In order to use the numerical approximation for the derivative, we need to perform the approximation for each of the x-values in the `x`-array. We access them by a `for`-loop through the 1000 elements in the `x`-array:

```
>> h = 0.001
>> df = zeros(1000,float)
>> for i in range(1000):
...     df[i] = (exp(-(x[i]+h)**2)-exp(-(x[i]-h)**2))/(2*h)
>> plot(x,df), show()
```

The resulting plot is shown in Fig. 2.3.

Even simple problems such as these are useful to implement as scripts saved in a file, since this makes debugging—the process of finding and removing errors in the script—simpler. If you make a small mistake, you have to retype all the commands when you operate on the command line, but if you use a script, you simply make a small change in the script, rerun, and that is it.

Summary

Using Python as a calculator:

- Direct calculations are done on the command line

  ```
  >> 10.0*sin(pi/3)+4.0**3
  ```

- Defining and reusing variables

  ```
  >> a = 2.0, b = 4.5, c = a**2 + b**2
  ```

- Vectorized plotting of functions

```
>> x=linspace(0,10,0.01), y=exp(-x)*sin(x),plot(x,y),show()
```

- Vectorized operations are denoted by a leading `.` and are done *element-wise*.

Functions and scripts:

- A script is a sequence of executable commands stored in a separate `.py`-file
  - All variables are available on from the command line afterwards
  - The script is run by typing `F5` in the editor
  - Scripts allow rapid rerunning a program after changes in parameters
- A function can be part of a script of stored in a separate `.py` file. It has the syntax

```
def myfunction(a,b,c):
v = a*b*c
d = v**2
y = 2.0*d
return y
```

- The name of the function (`myfunction`) should be the name of the `.py`-file. Variables defined inside the function, such as `v` and `d` are not available outside the function

Plotting:

- You plot two arrays `t` and `x` versus each other by

```
plot(t,x,'-b'), xlabel('t (s)'), ylabel('x (m)')
```

- Plotting markers and symbols are listed in Table 2.3.
- Plotting several data-sets in the same plot:

```
plot(t1,x1,'-b',t2,x2,'-r')
```

Table 2.3 Line markers, colors and plotting symbols

Colors		Lines		Symbols		Symbols	
b	blue	–	solid	.	point	v	triangle (down)
g	green	:	dotted	o	circle	^	triangle (up)
r	red	-.	dashdot	x	x-mark	<	triangle (left)
c	cyan	–	dashed	+	plus	>	triangle (right)
m	magenta	(none)	no line	*	star	p	pentagram
y	yellow			d	diamond	h	hexagram
k	black						
w	white						

or

```
plot(t1,x1,'-b')
hold('on'), plot(t2,x2,'-r'), hold('off')
```

- Combining several plots into one figure:

```
subplot(2,1,1), plot(t1,x1,'-b')
subplot(2,1,2), plot(t2,x2,'-r')
```

- Saving a figure to a file: either by using the save button from the figure window, or you can save a figure as a pdf from the command line by

```
savefig('myfigure.pdf')
```

where `myfigure.pdf` is the name of the generated file.

Loops:

- `for`-loops run a counter sequentially through a list of values:

```
for i in range(100):
  x[i] = sin(i/100.0)
```

- `while`-loops run until a given expression is true

```
i = 0 while (i<100):
  i = i + 1
  x[i] = sin(i/100.0)
```

Expressions:

- `if`-statements are used to run a sequence of commands given a particular expression is true:

```
if (x>10.0):
  y =  10.0
else:
  y = -10.0
```

- Expressions return true (1) or false (0) and can be joined using logical operators such as *or* and *and* as shown in Table 2.4

Table 2.4 Expressions and operators used in Python

Expression	Name	Example	Operator	Name	Example
`==`	equal	`(x==0.0)`	`and`	logical AND	`(x==0.0)and (y>0.0)`
`!=`	not-equal	`(x!=0.0)`	`or`	logical OR	`(x!=0.0)or (y>0.0)`
`>=`	greater than or equal	`(x>=0.0)`			
`<=`	less than or equal	`(x<=0.0)`			
`>`	greater than	`(x>0.0)`			
`<`	less than	`(x<0.0)`			

Exercises

2.1 Seconds.
(**a**) Write a script that calculates the number of seconds, s, given the number of hours, h, according to the formula $s = 3600\,h$.
(**b**) Use the script to find the number of seconds in 1.5, 12 and 24 h.

2.2 Spherical mass.
(**a**) Write a script that calculates the mass of a sphere given its radius r and mass density ρ according to the formula $m = (4\pi/3)\ \rho r^3$.
(**b**) Use the script to find the mass of a sphere of steel of radius $r = 1$ mm, $r = 1$ m, and $r = 10$ m.

2.3 Angle.
(**a**) Write a function that for a point (x, y) returns the angle θ from the x-axis using the formula $\theta = \arctan(y/x)$.
(**b**) Find the angles θ for the points $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$.
(**c**) How would you change the function to return values of θ in the range $[0, 2\pi]$?

2.4 Unit vector.
(**a**) Write a function that returns the two-dimensional unit vector, (u_x, u_y), corresponding to an angle θ with the x-axis. You can use the formula $(u_x, u_y) = (\cos\theta, \sin\theta)$, where θ is given in radians.
(**b**) Find the unit vectors for $\theta = 0, \pi/6, \pi/3, \pi/2, 3\pi/2$.
(**c**) Rewrite the function to instead take the argument θ in degrees.

2.5 Plotting the normal distribution. The normal distribution, often called the Gaussian distribution, is given as:

$$P(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} , \tag{2.7}$$

where μ is the average and σ is the standard deviation.
(**a**) Make a function `normal(x,mu,sigma)` that returns the normal distribution value, $P(x, \mu, \sigma)$ as given by the formula.
(**b**) Use this function to plot the normal distribution for $-5 < x < 5$ for $\mu = 0$ and $\sigma = 1$.
(**c**) Plot the normal distribution for $-5 < x < 5$ for $\mu = 0$ and $\sigma = 2$ and for $\sigma = 0.5$ in the same plot.
(**d**) Plot the normal distribution for $-5 < x < 5$ for $\sigma = 1$ and $\mu = 0, 1, 2$ in three subplots above each other.

2.6 Plotting $1/x^n$. The function $f(x; n)$ is given as $f(x; n) = x^{-n}$.
(**a**) Make a function `fvalue(x,n)` which returns the value of $f(x; n)$.
(**b**) Use this function to plot $1/x$, $1/x^2$ and $1/x^3$ in the same plot for $-1 < x < 1$.

2.7 Plotting $\sin(x)/x^n$. The function $g(x;n)$ is given as:

$$g(x;n) = \frac{\sin(x)}{x^n} \ . \tag{2.8}$$

(**a**) Make a function `gvalue(x,n)` which returns the value of $g(x;n)$.
(**b**) Use this function to plot $\sin(x)/x$, $\sin(x)/x^2$ and $\sin(x)/x^3$ in the same plot for $-5 < x < 5$.
(**c**) Use the help function to find out how to place legends for each of the plots into the figure.

2.8 Logistic map. The iterative mapping $x(i+1) = r\,x(i)\ (1 - x(i))$ is called the logistic map.
(**a**) Make a function `logistic(x,r)` which returns the value of $x(i+1)$ given $x(i)$ and r as inputs.
(**b**) Write a script with a loop to calculate the first 100 steps of the logistic map starting from $x(1) = 0.5$. Store all the values in an array `x` with $n = 100$ elements and plot x as a function of the number of steps i:
(**c**) Explore the logistic map for $r = 1.0$, 2.0, 3.0 and 4.0.

2.9 Euler's method. In mechanics, we often use Euler's method to determine the motion of an object given how the acceleration depends on the velocity and position of an object. For example, we may know that the acceleration $a(x,v)$ is given as $a(x,v) = -kx - cv$. If we know the position x and the velocity v at a time $t = 0$: $x(0) = x_0 = 0$ and $v(0) = v_0 = 1$, we can use Euler's method to find the position and velocity after a small timestep Δt:

$$v_1 = v(t_0 + \Delta t) = v(t_0) + a(v(t_0), x(t_0))\Delta t \tag{2.9}$$
$$x_1 = x(t_0 + \Delta t) = x(t_0) + v(t_0)\Delta t \tag{2.10}$$
$$v_2 = v(t_1 + \Delta t) = v(t_1) + a(v(t_1), x(t_1))\Delta t \tag{2.11}$$
$$x_2 = x(t_1 + \Delta t) = x(t_1) + v(t_1)\Delta t \tag{2.12}$$

and so on. We can therefore use this scheme to find the position $x(t)$ and the velocity $v(t)$ as function of time at the discrete values $t_i = i\Delta t$ in time.
(**a**) Write a function `acceleration(v,x,k,C)` which returns the value of $a(x,v) = -kx - Cv$.
(**b**) Write a script that calculates the first 100 values of $x(t_i)$ and $v(t_i)$ when $k = 10$, $C = 5$, and $\Delta t = 0.01$. Plot $x(t)$, $v(t)$, and $a(t)$ as functions of time.
(**c**) What would you need to change to instead find $x(t)$ and $v(t)$ is the acceleration was given as $a(v,x) = k\sin(x) - Cv$?

2.10 Throwing two dice. You throw a pair of six-sided dice and sum the number from each of the dice: $Z = X_1 + X_2$, where Z is the sum of the results from dice 1, X_1, and dice 2, X_2. If we perform this experiment many times (N), we can find the average and standard deviation from standard estimators from statistics. The average, $\langle Z \rangle$, of Z is estimated from:

$$\langle Z \rangle = \frac{1}{N} \sum_{j=1}^{N} Z_j \ , \tag{2.13}$$

and the standard deviation, ΔZ, is estimated from:

$$\Delta Z = \frac{1}{N-1} \left(\sum_{j=1}^{N} (Z_j - \langle Z \rangle \right)^2 \ . \tag{2.14}$$

(**a**) Write a function that returns an array of N values for Z.
(**b**) Write a function that returns an estimate of the average of an array `z` using the formula provided.
(**c**) Write a function that returns an estimate of the standard deviation of an array `z` using the formula provided.
(**d**) Find the average and standard deviation for $N = 100$ throws of two dice.

2.11 Reading data. The file trajectory.dat[4] contains a list of numbers:

```
t0 x0 y0
t1 x1 y1
.. .. ..
tn xn yn
```

corresponding to the time `t(i)` measured in seconds, and the x and y positions `x(i)` and `y(i)` measured in meters for the trajectory of a projectile.
(**a**) Read the data file into the arrays ‘t, `x`, and `y`.
(**b**) Plot the x and y positions as function of time in two plots above each other.
(**c**) Plot the (x, y) position of the object in a plot with x and y on the two axes.

2.12 Numerical integration of a data-set. The file velocityy.dat[5] contains a list of numbers:

```
t0 v0
t1 v1
.. .. ..
tn vn
```

corresponding to the time `t(i)` measured in seconds, and the velocity `y(i)` measured in meters per second for the trajectory of a projectile.
(**a**) Read the data file into the arrays `t`, and `v`.
(**b**) Plot $v(t)$ as function of time.

---

[4] http://folk.uio.no/malthe/mechbook/trajectory.dat.
[5] http://folk.uio.no/malthe/mechbook/velocityy.dat.

For a data-set `t(i)`, `v(i)`, you can estimate the function corresponding to the integral of $v(t)$ with respect to t at the times t_i using the iterative scheme:

$$y(t_1) \simeq y(t_0) + v(t_0)\,(t_1 - t_0) \tag{2.15}$$
$$y(t_2) \simeq y(t_1) + v(t_1)\,(t_2 - t_1) \tag{2.16}$$
$$\ldots \simeq \ldots \tag{2.17}$$
$$y(t_n) \simeq y(t_{n-1}) + v(t_{n-1})\,(t_n - t_{n-1}) \tag{2.18}$$

where $v(t_i)$ =‘v(i)’ and t_i =‘t(i)’ You can assume that the motion starts at $y(t_0) = 0.0$m at $t = t_0$.
(**c**) Write a script to calculate the time integral $y(t_i)$ of the dataset using this formula. Implement using a `for`-loop.
(**d**) Plot the position $y(t)$ and the derivative $v(t)$ as functions of time in two plots above each other.

Chapter 3
Units and Measurement

In physics we study nature quantitatively—we describe nature with numbers. But not with numbers alone. We need to relate numbers to the physical world using measurement units. For example, you could measure the width of your desk using a pencil as in Fig. 3.1, finding that the width is 6 times the length of your pencil and 2.73 times the length of your shoe. The number alone, 6 or 2.73, does not make any sense in the physical world without the unit: a pencil or a shoe. Using a pencil or a shoe may be practical for you, but not if you want to communicate your result to someone else. Therefore we need standardized units.

3.1 Standardized Units

Here we use the standard units provided by SI[1]—the Systeme International—that provide a few precisely defined *base units*. For example, the standard unit for length in the SI system is called 1 m. To ensure that this standard is the same everywhere there is a standard meter bar, made of platinum-iridium, which is kept at the International Bureau of Weights and Measures outside Paris. Accurate copies of this bar have been sent to various national standards laboratories around the world.

All other units are derived from the base units in Table 3.1. For example, the unit for force is Newton and the unit for energy is Joule. Both are defined as combinations of the base units in the SI system:

$$1\,\mathrm{Newton} = 1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}\ , \tag{3.1}$$

$$1\,\mathrm{Joule} = 1\,\mathrm{J} = 1\,\mathrm{N\,m} = 1\,\mathrm{kg\,m\,s^{-2}\,m} = 1\,\mathrm{kg\,m^2\,s^{-2}}\ . \tag{3.2}$$

[1] http://en.wikipedia.org/wiki/International_System_of_Units.

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_3

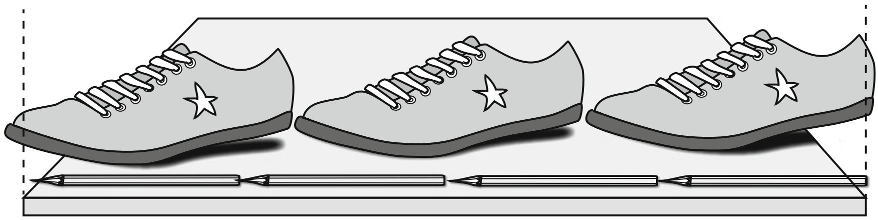

Fig. 3.1 Measuring the width of a table with a pencil and a shoe

Table 3.1 SI units and standard prefixes

SI units			Standard prefixes		
Quantity	Unit	Symbol	Factor	Prefix	Symbol
Length	meter	m	10^{12}	tera-	T
Time	second	s	10^{9}	giga-	G
Mass	kilogram	kg	10^{6}	mega-	M
Amount of substance	mole	mol	10^{3}	kilo-	k
Electric current	ampere	A	10^{-1}	deci-	d
Temperature	kelvin	K	10^{-2}	centi-	c
Luminous intensity	candela	cd	10^{-3}	milli-	m
			10^{-6}	micro-	μ
			10^{-9}	nano-	n
			10^{-12}	pico-	p
			10^{-15}	femto-	f

This high level of precision may seem unnecessary, but units were no joke for the NASA teams responsible for the safe landing of the Mars Climate Orbiter.[2] The Orbiter was launched in December 1998, and NASA lost contact with it on September 23rd, 1999. The Orbiter was last heard from as it approached Mars way too close to the surface. It turned out that the two groups in NASA working on the approach of the Orbiter used different measurement units. One group used English units (feet and pounds) and the other group used metric units (meters and kilograms), and they had not communicated this difference clearly. The result was devastating for the Orbiter and now serves as a warning for all of us: Keep track of your units!

[2] http://en.wikipedia.org/wiki/Mars_Climate_Orbiter.

Numerical Notation

Measured quantities in the SI system may be very large or very small. For example, the velocity of light in the SI system is:

$$c = 299792458\,\mathrm{m/s}\;, \tag{3.3}$$

and the length of the E. coli bacteria is

$$l = 0.000001\,\mathrm{m}\;. \tag{3.4}$$

A more practical way to represent these numbers is to use scientific notation, where we use a base of 10. The velocity of light is then:

$$c = 2.9979248 \;\times\; 10^8\,\mathrm{m/s}\;, \tag{3.5}$$

and the length of the E. coli bacteria is

$$l = 1 \;\times\; 10^{-6}\,\mathrm{m}\;. \tag{3.6}$$

We no longer need to count the number of digits to see the order of magnitude of the number. This notation is also used by Python where we give a number by a decimal number followed by `e` and the number x of tens, corresponding to the exponent x in 10^x:

```
>>> l = 0.000001
>>> l
9.9999999999999995e-07
```

(Notice also the problem with numerical representation of numbers in this example). Similarly, we use scientific notation when we input numbers in Python

```
>>> u = 1.25e6
>>> u
1250000.0
```

Prefixes

To make life simpler, there are prefixes for various standard factors so that we do not have to write out the power of tens each time. Instead of writing

$$0.001\,\mathrm{m} = 1.0 \times 10^{-3}\,\mathrm{m}\;, \tag{3.7}$$

we write

$$0.001\,\mathrm{m} = 1\,\text{millimeter} = 1\,\mathrm{mm}\;. \tag{3.8}$$

Similarly, we write

$$1\,\mathrm{nm} = 1 \times 10^{-9}\,\mathrm{m} \;, \tag{3.9}$$

and the well-known kilometer:

$$1\,\mathrm{km} = 1 \times 10^{3}\,\mathrm{m} = 1000\,\mathrm{m} \;. \tag{3.10}$$

We need a large span of prefixes because our physical world spans over a large range of scales.

3.2 Changing Units

There are many different units used to describe real world quantities: We may measure speed in meters per second, kilometers per hour or feet per second. How do we translate between them? Conversion is done by substituting equivalent quantities represented by different units.

The method is best demonstrated by example. For example, the speedometer in your car reports the speed to be 60 kilometers per hour (km/h). How can you convert this to meters per second (m/s)? This is done by replacing km with meters and hours with seconds by rewriting the units: $1\,\mathrm{km} = 1000\,\mathrm{m}$ and $1\,\mathrm{h} = 3600\,\mathrm{s}$:

$$60\,\frac{\mathrm{km}}{\mathrm{h}} = 60\frac{1000\,\mathrm{m}}{3600\,\mathrm{s}} = 16.6\,\frac{\mathrm{m}}{\mathrm{s}} \;. \tag{3.11}$$

What about the reverse scheme? If the wind is blowing 20 meters per second (m/s), what does this correspond to in kilometers per hour (km/h)? We use the same technique as above, although we need an intermediate step to write 1 m in terms of km and 1 s in terms of hours:

$$1\,\mathrm{km} = 1000\,\mathrm{m} \;\Rightarrow\; \frac{1}{1000}\,\mathrm{km} = 1\,\mathrm{m} \;; , \tag{3.12}$$

and similarly:

$$1\,\mathrm{h} = 3600\,\mathrm{s} \;\Rightarrow\; \frac{1}{3600}\,\mathrm{h} = 1\,\mathrm{s} \;. \tag{3.13}$$

We use these expressions to convert from m/s to km/h:

$$20\,\frac{m}{s} = 20\frac{\frac{1}{1000}\,\mathrm{km}}{\frac{1}{3600}\,\mathrm{h}} = 20\,\frac{3600}{1000}\,\frac{\mathrm{km}}{\mathrm{h}} = 72\,\frac{\mathrm{km}}{\mathrm{h}} \;. \tag{3.14}$$

3.3 Uncertainty and Significant Digits

Measured quantities are never exact, but have some uncertainty, which depends on the measurement method. We specify the uncertainty of a measurement by giving a range of values for the measured quantity. If a length is given as 25.0 ± 0.5 m, it means that length most probably is between $25.0 - 0.5$ m and $25.0 + 0.5$ m. (For quantities with a known distribution of values, we typically describe the range by the standard deviation of the measured quantity).

In practice we use an even briefer way of specifying the uncertainty—by only giving the *significant digits* when we write down the numerical value of a quantity. For a distance of 25.0 ± 0.5 m it does not make sense to provide many more digits for the length. Writing 25.0000 ± 0.5 m clearly overstates the precision, since we do not really know whether the number is 24.5 m or 25.5 m. Instead we use the number of digits to indicate the uncertainty. We only provide the number of digits we are certain of—the significant digits. Instead of writing $d = 25.0 \pm 0.5$ m we write $d = 25$ m. This would imply the same: we do not really know whether it is 25.5 m or 24.6 m, but we know that it is not 22 m. Standard practice is that the last digit provided may be uncertain. If you write $d = 25$ m is means that the value could be $d = 24$ m or $d = 26$ m, and if you write $d = 25.0$ m it means that the value could be $d = 25.05$ m, but is probably not $d = 25.15$ m.

This implied uncertainty is why you should never report the full numerical values you get from your calculator or your program. Your program returns a lot of digits—as many digits as it stores—but these digits may not be significant. You therefore need to ensure that you report only the number of significant digits. But how do you know that? You know it because the uncertainty of the result of your calculations will depend on the uncertainty of the numbers you put in. Your calculations cannot improve the uncertainty! The results you report must therefore always reflect the number of significant digits in the data you start with. And it is the number with the largest uncertainty that determines the final uncertainty.

You will learn more formally about how to handle uncertainties in complex calculations in your laboratory courses, but for now you can use the following rules of thumb for handling the number of significant digits:

Multiplication: The number with the least number of significant digits determines the number of significant digits of the result

Addition: It is the position of the decimal point in each of the numbers that determines the uncertainty.

For *addition*, the position of the decimal point and not the number of significant digits determines the uncertainty of the result:

$$3.4\,\text{mm} + 10\,\text{mm} = 13\,\text{mm} , \tag{3.15}$$

$$1000.00\,\text{m} + 5\,\text{m} = 1005\,\text{m} , \tag{3.16}$$

$$1\,\mathrm{km} + 10\,\mathrm{m} = 1\,\mathrm{km} + 0.010\,\mathrm{km} = 1\,\mathrm{km}\ , \tag{3.17}$$

$$1.000\,\mathrm{km} + 10\,\mathrm{m} = 1.000\,\mathrm{km} + 0.010\,\mathrm{km} = 1.010\,\mathrm{km}\ , \tag{3.18}$$

where you see how we lose small numbers due to uncertainty in the addends.

For *multiplication*, the number of significant digits in the result cannot be larger than in any of the factors:

$$2.10101\,\mathrm{N} \times 4.0\,\mathrm{m} = 8.4\,\mathrm{Nm}\ , \tag{3.19}$$

$$2\,\mathrm{kN} \times 4.400\,\mathrm{m} = 9\,\mathrm{Nm}\ , \tag{3.20}$$

$$2\,\mathrm{kN} \times 4.000\,\mathrm{m} \times 2.600\,\mathrm{s} = 2 \times 10^1\,\mathrm{Nms}\ . \tag{3.21}$$

3.4 Numerical Representation

Several challenges arise when we use a computer to solve physics problems. First, the computer only stores numbers and not the units. Therefore you need to keep track of the units. Second, your programs always provide more digits than the number of significant digits, so you have to keep track of the number of significant digits. Third, the computer introduces new errors by itself—errors due to the digital representation of numbers, and errors that are inherent in the algorithms we use to solve a problem.

Units on the Computer

There are no units in your program, so you need to keep track of the units yourself. If you calculate the velocity of a car driving 2.0 km in 10.0 s, you find:

```
>>> x = 2.0
>>> t = 10.0
>>> v = x/t
>>> v
0.20000000000000001
```

It is your job to know that the units for the answer is m/s. Because you could have done exactly the same calculation with $x = 2.0\,\mathrm{km}$ and $t = 10.0\,\mathrm{s}$, and the answer would instead be 0.2 km/s. Or you could have done the calculation $x = 2.0\,\mathrm{nm}$ and $t = 10.0\,\mathrm{ms}$, and the result would have been $v = 0.2\,\mathrm{nm/ms}$.

How to ensure that you keep track of the units correctly? We advice you to write the units into your code when you define numbers. In your script, you could write:

```
x = 2.0 # m
t = 10.0 # s
```

This does not solve the problem, but this is a habit that makes it easier for you to spot the units.

In addition, we advice you to always use units on the axes in your plots. Indeed, we argue that a plot in physics is not complete without units on the axes. Make this a habit from the start!

Finally, as you become more proficient in numerical methods, you should also learn how to rewrite your equations in non-dimensional form.

Too Many or Too Few Digital Digits

Your program tends to return a long sequence of numbers irrespective of the number of digits you put in: It does not care about significant numbers as do we:

```
>>> x = 10.0;
>>> t = 3.0;
>>> x/t
3.3333333333333335
```

It is therefore always your job not to report the numbers you get out directly, but to apply the rules of the number of significant digits before you report the calculated value.

Digital Representation of Numbers

The numbers you use in your program can be of various *types* and are stored digitally in various ways.

Numbers in computers are stored in units of 4 or 8 bytes, which is the size most efficiently handled by the hardware. One byte corresponds to an 8-bit binary number. One bit is a binary number, which means that it is either 0 or 1. With 8 bits we can make 2^8 different numbers. For example, we could enumerate the numbers from 0 to $255 = 2^8 - 1$. You may use binary as a data type in your programs, but you will usually use either integers, called *integers*, or real numbers, called *floating point numbers*.

Integers

Integers are usually either represented by 4 or 8 bytes of computer memory. This means that there is a maximum number that can be represented. If you use a 4-byte integer the integers can range from $-2^{31} = -2\,147\,483\,648$ to $2^{31} - 1 = 2\,147\,483\,647$. However, in Python your programming language will sort out problems if you exceed this number by automatically switching to a data type such as a 8 byte integer.[3]

[3] In other programming languages, type switching is not standard, and you may "run out of integers" with surprising results.

Notice that if you define two numbers as integers, for example, by introducing them as integers when you assign values to the variables, the operations between two such variables will be integer operations. For example, division will be integer division, and will return an integer:

```
>>> a = 4
>>> b = 3
>>> a/b
1
```

Therefore, you should make it a habit of adding a `.0` to all numbers when you assign variables, unless you are sure you want to define an integer.

Floating-Point Numbers

While floating-point numbers are supposed to represent real numbers, they also have a finite digital precision, corresponding to the number of possible values that can be spanned by the 4 or 8 bytes used to represent the floating-point number.

Floating point numbers are represented by the (significant) digits, and an exponent, using the method that we called scientific notation.

For a 4-byte floating-point number, the IEEE[4] standard determines that 23 bits are used for the digits and 9 bits are used for the exponents - including the signs for both. 4-byte floating point numbers therefore have 6-9 (significant) digits, and cover a range from -3.4×10^{38} to -1.4×10^{-45} for negative numbers, and from 1.4×10^{-45} to 3.4×10^{38} for positive numbers.

However, you will not encounter 4-byte numbers when programming in Python since by default the floating-point numbers are 8-byte. In this case, numbers have 15-17 (significant) digits, and cover a range from -1.8×10^{308} to -5×10^{-324} for negative numbers, and from 5×10^{-324} to 1.8×10^{308} for positive numbers. With 8-byte floating-point numbers you will seldom encounter cases where you are limited by the numerical precision. For all practical purposes you will be limited by the number of significant digits in your measurements. While the number of significant digits usually does not pose a practical problem, we still need to remember that the floating-point numbers only have a given resolution. Not all numbers are possible to represent, and rounding errors may pop up in unexpected places. For example:

```
>>> l = 0.0000001
>>> l
9.9999999999999995e-08
```

shows that even when you just enter a number, it may not be stored as exactly the same number as you entered, however, the error, the uncertainty, is very small. This has few practical consequences. But there is one practical consequence you should be aware of: You should not test if a floating-point number is equal to a particular value. For example, you are usually advised not to test if a variable is exactly equal to zero, but rather test when it is very close to zero or when the variable changes sign.

[4] http://ieee.org.

For example, if you model a falling ball as a function of time, you may experience that it never reaches precisely zero height, it may be slightly above 0 at one time and then slightly below 0 at the next timestep—independently of how small you make your time steps. Therefore, be careful when checking if a value is exactly equal to zero!

Numerical Errors

We will address various numerical methods throughout this book. You should be aware that the numerical methods themselves may also introduce errors, and that these errors in some cases can be non-negligible. This may be because the methods are unsuited for the problems we address, or it may, for example, be that you use too large an integration step.

For most numerical methods from calculus, such as for numerical derivation and integration, the error from the numerical calculation depends on the size of the integration step, which usually corresponds to the time step. Typically, the error decreases with the integration step, and you can improve your error by decreasing your integration step. But only up to a certain limit, beyond which numerical rounding errors dominate. Trying to get very precise results be choosing very small integration steps can therefore sometimes lead to large errors. We will address these features in detail when we discuss the various numerical methods.

You must always take extra care when applying a numerical method to analyze your problem because:

- the method you apply may have an error that may affect the number of significant digits in your answer
- the method you apply may be unstable, producing errors that are orders of magnitude off the correct result
- you may have implemented the method incorrectly, producing a result that is significantly wrong

We therefore always use our physical intuition and insight when we analyze our results. You need to check whether the results are reasonable compared with your intuition. And you also need to check whether the results violate basic physical principles, such as the conservation laws of mass and momentum that you will learn throughout this book. However, by ensuring that you always test your methods against standardized problems with know solutions before venturing into the unknown, numerical methods provide a robust, versatile set of tools that allows us to solve practically any problem in mechanics with the precision we want.

Summary

Standard units:

- In physics all quantities are measured in physical units, and we always need to include the units when we present a quantity
- There is an international standard set of units, the SI units, which is based on a few base units: *meter*, *second*, *kilogram*, *mol*, *ampere*, *kelvin*, and *candela*
- All other units are derived from the base units. For example, the unit for force, Newton, is defined as $1\,\text{N} = 1\,\text{kg}\,\text{m}\,\text{s}^{-2}$.
- We use scientific notation to present numbers, by giving the base number between 0 and 10, and an exponent of 10: $998.23\,\text{m} = 9.9823 \times 10^2\,\text{m}$,
- There are standard prefixes for the most common orders of ten: so that $9.0 \times 10^{-9}\text{s} = 9.0\text{ns}$. (See Table 3.1).

Unit conversion:

- Units are treated as ordinary mathematical symbols
- We can convert between units by replacing equivalent units, for example, we can replace $1000\,\text{m}$ by $1\,\text{km}$, or we can replace $1\,\text{h}$ by $3600\,\text{s}$: $72\,\text{km/h} = 72\,1000\,\text{m}/\,(3600\,\text{s}) = (72/3.6)\,\text{m/s} = 20\,\text{m/s}$
- We can similarly convert by replacing $1\,\text{m}$ by $(1/1000)\,\text{km}$ and $1\,\text{s}$ by $(1/3600)\,\text{h}$: $20\,\text{m/s} = 20\,(1/1000)\,\text{km}/\,((1/3600)\,\text{h}) = 72\,\text{km/h}$

Uncertainty:

- Measuring a physical quantity results in uncertainty. We specify the uncertainty by a range: $x = 20.0 \pm 0.5\,\text{m}$
- Number are given so that the last digit may be uncertain. We call this number of digits the *significant number of digits*.
- If you multiply two numbers, the number of significant digits in the answer is determined by the factor with the least number of significant digits $9.8 \times 10000.0 \times 2 = 2 \times 10^5$
- If you add two numbers, the number of significant digits in the answer depends on the position of the decimal place: $1000.0 + 2 = 1002$ and $1000 + 0.1 = 1000$

Digital representation of numbers:

- Numerical quantities are without units, so you need to keep track of them
- Your answers and plots should include units
- Numbers are represented as integers or floating point number in the computer
- Floating point numbers have a limited resolution, which may lead to rounding errors
- Numerical algorithms may introduce errors that are significant—error analysis is important.

Exercises

3.1 Kilometers per hour. A car is driving at 144 km/h. Find the velocity in m/s.

3.2 Miles per hour. You car speedometer is showing both km/h and mph, miles per hour. 1 mile is 1609.34 m.
(**a**) If your speedometer is showing 70 km/h. What does it show in mph?
(**b**) If your speedometer is showing 55 mph. What does it show in km/h?

3.3 Acceleration of gravity. The acceleration of gravity is approximately $g = 9.8\,\mathrm{m/s^2}$.
(**a**) Find the acceleration of gravity in feet per second squared, ft/s^2. 1 foot is 0.3048 m.
(**b**) Find the acceleration of gravity in kilometers per hour squared, $\mathrm{km/h^2}$.

3.4 Bacterial volume. A bacteria is like a cylinder with length 4 μm, and radius 1 μm.
(**a**) Find its volume in μm^3.
(**b**) Find its volume in m^3.
(**c**) Find its volume in liters.

3.5 Ruler length. You have a platinum-iridium ruler that you have measured to be 0.11236 m using a high precision method. You use the ruler to measure the length of your desk, and find that the ruler fits about 20 times across the desk. What is the length of your desk?

3.6 Sphere mass and volume. A small steel sphere has a radius of 1.2 mm.
(**a**) What is the volume of the sphere?
(**b**) The density of the particular steel alloy used is $\rho = 7782\,\mathrm{kg/m^3}$. What is the mass of the sphere?

3.7 Laserlength. You use a laser distance measurer to measure the distance from one wall to another in your house. It reads 11.2 m. As you walk across to the other wall, you see that there is a small protrusion from the wall. Using your tape measure, you find that the protrusion is 5 mm high. What is the distance from the other wall to the protrusion?

3.8 Salmon speed. You have designed a special circuit to measure the swimming speed of a salmon. The circuit has a length of 62.8 m. You measure the time a salmon takes to swim one lap to be 20.6 s.
(**a**) What is the swimming speed of a salmon?
Your assistant insists that you would get better precision if you instead measured the time the salmon took to swim 10 rounds. You find that the salmon uses 206.0 s to swim 10 rounds.
(**b**) What is the speed of the salmon?
(**c**) Does this produce better accuracy? Can you give other examples of situations where this strategy would improve the accuracy?

Chapter 4
Motion in One Dimension

As a professional physicist you will be expected to be able to determine how things move: What is the path of a proton through a curved particle accelerator? What is the motion of a passenger in a car during a collision? How does a blood cell move through the micro-capillaries in your body? Professionally and privately, you will be expected to be able to solve any such problem your friends or your employer may come up with. How can you pull it off?

Fortunately, there is a simple method to determine the motion of an object. Objects move due to the forces acting on them. As soon as you have figured out what forces are acting on them, and you have found a model that predicts the magnitude and direction of the force during the motion, you can find the acceleration of the object. From the acceleration you can determine the motion of the object given its starting position and velocity. You will work through this procedure repeatedly over the next chapters, gradually filling all the concepts with meaning, until the procedure becomes a natural part of your way of thinking.

In this chapter we concentrate on developing our intuition of motion, on finding methods to formulate mathematical equations that determine the motion, and on developing analytical and numerical methods to solve the equations of motion.

You will learn to describe the motion of an object by its position as a function of time. We introduce the velocity and the acceleration of an object, which are the first and second time-derivatives of the position of the object. We also show how to find expressions for the motion from the velocity or acceleration—finding the equations of motion for the object.

4.1 Description of Motion

In a fantastic race in the 100 m finals of the 2008 Olympic Games in Beijing, Usain Bolt set a new world record of 9.69 s. He even took the time to celebrate his victory over the last 20 meters of the race. But did this affect his winning time? Could he have run even faster?

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_4

In order to answer such a question, we need a quantitative description of the race. We already know something: He ran 100 m in 9.69 s. But we want more detail—a finer resolution of the motion. We want to know where he was at any intermediate time from he started until he finished the race.

Motion Diagram

The first few seconds of the race are illustrated by the four pictures in Fig. 4.1. How can we describe the motion of Usain Bolt in lane four? One method is to define his position by the front of his chest. For each image, we draw a dot on the ground directly below his chest, resulting in a sequence of dots along lane four. We can now describe the race by measuring the distance, x, from the starting line to each dot—giving us a sequence of *positions*, x_i, at times t_i, for $i = 0, 1, 2, \ldots$.

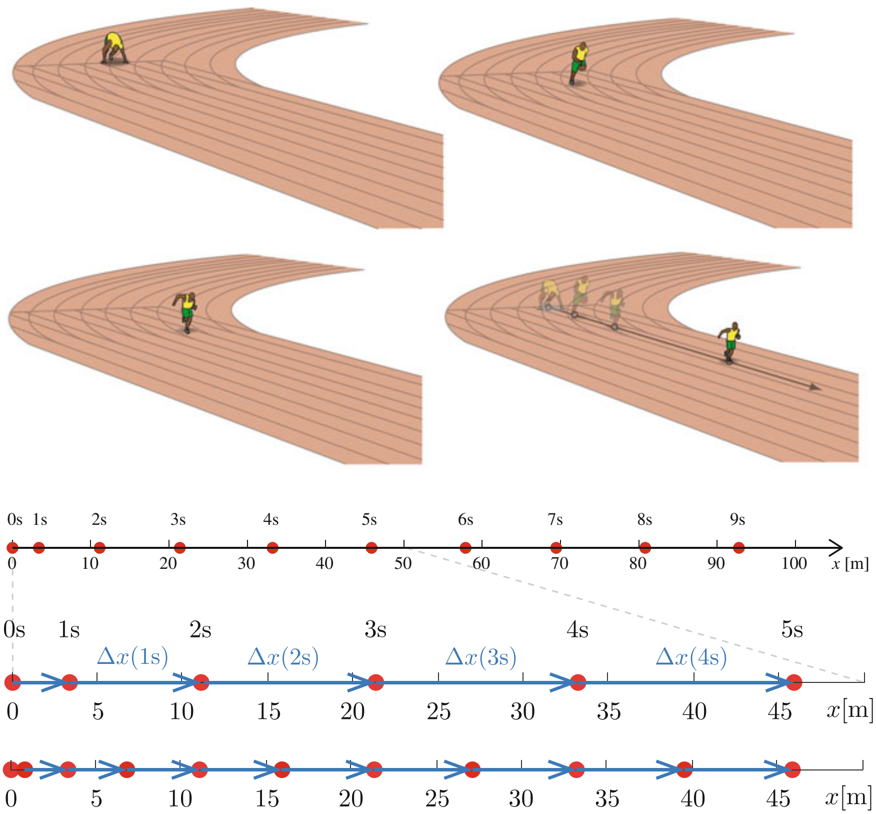

Fig. 4.1 *Top* Illustrations from the 100 m final in the 2008 Olympic Games in Beijing, showing the position of the Usain Bolt during the first three seconds. The dots in the 3 s image illustrate the position of the runner in lane 4 after 0, 1, 2, and 3 s. (*Bottom*) The position $x(t_i)$ of the runner is shown at 1 and 0.5 s intervals. Displacements Δx are drawn in *blue*

Table 4.1 Data from Usain Bolt's race

i	0	1	2	3	4	5	6
t_i (s)	0.0	1.0	2.0	3.0	4.0	5.0	6.0
x_i (m)	0.0	3.4	11.1	21.3	33.2	45.8	57.9

We plot a point at the position x_i along the x-axis to illustrate the motion in a *motion diagram* (Fig. 4.1, Table 4.1):

A **motion diagram** illustrates the motion by a sequence of positions x_i at subsequent times t_i for $i = 0, 1, 2, \ldots$, preferably at times $t_i = t_0 + i\Delta t$, where Δt is the time interval.

Position and Time

From Fig. 4.1 we see that the runner is at $x(0\,\text{s}) = 0.0\,\text{m}$ when $t = 0\,\text{s}$ and at $x(3\,\text{s}) = 21.3\,\text{m}$ when $t = 3\,\text{s}$. Even though we have only measured the position at discrete times t_i, we expect the position of the runner to vary continuously with time, as illustrated by the plot of $x(t)$ in Fig. 4.2. This is indeed how we characterize motion:

The motion of an object is described by the **position**, $x(t)$, as a function of time, t, measured in a given reference system.

Reference System and Origin

We have chosen to measure the position x along the lane. We call this direction the x-axis. The position x is measured from the starting line, which we call the origin—the point where x is zero. The choice of an origin and an axis is called a *reference system*. The axis has a direction which tells us in what direction x is increasing—this is indicated by the arrow on the axis. The axis is directed from the starting line to the finishing line, so that the position of the runner increases during the race.

You are free to choose the axes and the origin of your reference system as you like, but we usually try to choose so that measurements become simple, as we have done here.

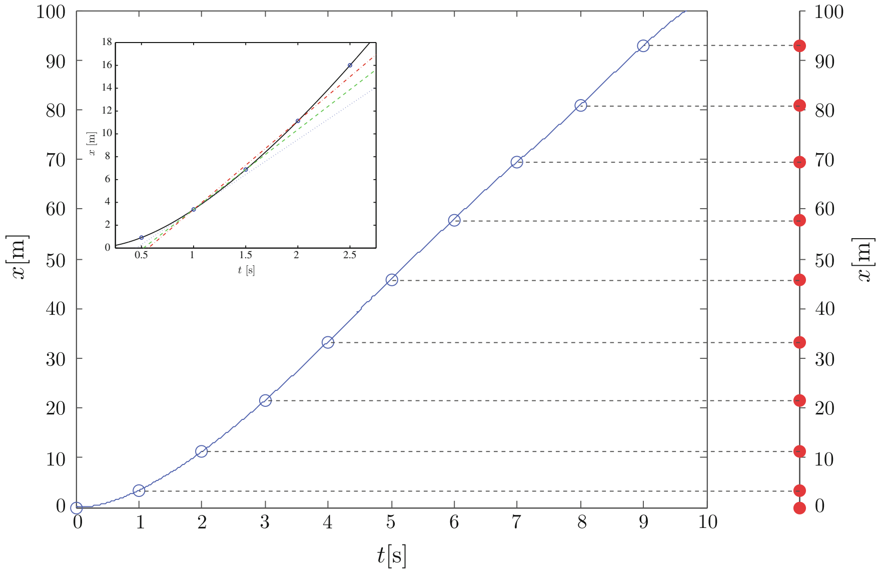

Fig. 4.2 A plot of the position x as a function of time for Usain Bolt. The circles along the curve show the position at time intervals of 1 s, corresponding to the positions in the motion diagram. The correspondence between the two representations of the motion is shown by inserting a rotated motion diagram to the right of the plot. (Inset) A magnification of $x(t)$. The average velocities at $t = 1$ s for time intervals $\Delta t = 1$ s and $\Delta t = 0.5$ s are illustrated by the slopes of the red and green lines respectively. The instantaneous velocity is illustrated by the slope of the dotted blue line, which corresponds to the slope of the tangent to the curve at $t = 1$ s.

Velocity

The motion diagram in Fig. 4.1 visualizes the change in position over a time interval Δt. The change in position from time $t = 1$ s to $t = 2$ s is:

$$x(2\,\mathrm{s}) - x(1\,\mathrm{s}) = 11.1\,\mathrm{m} - 3.4\,\mathrm{m} = 7.7\,\mathrm{m} \tag{4.1}$$

We call this change the *displacement*, $\Delta x(1\,\mathrm{s})$:

The **displacement** $\Delta x(t_1)$ over the time interval from $t = t_1$ to $t = t_1 + \Delta t$ is defined as:

$$\Delta x(t_1) = x(t_1 + \Delta t) - x(t_1)\ . \tag{4.2}$$

The displacement is read directly from the motion diagram as the length of the line from $x(1\text{ s})$ to $x(2\text{ s})$. The displacement has a direction—it is the displacement *from* $x(t_i)$ *to* $x(t_i + \Delta t)$—and it is therefore drawn as an arrow in Fig. 4.1.

The first few displacements in Fig. 4.1 are increasing. This means that he is running faster. But how fast is he running? This cannot be described by displacement alone, because the displacements become smaller when we decrease the time interval as shown in Fig. 4.1. It is the displacement per time that describes how fast he is running:

The **average velocity** from $t = t_1$ to $t = t_1 + \Delta t$ is:

$$\bar{v}(t_1) = \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} = \frac{\Delta x(t_1)}{\Delta t} . \tag{4.3}$$

The average velocity has units meters per second, m/s.

The average velocities for the runner in Fig. 4.1 at $t = 1$ s and $t = 2$ s over the time interval $\Delta t = 1$ s are:

$$\bar{v}(1\text{ s}) = \frac{7.7\text{ m}}{1\text{ s}} = 7.7\text{ m/s} , \tag{4.4}$$

$$\bar{v}(2\text{ s}) = \frac{10.2\text{ m}}{1\text{ s}} = 10.2\text{ m/s} , \tag{4.5}$$

However, if we calculate the average velocity from the bottom-most diagram in Fig. 4.1, the time interval is $\Delta t = 0.5$ s, and the velocities are:

$$\bar{v}(1\text{ s}) = \frac{3.5\text{ m}}{0.5\text{ s}} = 7.0\text{ m/s} , \tag{4.6}$$

$$\bar{v}(2\text{ s}) = \frac{4.9\text{ m}}{1\text{ s}} = 9.8\text{ m/s} , \tag{4.7}$$

We see that the average velocities depend on the time interval Δt! We can understand this from the inset in Fig. 4.2. First, we notice that we can read the average velocity $\bar{v}(1\text{ s})$ directly from the curve, $x(t)$, as the slope of the curve from the point $x(1\text{ s})$ to the point $x(1\text{ s} + \Delta t)$. From the figure, we see that $\bar{v}$ changes slightly as we change the time interval from $\Delta t = 1$ s to $\Delta t = 0.5$ s because the function $x(t)$ is curving. However, we also see that when the time interval Δt becomes smaller and smaller, the average velocity approaches a specific value given as the slope of the curve in the point $t = 1$ s. We call the velocity in this limit the *instantaneous velocity* at the time t, $v(t)$:

The **instantaneous velocity** is defined as the time derivative of the position:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \ . \tag{4.8}$$

In the following, whenever we use the term velocity, we will mean the instantaneous velocity.

Notation for Time Derivatives

Notice that the notation $x'(t)$ for the derivative that you may be used to from calculus, is not commonly used in physics. This is to avoid confusion with x', which is often used to represent a length in a coordinate system called the "marked" coordinate system. The notation $x'(t)$ can therefore be ambiguous: it may be interpreted as the position x' as a function of time, or as the time derivative of the position x. Instead, we denote the time derivative of a quantity by the placing a dot over it. The velocity is therefore often written as:

$$v(t) = \frac{dx}{dt} = \dot{x} \ . \tag{4.9}$$

Visualizing the Velocity $v(t)$

The velocity $v(t)$ represents the slope of the curve, $x(t)$. In many cases it may be useful to visualize the motion by looking at both the plot of $x(t)$ and the plot of $v(t)$, as shown in Fig. 4.3. In this case, it is evident that the velocity is changing throughout the motion. Initially, the velocity is increasing as the runner sprints out

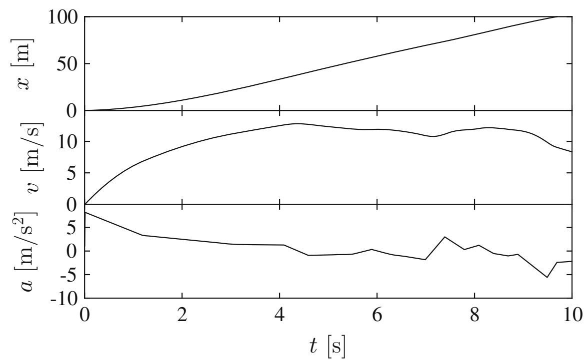

Fig. 4.3 A plot of the position $x(t)$, velocity, $v(t)$, and acceleration, $a(t)$, as a function of time for Usain Bolt

from the starting line. In the middle of the race the velocity is approximately constant, while at the end of the race, the runner is slowing down, and the velocity is falling.

Acceleration

The velocity may also vary throughout the motion. From Fig. 4.3 we see that the runner starts at rest and increases his velocity with time. Just as we introduced the velocity to characterize the rate of change of position, we introduce the acceleration to characterize the rate of change of the velocity:

The **average acceleration** over a time interval Δt from t to $t + \Delta t$ is:

$$\bar{a}(t) = \frac{v(t + \Delta t) - v(t)}{\Delta t} \ . \tag{4.10}$$

The instantaneous acceleration is the limit of the average acceleration when the time interval approaches zero:

The **instantaneous acceleration** is defined as:

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} = \dot{v} \ . \tag{4.11}$$

When we use the term acceleration we mean the instantaneous acceleration.

The acceleration can be found as the slope of the $v(t)$ curve. Figure 4.3 shows a plot of $a(t)$ together with both position $x(t)$ and velocity $v(t)$. Notice that the acceleration curve is "noisy" and consists of clear steps. This is not a physical effect, but rather an effect of how the data was gathered and interpolated. Real data often have noise from various sources—so you should expect noisy curves when you look at real systems. (You can learn more about how this data was measured in boltdatabox[1]).

Because the velocity is given as the time derivative of the position $x(t)$, we can also write the acceleration as the time derivative of the position $x(t)$ by inserting (4.9) into (4.11):

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2} \ . \tag{4.12}$$

[1] http://folk.uio.no/malthe/mechbook/boltdatabox.

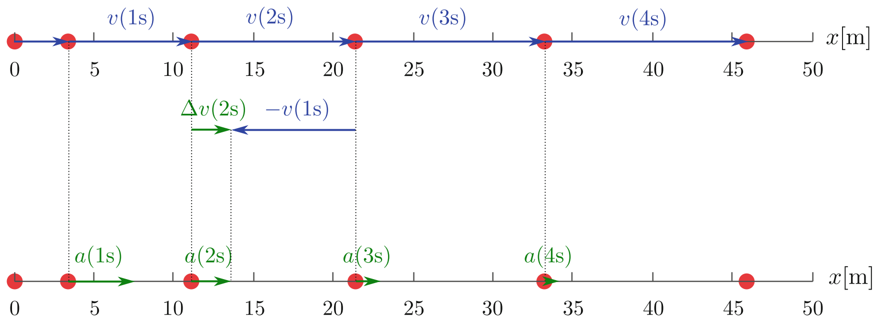

Fig. 4.4 Motion diagram for Usain Bolt. The *top* figure shows the velocities at time intervals of 1 s. The displacements are interpreted as velocities. The *top* figure shows how the change in velocity at $t = 2\,\mathrm{s}$ is constructed from the velocity at $t = 1\mathrm{s}$ and the velocity at $t = 2\,\mathrm{s}$. The resulting difference, $\Delta v(2\,\mathrm{s})$ is interpreted as the average acceleration. The *bottom* figure shows the accelerations estimated from the motion diagram

Using the dot-notation, we can write this as:

$$a(t) = \dot{v}(t) = \ddot{x}(t) \ , \tag{4.13}$$

or in shorthand

$$a = \dot{v} = \ddot{x} \ . \tag{4.14}$$

Interpretation of Motion Diagrams

It is often difficult to obtain a good intuition for acceleration, in particular for two- and three-dimensional motions, but sometimes also for one-dimensional motions. Experience shows that motion diagrams are useful tools for developing a good intuition for accelerations—this is why we include them here.

As long as all the time intervals in a motion diagram are identical, the displacements in the motion diagram may be interpreted as average velocities. In Fig. 4.4 the displacements and therefore the average velocities, are initially increasing, until at $t = 4\,\mathrm{s}$ they are approximately constant. The change in average velocity from $t = 1\,\mathrm{s}$ to $t = 2\,\mathrm{s}$ is:

$$\Delta\bar{v}(1\,\mathrm{s}) = \bar{v}(2\,\mathrm{s}) - \bar{v}(1\,\mathrm{s}) = 5\,\mathrm{m/s} \tag{4.15}$$

We introduce the average acceleration[2] as:

[2]The attentive reader may realize that the average acceleration should really be defined in terms of the change in instantaneous velocity: $\bar{a} = (v(t + \Delta t) - v(t))/\Delta t$ and not in terms of the average velocity as done here. However, this small difference in definitions becomes insignificant when the time interval becomes sufficiently small.

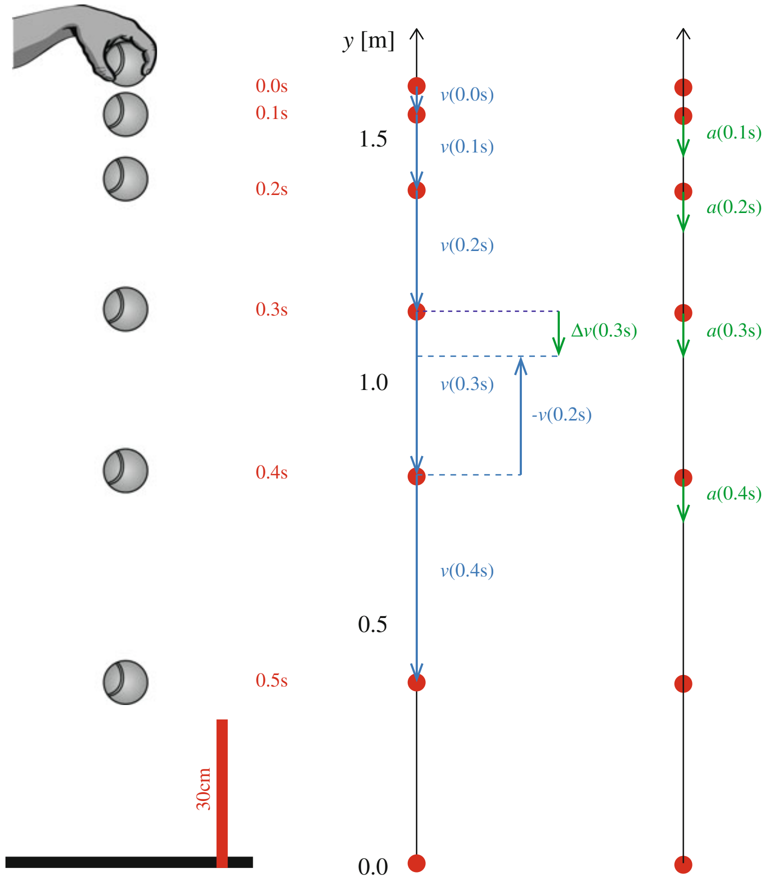

Fig. 4.5 *Left* Digital images from a falling tennis ball—we have made an artistic rendering of the ball for clarity. *Right* Motion diagram for the tennis ball. The *left* diagram shows the positions and velocities, and the *right* diagram illustrates the accelerations

$$\bar{a} = \frac{\Delta \bar{v}}{\Delta t} \tag{4.16}$$

The average acceleration can be constructed geometrically from the motion diagram by subtracting two subsequent (average) velocities in the diagram, as illustrated in Fig. 4.4.

4.1.1 Example: Motion of a Falling Tennis Ball

This example demonstrates how we can find the velocity and acceleration from the motion diagram of a falling tennis ball, both by hand calculation, using Python and from a mathematical model of the motion.

Table 4.2 Table with calculated values

i	t_i (s)	y_i (m)	Δy_i (m)	$\bar{v}_i$ (m/s)	$\bar{a}_i$ (m/s^2)
1	0.0	1.60	−0.05	−0.5	
2	0.1	1.55	−0.15	−1.5	−10.0
3	0.2	1.40	−0.24	−2.4	−9.0
4	0.3	1.16	−0.34	−3.4	−10.0
5	0.4	0.82	−0.43	−4.3	−9.0
6	0.5	0.39			

Motion diagram: The motion of a falling tennis ball was captured with a digital camera. The first few images were combined into one picture as shown in Fig. 4.5. From the sequence of images, we measure the vertical position of the ball by comparing the height of the center of the ball to the ruler seen in the images. The positions are shown in Table 4.2.

We draw the motion diagram by marking the positions y_i with dots along the vertically oriented y-axis as illustrated in Fig. 4.5. We illustrate the velocities by the displacements, which are drawn as arrows from point to point. The average velocities can be calculated from the data: For each i in Table 4.2 we calculate the average velocity from t_i to t_{i+1} using:

$$\bar{v}_i = \frac{y_{i+1} - y_i}{\Delta t} \ . \tag{4.17}$$

The corresponding results are shown in the table. However, we cannot use this method to find a value for $i = 5$ since we do not know y_6. We find that all the velocities are negative. Since we have chosen the positive direction to be up (the arrow on the y-axis points upward) this means that the ball is falling down—as expected.

The velocities are increasing in magnitude since the ball is accelerating downward. We estimate the average accelerations by

$$\bar{a}_i = \frac{\bar{v}_i - \bar{v}_{i-1}}{\Delta t} \ , \tag{4.18}$$

and the results are shown in Table 4.2. For the accelerations, we cannot find a value for $\bar{a}_i$ for $i = 0$ or for $i = 5$, since the velocity are not defined at $i = -1$ or at $i = 5$. If you look at Fig. 4.5 you can also see how to construct the accelerations directly from the motion diagram.

The data shows that the acceleration is approximately constant $a \simeq -9.5 \pm 0.5\,\text{m/s}^2$ throughout the fall. This experiment therefore tells us that a tennis ball falls with a constant acceleration—which is close to what you may recognize as the acceleration of gravity, $g = 9.8\,\text{m/s}^2$.

Mathematical model: A physicist friend of yours tells you that there is a mathematical model for the motion of a falling tennis ball when there is no air resistance

$$y(t) = y_0 - \frac{1}{2}gt^2 , \tag{4.19}$$

where $g = 9.8\,\mathrm{m/s^2}$ is a constant and y_0 is the position of the tennis ball at $t = 0\,\mathrm{s}$. Let us see how this model matches up with the observed data.

We calculate the position of the ball for various times. From the experimental data, we see that $y(0\,\mathrm{s}) = 1.6\,\mathrm{m}$. We use Python as a calculator to find the positions for all the times in Table 4.2 with a single line of code:

```
>> g = 9.8
>> t = array([0.0, 0.1, 0.2, 0.3, 0.4, 0.5])
>> y = 1.6 - 0.5*g*t**2
>> print y
[ 1.6    1.551  1.404  1.159  0.816  0.375]
```

Notice that the command `t**2` tells Python to apply the operation for each element in the array `t`, generating an `y`-array of 6 elements. This vectorized notation allows us write the code in a similar way to the mathematics. We can output the data in a form that looks more like Table 4.2:

```
>> transpose([t,y])
array([[ 0.    ,  1.6  ],
       [ 0.1   ,  1.551],
       [ 0.2   ,  1.404],
       [ 0.3   ,  1.159],
       [ 0.4   ,  0.816],
       [ 0.5   ,  0.375]])
```

where the `transpose()` means transpose. Without it, the table would have been oriented differently. Try it!

The resulting values for $y(t)$ are similar to the experimental data, but in the experiment the ball falls a bit slower than in the mathematical model: In the experiment the ball is at $y = 0.39\,\mathrm{m}$ at $t = 0.5\,\mathrm{s}$, whereas the mathematical model predicts $y = 0.375\,\mathrm{m}$.

We can compare the results better by studying the velocities and accelerations. In the mathematical model, we know $y(t)$, and we can calculate the *instantaneous* velocity and acceleration by applying the definitions directly. The velocity of the ball is defined as:

$$v = \frac{dy}{dt} , \tag{4.20}$$

and if we insert $y(t)$ from (4.19) we get

$$v = \frac{d}{dt}\left(y_0 - \frac{1}{2}gt^2\right) = -gt . \tag{4.21}$$

Similarly, the acceleration is defined as

$$a = \frac{dv}{dt} , \tag{4.22}$$

where we insert $v(t)$ from (4.21) and get

$$a = -g = -9.8\text{m/s} . \tag{4.23}$$

The acceleration in the mathematical model is a constant. But we cannot really compare with the experimental data, since they have too low precision. We need better data!

High resolution data: To study the process in more detail, the motion of the falling tennis ball was also recorded by a motion detector placed directly above the ball. The detector provides the vertical position y of the ball, but at a much higher time resolution that the images: The detectors measures y at a time interval of Δt = 0.001s. The data is stored in the file fallingtennisball02.d.[3] The first few lines of the file looks like:

```
0.0000000000000000e+00   1.6000000000000001e+00
1.0000000000000020e-03   1.5999950510001959e+00
2.0000000000000044e-03   1.5999803020031378e+00
3.0000000000000070e-03   1.5999557530158828e+00
...                      ...
```

where each line contains the time t_i in seconds and the position y_i in meters (given in scientific notation, but with no unit). We read the data-set from file, using `loadtxt`

```
t,x = loadtxt('fallingtennisball02.d',usecols=[0,1],unpack=True);
```

This generates the arrays `t`, and `x`.

We see what is in the data-set by plotting the position as a function of time, $y(t)$, using:

```
plot(t,x)
xlabel('t [s]')
ylabel('x [m]')
```

What does the resulting plot in Fig. 4.6a show? From the plot, we see that the ball falls down, bounces up from the surface to reach a lower height than the first time, and so on. The first 0.5 s of the motion resembles what we found by analyzing the images: the position decreases with time. And we see that ball is falling faster with time—it accelerates. But it is difficult to see details of the motion directly from this plot. Could you say if the acceleration is constant or not for the first 0.5 s from this plot? To gain more insight, we need to analyze the velocity and acceleration of the ball.

Numerical derivatives: Because we do not know $y(t)$ for all values of t, but only the measured values of $y(t_i)$, we cannot find an exact, analytical expression for the derivative of $y(t)$ as we did when we had a mathematical model. However, we can follow the procedure we used for the image data in (4.17): We can approximate the instantaneous velocity by the average velocity from t_i to $t_i + \Delta t$:

$$\frac{dy}{dt} = v(t_i) \simeq \bar{v}(t_i) = \frac{y(t_i + \Delta t) - y(t_i)}{\Delta t} . \tag{4.24}$$

[3] http://folk.uio.no/malthe/mechbook/fallingtennisball02.d.

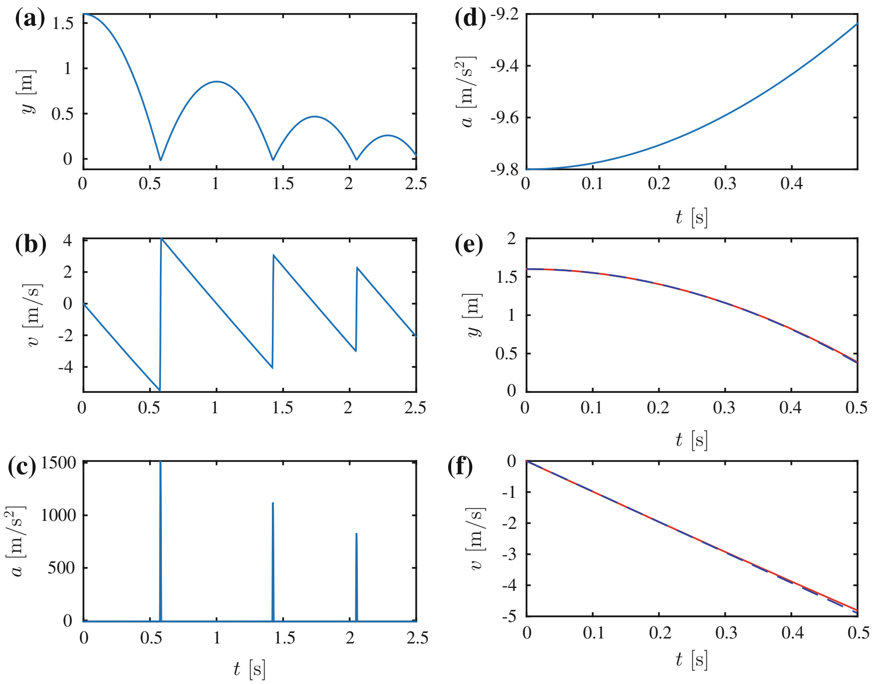

Fig. 4.6 **a**, **b**, **c** Plot of the position $y(t)$, velocity, $v(t)$, and acceleration, $a(t)$, of the ball as a function of time t. **d** Plot of $a(t)$ for $t < 0.5$s. **e**, **f** Comparison of $y(t)$ and $v(t)$ for the experimental data (*red, solid line*) and the mathematical model (*blue, dashed lined*)

The average velocity is an example of a *numerical derivative* of the position—a numerical method to calculate the derivative. This method is easily implemented numerically by directly converting the mathematical formula to code:

```
v[i] = (v[i+1]-v[i])/dt
```

We need to apply this rule to each element i from 0 to $n - 2$, where n is the number of data points $y(t_i)$. This is done using a `for`-loop:

```
n = len(x);
dt = t[1] - t[0];
v = zeros(n-1,float);
for i in range(n-1):
   v[i] = (x[i+1] - x[i])/dt;
```

Here, we find `n`, the number of elements in the `y`-array, and the time difference `dt`, which we calculate from the first two times since the time intervals are regular. We also prepare an empty array `v`, which we will fill with velocities. But why do we only make it $n - 1$ elements long? Because the formula `v[i] = (y[i+1] - y[i])/dt`, cannot be applied to the last element in the array, since we would then have no data for $i + 1$. (We saw the same in Table 4.2). For the same reason, we must stop the loop at $n - 2$.

Similarly, we find the acceleration by using the numerical derivative of the velocity:

$$a(t_i) \simeq \bar{a}(t_i) = \frac{v(t_i) - v(t_{i-1})}{\Delta t} . \quad (4.25)$$

We apply this mathematical definition of the derivative directly to the data:

```
a = zeros(n-1,float);
for i in range(1,n-1):
   a[i] = (v[i] - v[i-1])/dt;
```

For the acceleration, the formula `v[i] = (v[i] - v[i-1])/dt` cannot be applied to the first element in the array, since we have no data for $i = -1$. The loop therefore starts at $i = 1$ (Again, this is the same as in Table 4.2).

Plotting: We plot $x(t)$, $v(t)$, $a(t)$ by:

```
subplot(3,1,1)
plot(t,x)
ylabel('x [m]')
subplot(3,1,2)
plot(t[0:n-1],v)
ylabel('v [m/s]')
subplot(3,1,3)
plot(t[1:n-1],a[1:n-1])
xlabel('t [s]')
ylabel('a [m/s^2]')
```

Here we have used the `subplot` command to generate a set of plots. (Consult Python to find out how the plots are numbered using `help(subplot)`.) Notice that the velocity is only defined for i from 0 to $n - 2$. We therefore only include the corresponding values of t_i in the plot. Similarly, the acceleration is defined from 1 to $n - 2$, and we only plot the corresponding values of t_i.

Plotting parts of the data: It is difficult to see the acceleration of the ball while it is falling from Fig. 4.6c. How can we plot only the first 0.5 s of the motion? We find the value for i where t_i goes from begin smaller than 0.5 to larger than 0.5 using `find`:

```
imax = max(find(t<=0.5))
plot(t[1:imax],a[1:imax])
xlabel('t [s]')
ylabel('a [m/s^2]')
```

and plot $a(t)$ for this range of t-values in Fig. 4.6d. (You could also have made this plot by using the zoom button in the plotting window). The acceleration is clearly *not* a constant in this case. It starts at $-9.8\ \mathrm{m/s^2}$, but its magnitude becomes smaller with time. (This is due to air resistance).

Comparison with mathematical model: How large are the differences between the experimental data and the mathematical model for motion without air resistance? A good way to compare, is to plot the model in the same plot as the data. The model was:

$$y(t) = y_0 - \frac{1}{2}gt^2 \text{ and } v(t) = -gt . \quad (4.26)$$

We implement these formulas directly in the program, and plot both data and model:

```
g = 9.8 # m/s^2
y0 = 1.6 # m
vt = -g*t
yt = y0 - 0.5*g*t**2
subplot(2,1,1)
plot(t[0:imax],y[0:imax],'-r');
hold('on')
plot(t[0:imax],yt[0:imax],'--b');
hold('off')
xlabel('t [s]')
ylabel('y [m]')
subplot(2,1,2)
plot(t([0:imax],v[0:imax],'-r');
hold on
plot(t[0:imax],vt[0:imax],'--b');
hold('off')
xlabel('t [s]')
ylabel('v [m/s]')
```

We use `hold on` to get both plots in the same figure (see Fig. 4.6e, f. Here we notice that the differences in $y(t)$ and $v(t)$ are more difficult to spot. Using the acceleration for comparisons was therefore a better approach to spot the differences. And an approach with a sound, physical basis, since we will later learn that differences in physics appear in differences in the accelerations.

Further work: We leave it to you to look more carefully at what happens during the bounce. Can you zoom in on the relevant area?

Comment: Notice that the data in this example were based on numerical results and not experimental data in order to get clear results. Experimental data will typically contain significant noise, which we did not want to include here. The program used to generate the data-set is makefallingtennisball.m.[4]

4.2 Calculation of Motion

Mechanics is about the motion of objects. Usually, we do not know the position as a function of time. Instead, we want to determine the motion based on measurements of the acceleration (or velocity); based on a mathematical expression for the acceleration; or based on a differential equation for the position. We therefore need tools to do the opposite of what we did above: We need tools to find the motion, $x(t)$, from the acceleration, $a(t)$, of an object (Table 4.3).

Discrete Integration

As lead developer of "The Rocket", a new roller-coaster ride at a major theme-park, you have fitted an accelerometer into a test-cart. The accelerometer records the

[4]http://folk.uio.no/malthe/mechbookmakefallingtennisball.m.

Table 4.3 Data from the motion of "The Rocket"

i	0	1	2	3	4	5
t_i (s)	0.0	0.1	0.2	0.3	0.4	0.5
a_i (m/s^2)	0.00	1.43	2.80	4.13	5.62	7.21

acceleration of the cart at regular time intervals of 0.1 s. How can you use this data to determine the velocity and position of the test cart?

The problem is how to find the sequence of positions, $x(t_i)$, from the sequence of accelerations, $a(t_i)$? This is the reverse of what we have been doing so far, where we have estimated first the velocities and then the accelerations from the positions using numerical derivatives. Can we simply use the methods we have developed for numerical derivatives "in reverse"? The average acceleration from $t_1 = 0.0$ s to $t_2 = 0.1$ s is

$$\bar{a}(t_i) = \frac{v(t_i + \Delta t) - v(t_i)}{\Delta t} \; . \tag{4.27}$$

(So far this is an *exact* result—we have not done any approximations yet). We can "reverse" (4.27) to find an equation for the velocity at the time $t = t_i + \Delta t$:

$$\begin{aligned} v(t_i + \Delta t) - v(t_i) &= \bar{a}(t_i)\,\Delta t \\ v(t_i + \Delta t) &= v(t_i) + \bar{a}(t_i)\,\Delta t \end{aligned} \tag{4.28}$$

This method would allow us to step one step forward in time from the time $t = t_i$ to the time $t = t_i + \Delta t$, if only we knew the average acceleration of the time interval. Unfortunately, the accelerometer does not give the average, but rather the instantaneous acceleration of the cart, $a(t_i)$. Let us ignore this distinction and approximate the average acceleration over the time interval by the instantaneous acceleration at the beginning of the time interval:

$$\bar{a}(t_i) \simeq a(t_i) \; , \tag{4.29}$$

We are now in a position to use (4.28) to step forward in small steps of Δt, calculating the changes in the velocities of the cart as we go. However, finding the velocities only takes us part of the way—we also need to determine the positions, $x(t_i)$, of the cart, from the velocities, $v(t_i)$, calculated using (4.28). This time, we "reverse" the numerical derivative of the position:

$$\begin{aligned} x(t_i + \Delta t) - x(t_i) &= \bar{v}(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + \bar{v}(t_i)\,\Delta t \; . \end{aligned} \tag{4.30}$$

Where we again assume that the average velocity is approximately the same as the velocity we calculated in (4.28): $\bar{v}(t_i) \simeq v(t_i)$. We are now ready to use (4.28) and

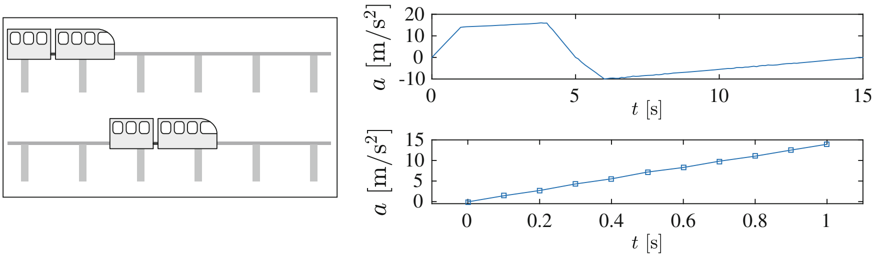

Fig. 4.7 Illustration of the motion of "The Rocket". The accelerations are illustrated for the whole time interval (*top figure*) and the time-resolution is shown by the squares representing the measurement points (*bottom figure*)

(4.30) to move forwards in steps of Δt. However, since these methods only give the increments in the velocity and the position, we need to know the first velocity of the cart, $v(t_0) = v_0$ and where the cart starts from, $x(t_0) = x_0$. This is called the initial conditions of the problem.[5]

We are now ready to find the velocities and positions, starting at the time $t = t_0 = 0.0$s:

1. At $t = t_0 = 0.0$ s, the velocity and position of the cart is given $v(t_0) = v(0.0\,\text{s}) = 0.0$ m/s, $x(t_0) = x(0.0\,\text{s}) = 0.0$ m.
2. At $t = t_0 + \Delta t = 0.1$ s, the velocity of the cart is:

$$v(0.1\,\text{s}) \simeq v(0.0\,\text{s}) + a(0.0\,\text{s})\,\Delta t = 0.5\,\text{m/s}\;, \tag{4.31}$$

where the acceleration $a(0.0\,\text{s}) = 5.0\,\text{m/s}^2$ is listed in the table Fig. 4.7. The position of the cart is:

$$x(0.1\,\text{s}) \simeq x(0.0\,\text{s}) + v(0.0\,\text{s})\,\Delta t = 0.0\,\text{m} \tag{4.32}$$

3. At $t = t_1 + \Delta t = 0.2$ s, the velocity of the cart is:

$$v(0.2\,\text{s}) \simeq v(0.1\,\text{s}) + a(0.1\,\text{s})\,\Delta t = 0.9\,\text{m/s};\;, \tag{4.33}$$

where the acceleration $a(0.1\,\text{s}) = 7.0$ m/s is listed in the table in Fig. 4.7. The position of the cart is:

$$x(0.2\,\text{s}) \simeq x(0.1\,\text{s}) + v(0.1\,\text{s})\,\Delta t = 0.05\,\text{m}\;, \tag{4.34}$$

[5]Notice that we need two initial conditions, $v(t_0) = v_0$ and $x(t_0) = x_0$ to determine the position from the acceleration: This is because we need to first calculate the velocities, and this requires an initial condition of the velocities, and then calculate the position, and this also requires an initial condition, this time on the position.

where the velocity $v(0.1\,\text{s}) = 0.5\,\text{m/s}$ was found in the previous step of the calculation.

This method is called *Euler's method* for numerical integration, and it is sufficiently flexible and robust to solve most problems presented in this book!

In **Euler's method** we find the position, $x(t_i)$, and velocity, $v(t_i)$, of an object as a function of time by a stepwise summation of the acceleration, $a(t_i)$, and the velocity, $v(t_i)$:

$$\begin{aligned} v(t_0) &= v_0 \\ x(t_0) &= x_0 \\ &\dots \\ v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i)\,\Delta t \end{aligned} \tag{4.35}$$

We apply this method to find the position and velocities for the motion of "The Rocket". The accelerations for the cart are stored in the file therocket.dat,[6] where each line contains a time (in seconds) and an acceleration (in m/s^2):

```
0.0000000e+000  2.7316440e-001
1.0000000e-001  1.4411079e+000
2.0000000e-001  2.6693138e+000
3.0000000e-001  4.2383806e+000
```

We read the data into Python find the time-step Δt from $t_2 - t_1$, and apply Euler's algorithm from (4.35) for each i starting from the initial condition $x(t_0) = 0\,\text{m}$ and $v(t_0) = 0\,\text{m/s}$ using a `for`-loop.

```
t,a = loadtxt('rocket.dat',usecols=[0,1],unpack=True);
dt = t[1] - t[0];
n = length(t);
x = zeros(n,float);
v = zeros(n,float);
x[0] = 0.0; # Initial value
v[0] = 0.0; # Initial value
for i in range(1,n):
  v[i] = v[i-1] + a[i-1]*dt;
  x[i] = x[i-1] + v[i-1]*dt;
```

The resulting position and velocity plots are shown in Fig. 4.8.

The procedure presented here covers the most important topic in kinematics: How to determine the motion of an object given the acceleration of the object. This is important because you will later learn that the physics of a problem—the interactions between the object and other objects—gives the acceleration of the object. Given the acceleration it will be up to you to determine the motion—and you can do this using the methods provided here: Either by using Euler's method (or more advanced

[6] http://folk.uio.no/malthe/mechbook/therocket.dat.

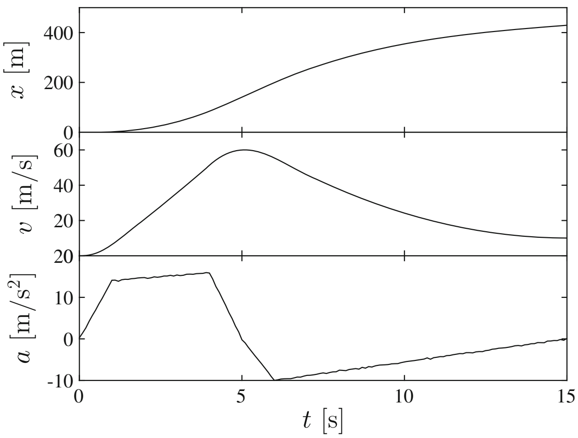

Fig. 4.8 Illustration of the motion of "The Rocket", showing the measured acceleration, and the calculated velocity and position

techniques) to solve the problem numerically, or by finding a solution to the problem based on the specialized techniques you have learned in calculus.

Formal Integration

A more formal formulation of the problem is to assume that we know the acceleration $a(t)$ of an object as a function of time. How do we find the position and velocity of the object in this case?

Again, we realize that we have already solved the "reverse" problem—we know that the acceleration is the time derivative of the velocity and that the velocity is the time derivative of the position. We find the velocity by integrating the definition of acceleration:

$$a(t) = \frac{dv}{dt} \Rightarrow \int_{t_0}^{t} a(t)\, dt = \int_{t_0}^{t} \frac{dv}{dt}\, dt = v(t) - v(t_0) \ , \tag{4.36}$$

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)\, dt \ . \tag{4.37}$$

When we know the velocity as a function of time, we can find the position by integrating the velocity, starting from the definition of velocity:

$$v(t) = \frac{dx}{dt} \Rightarrow \int_{t_0}^{t} v(t)\, dt = \int_{t_0}^{t} \frac{dx}{dt}\, dt = x(t) - x(t_0) \tag{4.38}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t)\, dt \ . \tag{4.39}$$

If we insert $v(t)$ from (4.37), we get:

$$
\begin{aligned}
x(t) &= x(t_0) + \int_{t_0}^{t} [v(t_0) + \int_{t_0}^{t} a(t)\,dt]\,dt \\
&= x(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt \ . \qquad (4.40)
\end{aligned}
$$

These equations constitute the **integration method** to find the position $x(t)$ and velocity $v(t)$ given the acceleration $a(t)$ of an object:

$$
v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt \ , \qquad (4.41)
$$

$$
x(t) = x(t_0) \ + \int_{t_0}^{t} v(t)\,dt = x(t_0) \ + \ v(t_0)(t - t_0) + \int_{t_0}^{t} [\int_{t_0}^{t} a(t)\,dt]\,dt \ . \qquad (4.42)
$$

There is no need to memorize these equations. They follow from your knowledge of calculus. You only need to remember the definitions of the velocity as the time derivative of the position, and the acceleration as the time derivative of the velocity.

We can apply this method to find the motion for constant acceleration, $a(t) = a_0$, with initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$:

$$
v(t) = v(t_0) + \int_{t_0}^{t} a_0\,dt = v0 + a_0\,(t - t_0) \ . \qquad (4.43)
$$

and

$$
x(t) = x(t_0) + \int_{t_0}^{t} v(t)\,dt = x_0 + v_0\,(t - t_0) + \frac{1}{2} a_0\,(t - t_0)^2 \ . \qquad (4.44)
$$

Differential Equations

Usually, we do not have a set of measurements or a mathematical expression for the acceleration. Instead, we find an expression for the acceleration based on a physical model of the forces acting on the object, and from the forces we find the acceleration. Given this expression for the acceleration, we determine the velocity and position of the object. But this sounds exactly like what we did above? We integrate the acceleration to find the velocity, and then integrate again to find the position. Unfortunately, direct integration only works if the acceleration is *only* a function of time. In most cases, we do not have an expression of the acceleration as a function of time, but instead we know how the acceleration varies with velocity and position. For example, a tiny grain of sand sinking in water has an acceleration on the form:

$$\frac{d^2x}{dt^2} = a = -a_0 - c \cdot v \;, \tag{4.45}$$

where the acceleration depends on the velocity of the grain! And a ball suspended in a vertical spring has an acceleration:

$$\frac{d^2x}{dt^2} = a = -C \cdot x \;, \tag{4.46}$$

that depends on the position of the ball. Such problems cannot be solved by direct integration, because the function $x(t)$ and its derivatives occur on both sides of the equation. Such equations are called differential equations. Finding analytical solutions of differential equations require some skill and experience, but, fortunately, we can solve them numerically in exactly the same way we did above.

Numerical Solution

In most mechanics problems, we want to find the position, $x(t)$, which satisfies an equation on the form:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right) \;,\; v(t_0) = v_0 \;,\; x(t_0) = x_0 \;, \tag{4.47}$$

We find the solution by moving forwards in time in small increments Δt. We start from the initial values $x(t_0) = x_0$ and $v(t_0) = v_0$. We find the velocity and position after a small time-step Δt using Euler's method (4.28):

$$v(t_0 + \Delta t) \simeq v(t_0) + a(t_0, x(t_0), v(t_0))\, \Delta t \;, \tag{4.48}$$

$$x(t_0 + \Delta t) \simeq x(t_0) + v(t_0)\, \Delta t \;, \tag{4.49}$$

where $a(t_0, x(t_0), v(t_0))$ is the acceleration we get when we put the values at $t = t_0$ into the expression we have for the acceleration in (4.47). We can now continue to step forward in time, finding subsequent values $x(t_i)$ and $v(t_i)$ in steps of Δt. This method is called Euler's method. It is definitely not the best numerical method of integration—actually we strongly advice against using Euler's method. It's strength is rather in the simple, intuitive implementation. Surprisingly, changing the step in (4.49) to the following:

$$x(t_0 + \Delta t) \simeq x(t_0) + v(t_0 + \Delta t)\, \Delta t \;, \tag{4.50}$$

gives significantly better solutions for many problems. This improved method is called Euler-Cromer's method, and you can use this method safely for most problems you encounter.

In **Euler-Cromer's method** to solve the (second order) differential equation of motion:

$$\frac{d^2x}{dt^2} = a\left(t, x, \frac{dx}{dt}\right) \ , \ v(t_0) = v_0 \ , \ x(t_0) = x_0 \ , \tag{4.51}$$

we perform the following steps:

$$\begin{aligned} v(t_i + \Delta t) &\simeq v(t_i) + a(t_i, x(t_i), v(t_i))\, \Delta t \\ x(t_i + \Delta t) &\simeq x(t_i) + v(t_i + \Delta t)\, \Delta t \end{aligned} \tag{4.52}$$

4.2.1 Example: Modeling the Motion of a Falling Tennis Ball

This example demonstrates how we can calculate the motion of a falling tennis ball given an expression for the acceleration.

Background: In Sect. 4.1.1 we studied the motion of a falling tennis ball based on measurements of its motion. However, in physics we do not only want to observe motion, we want to predict it. We do this by first analyzing the problem to find the forces acting on the object, and from the forces we find a mathematical model of the acceleration of the object. (You will learn to do this in the next chapter. For now we will assume that the acceleration is given). From the acceleration, we find the position and velocity by analytical or numerical integration. We call this recipe the structured problem-solving approach.

System sketch: Your first step should always be to make a sketch the process. In physics, our sketches are vessels for our thoughts. A good, functional sketch is therefore an important part of solving a problem. While the left part of Fig. 4.9 has a nice artistic appeal and also illustrates the motion in detail, we do not encourage such detailed sketches. Instead, you should make a sketch that only focuses on the most important features of the process, as in the rightmost figure. Here we illustrate *the object* (the tennis ball), *its surroundings* (most importantly the floor), and *the coordinate system* with a clearly marked axis. We have also illustrated the initial position and velocity of the ball, and its position and velocity at a time t. Drawing a simplified illustration helps you discern the important from the unimportant, and it helps you convert a physical situation into a mathematical problem: The figure shows the axis and the position of the ball, $y(t)$, and nothing else.

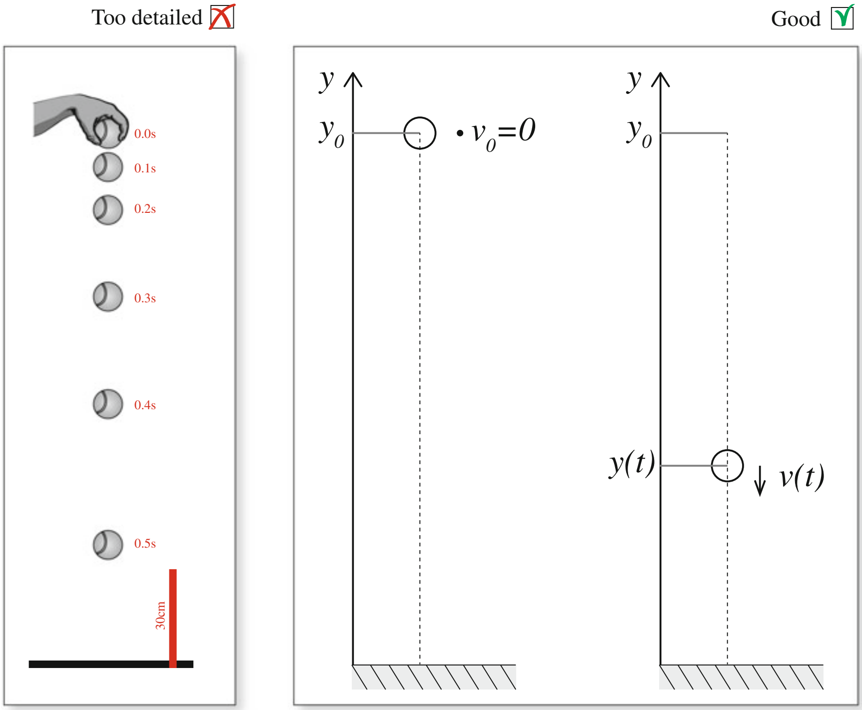

Fig. 4.9 *Left* Too detailed illustration. (*Right*) Correct, simple sketch

Simplified model: From an analysis of the physics of the system, we have found that the acceleration of the ball is a constant:

$$a = -g = -9.8\,\text{m/s}^2\ . \tag{4.53}$$

(You will learn where this model comes from later. Now we only want to address the consequences of such a model). In addition, we know that the ball starts from rest at the position $y_0 = 2.0\,\text{m}$ at the time $t_0 = 0\,\text{s}$:

$$y(0\,\text{s}) = 2.0\,\text{m}\ ,\ \ v(0\,\text{s}) = 0\,\text{m/s}\ . \tag{4.54}$$

We have now formulated a mathematical description of the problem we want to solve:

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -g\ ,\ \ v(0) = v_0\ ,\ \ y(0) = y_0\ . \tag{4.55}$$

Solving this equation means to find the velocity $v(t)$ and the position $y(t)$ of the ball for any time t. We call this *the modeling step*, finding the mathematical problem to solve, and the next step is *to solve* this problem—to find $v(t)$ and $y(t)$.

Solving the simplified model: Since the acceleration is given and a constant, we can find the velocity by direct integration of the acceleration:

$$\frac{dv}{dt} = -g ; \tag{4.56}$$

$$\int_{t_0}^{t} \frac{dv}{dt}\, dt = \int_{t_0}^{t} -g\, dt \ , \tag{4.57}$$

$$v(t) - \underbrace{v(t_0)}_{=0\,\mathrm{m/s}} = -g\, t + g \underbrace{t_0}_{=0\,\mathrm{s}} \ , \tag{4.58}$$

which gives

$$v(t) = -gt \ . \tag{4.59}$$

Similarly, we find the position by integrating the velocity:

$$\frac{dy}{dt} = v(t) \ , \tag{4.60}$$

$$\int_{0}^{t} \frac{dy}{dt}\, dt = \int_{0}^{t} v(t)\, dt \ , \tag{4.61}$$

$$y(t) - y(0) = \int_{0}^{t} -gt\, dt = -\frac{1}{2} g t^2 \ , \tag{4.62}$$

which gives

$$y(t) = y(0) - \frac{1}{2} g t^2 \ . \tag{4.63}$$

Analysis of the simplified model: This is the complete solution to the problem. We know the position and velocity as a function of time. When you have this solution, you are prepared to answer any question about the motion. For example, you can find out when the ball hits the ground and you can find the velocity of the ball when it hits the ground. How would you do that? You need to translate the question into a mathematical problem. We do this by stating the condition "when the ball hits the ground" in mathematical terms: The ball hits the ground when its position is that of the ground, that is, when $y(t) = 0\,\mathrm{m}$. (Notice, we have ignored the extent of the ball here). We can use our solution in (4.63) to find the corresponding time:

$$y(t) = y(0) - \frac{1}{2} g t^2 = 0\,\mathrm{m} \ \Rightarrow \ t = \sqrt{\frac{2\, y(0)}{g}} \ . \tag{4.64}$$

A more realistic model: Unfortunately, data for the motion of the tennis ball, shown in Fig. 4.6b, show that the ball does not have a constant acceleration. This is due to air resistance—an effect not included in the simplified model. Fortunately, we have good models for air resistance. For a falling ball in air, a more realistic model that includes the effect of air resistance is:

$$a = -g - Dv|v| \; , \tag{4.65}$$

where $v = v(t)$ is the velocity of the ball, $g = 9.8\,\mathrm{m/s^2}$ is the same constant as above, and the constant D depends on details of the ball. For a tennis ball $D = 0.0245\,\mathrm{m^{-1}}$ is a reasonable value. (You will learn about the background for this model and how to determine values for D later). We can now formulate a mathematical problem:

$$a = \frac{dv}{dt} = -g - Dv|v| \; , \tag{4.66}$$

with initial conditions $v(0\,\mathrm{s}) = 0\,\mathrm{m/s}$ and $y(0\,\mathrm{s}) = 2.0\,\mathrm{m}$.

Solution of the realistic model: Our task is to solve this problem, which means to find $v(t)$ and $y(t)$ for the ball. This can be done either numerically or analytically. The numerical solution is straightforward, using the approach we have derived, but the analytical solution requires some knowledge of differential equations.

Numerical solution: We apply Euler-Cromer's method to find the positions and velocities by stepwise integration starting from the initial conditions. The integration step in Euler-Cromer's method is:

$$v(t_i + \Delta t) = v(t_0) + a(t_i, v_i, y_i)\,\Delta t \tag{4.67}$$

$$y(t_i + \Delta t) = y(t_0) + v(t_i + \Delta t)\,\Delta t \; , \tag{4.68}$$

where we insert the acceleration from (4.65):

$$a(t_i, v_i, y_i) = -g - Dv(t_i)|v(t_i)| \; , \tag{4.69}$$

This is implemented as follows: We define the physical constants and values given in the problem: g, D, $y(0)$ and $v(0)$:

```
D = 0.0245 # m^-1
g = 9.8    # m/s^2
y0 = 2.0
v0 = 0.0
```

We need to determine for how long we want to calculate the motion: What will be our maximum value of t? There are typically two strategies: We can make an initial guess for the duration of the simulation, or we can determine when the simulation should stop during the simulation. First, we make a guess for the duration of the simulation. Based on the existing data from Fig. 4.6 we guess that $t = 0.5\,\mathrm{s}$ is a reasonable simulation time:

```
time = 0.5
```

Next, we need to decide the time-step Δt. This needs to be small enough to ensure a good precision of the result, but not too small or the simulation takes too long. We try a value of $\Delta t = 0.00001$ s:

```
dt = 0.00001
```

Based on this, we calculate how many simulation steps we need, $n = t/\Delta t$, and generate arrays for the positions, velocities, accelerations and time for the simulation. All values are initially set to zero:

```
# Variables
n = ceil(time/dt)
y = zeros(n,float)
v = zeros(n,float)
a = zeros(n,float)
t = zeros(n,float)
```

Then we set the initial conditions:

```
# Initialize
y[0] = y0
v[0] = v0
```

Before, finally, the Euler-Cromer steps are implemented in an integration loop. The whole program is given in the following:

```
from pylab import *
D = 0.0245 # m^-1
g = 9.8    # m/s^2
y0 = 2.0
v0 = 0.0
time = 0.5
dt = 0.00001
# Variables
n = ceil(time/dt)
y = zeros(n,float)
v = zeros(n,float)
a = zeros(n,float)
t = zeros(n,float)
# Initialize
y[0] = y0
v[0] = v0
# Integration loop
for i in range(n-1):
    a[i] = -g -D*v[i]*abs(v[i])
    v[i+1] = v[i] + a[i]*dt
    y[i+1] = y[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
```

The resulting plots of $x(t)$, $v(t)$, and $a(t)$ are shown in Fig. 4.10.

Analysis of realistic model results: We can now use this result to answer questions like how long does it take until the ball hits the ground? Again, we answer the question by translating it into a mathematical question: The ball hits the ground when $y(t) = 0$ m. However, in this case, we must find the solution numerically. The simplest approach to this would be to find when y becomes zero during the simulation. It is tempting to do this by checking when $y(t) = 0$ m:

```
if (y[i]==0.0)
  print t[i]
```

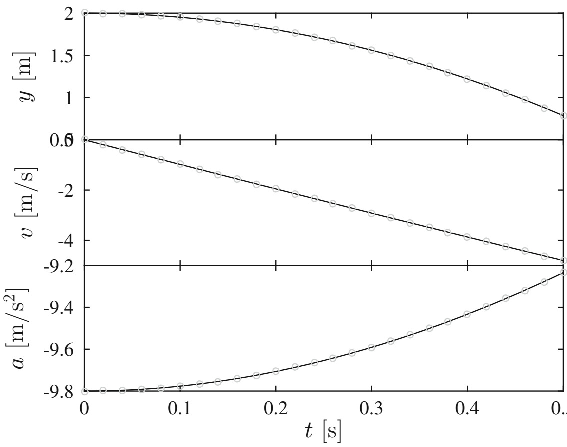

Fig. 4.10 Plots of $y(t)$, $v(t)$, and $a(t)$ calculated using the model for air resistance (*black line*) compared with analytical model (*gray circles*)

But this will not work, because $y(t_i)$ will usually not be zero for any i. Typically, the program will step right past $y = 0$ going from a small positive value at some t_i to a small negative value at t_{i+1}. We should instead find the first time $y(t)$ passes 0, that is, we should find the first t_{i+1} when $y(t_{i+1}) < 0$. Then we know that $y(t) = 0$ somewhere in the interval $t_i < t < t_{i+1}$. We can then estimate a precise value for t using interpolation, or we can simply use the value t_{i+1}, if we find that this gives us sufficient precision. This is implemented in the following modification to the program, where we have also stopped the calculation when the ball hits the ground:

```
for i in range(n-1):
    a[i] = -g -D*v[i]*abs(v[i])
    v[i+1] = v[i] + a[i]*dt
    y[i+1] = y[i] + v[i+1]*dt
    if (y[i+1]<0):
        break
    t[i+1] = t[i] + dt
print v[i+1]
plot(t[0:i-1],a[0:i-1])
xlabel('t [s]');
ylabel('a [m/s^2]');
```

where we have used `break` to stop the loop when the condition is met. Notice that we should now only plot the values up to `i`, because we have not calculated any more values. The values from $i + 2$ to n were set to zero initially for y, v, and a and will make your plot confusing if you include them. (Try it and see).

Test your understanding: What would happen if we considered that the ball had an initial velocity $v_0 = -2v_T$ when it started? Sketch the resulting position, velocity and acceleration as a function of time.

Analytical solution: The differential equation in (4.66) is one of a few equations we can solve analytically as long as the velocity does not change sign. When the ball is falling down, the velocity is negative, and we can replace $|v|$ by $-v$:

$$\frac{dv}{dt} = -g - Dv\,(-v) = -g + D\,v^2 \; . \tag{4.70}$$

This equation can be solved using separation of variables.

We separate the variables, so that all v's are on the left side and all t's are on the right:

$$\frac{dv}{g - Dv^2} = -1\,dt \; . \tag{4.71}$$

The differential equation can now be solved by integrating each side from $v_0 = 0$ m/s to v and from $t_0 = 0$ s to t:

$$\int_{v_0}^{v} \frac{dv}{g - Dv^2} = \int_{t_0}^{t} -1\,dt = -t \; , \tag{4.72}$$

The left-side integral can be solved using your knowledge from calculus (or by using the symbolic solver in Python) giving:

$$\int_{0}^{v} \frac{dv}{g - Dv^2} = \frac{1}{g} v_T \tanh^{-1}\left(\frac{v}{v_T}\right) \; , \tag{4.73}$$

where we have introduced the quantity $v_T = \sqrt{g/D}$ to simplify the notation. We notice that v_T has dimensions m/s, and we may therefore call it a velocity. We insert (4.73) back into (4.72), getting

$$v_T \tanh^{-1}(v/v_T) = -gt \;\Rightarrow v = v_T \tanh(-gt/v_T) \; . \tag{4.74}$$

We have now found the velocity on the form $v = v(t)$, and we can simply integrate the velocity from t_0 to t to find $y(t)$:

$$y(t) - y(t_0) = \int_{t_0}^{t} v(t)\,dt = \int_{0}^{t} v_T \tanh\left(-\frac{gt}{v_T}\right)\,dt \; . \tag{4.75}$$

This integral can solved by the symbolic integrator in Python giving:

$$y(t) = y(0) - v_T^2/g \log\cosh\frac{gt}{v_T} \; . \tag{4.76}$$

Figure 4.10 shows that the analytical solutions (given by circles) are identical to the numerical solutions (lines).

Symbolic solution: The differential equation in (4.66) can also be solved directly using the symbolic solver in Python. We can solve the differential equation for the velocity, $v(t)$:

$$\frac{dv}{dt} = -g + D\,v^2 \; , \; v(0) = 0 \; . \tag{4.77}$$

First, we define the variables g and D as symbolic variables, and the function v(t) as a symbolic functions:

```
>> from sympy import *
>> v = Function('v')
>> t = Symbol('t',real=True,positive=True)
>> g = Symbol('g',real=True,positive=True)
>> D = Symbol('D',real=True,positive=True)
```

Python can then solve the equation with the initial condition by

```
>> dsolve(Derivative(v(t),t)+g-D*v(t)**2,v(t))
-sqrt(1/(D*g))*log(-g*sqrt(1/(D*g)) + v(t))/2 +
...    sqrt(1/(D*g))*log(g*sqrt(1/(D*g)) + v(t))/2 == C1 - t
```

Then, we need to determine the value of the unknown constant from the initial condition, $v(0) = 0$, which sets C1 to zero. After some reorganization, we find

```
-(sqrt(g)*tanh(sqrt(D*g)*t))/sqrt(D)
```

which is the same answer as we found by our analytical solution. We can then find the position by symbolic integration of this equation:

```
>> integrate(-(sqrt(g)*tanh(sqrt(D*g)*t))/sqrt(D),t)
-sqrt(g)*(-t - log(tanh(sqrt(D)*sqrt(g)*t) - 1)/...
   (sqrt(D)*sqrt(g)))/sqrt(D)
```

Analysis of analytical solution: We can now use the analytical solution to solve problems of interest, such as finding out when the ball hits the floor, which occurs at $y(t) = 0$ m, that is, when

$$y(0) = v_T^2/g \log \cosh \frac{gt}{v_T} \Rightarrow \frac{gt}{v_T} = \cosh^{-1} \exp \frac{y(0)g}{v_T^2} \tag{4.78}$$

that is:

$$t = \frac{v_T}{g} \cosh^{-1} \exp \frac{y(0)g}{v_T^2} . \tag{4.79}$$

Summary

Motion: The motion of an object is described by:

- the position, $x(t)$, as a function of time, measured in a specified coordinate system
- the velocity $v(t) = dx/dt$
- the acceleration $a(t) = dv/dt = d^2x/dt^2$

Structured problem-solving approach:

- The structured problem-solving approach is illustrated in Fig. 4.11.

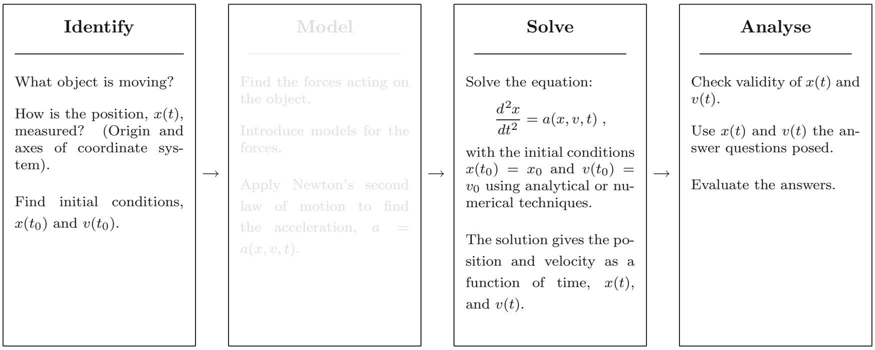

Fig. 4.11 Structured problem-solving approach. The second box, model, will be filled in Chap. 5

Solution methods: In the "Solver" we solve the equation:

$$\frac{d^2x}{dt^2} = a(t, x, \frac{dx}{dt}) \ .$$

with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$.

- *Numerically*, we solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$v(t_i + \Delta t) = v(t_i) + \Delta t \cdot a(x(t_i), v(t_i), t_i) \ ,$$

$$x(t_i + \Delta t) = x(t_i) + \Delta t \cdot v(t_i + \Delta t) \ .$$

- *Analytically*, when the acceleration, $a = a(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t)dt \ , \ x(t) = x(t_0) + \int_{t_0}^{t} v(t)dt \ ,$$

  A typical example is motion with constant acceleration.
- When the acceleration has a general form, $a = a(t, x, v)$, we need to solve the differential equation. In this case, there are no general approaches that always work. Instead, you must rely on your experience and your knowledge of calculus.

Exercises

Discussion Questions

4.1 Pedometer. Can you use the accelerometer in your phone as a pedometer? Explain.

4.2 Error in speedometer. If your speedometer overestimates your velocity by 10 percent, how will that affect your measurement of your cars acceleration?

4.3 Speed of the clouds. Is it possible to use your camera to measure the speed of the clouds? What would you need to know to do that?

4.4 The slow trip. Is is possible to go for a trip (in one dimension) where the total displacement is zero, but your average velocity is non-zero?

4.5 Driving backwards. You drive in a train that is subject to constant acceleration. Can the train reverse its direction of motion?

4.6 No motion. Is is possible to envision a motion where you for a period have no displacement, but non-zero velocity? (You may use an $x(t)$ plot for illustration).

4.7 Non-falling ball. You throw a ball downwards from a high building. Can you think of a situation where the ball would have an acceleration upwards? What would happen?

4.8 Travels by sea. A boat is sailing north. Is it possible for the boat to have a velocity toward the north, but still have an acceleration toward the south?

4.9 Acceleration during throw. You throw a ball upwards as far as you can. The ball reaches its maximum height far above you. When was the magnitude of the acceleration the largest? While in your hand while throwing it or during its subsequent motion through the air?

4.10 Passing objects. A disgruntled physics student drops his pc from a window onto the ground. (You should not try this at home). At the same time as she lets the pc go, another student throws a ball upwards. The ball reaches its maximum position at the exact height where the pc was released. At what height do the pc and the ball pass each other? At the midpoint, above the midpoint or below the midpoint? Do they have the same magnitudes of their velocities at this point?

Problems

4.11 Space shuttle launch. When the space shuttle is lifting off, the vertical positions for the first 10 s in 1 s intervals are given as

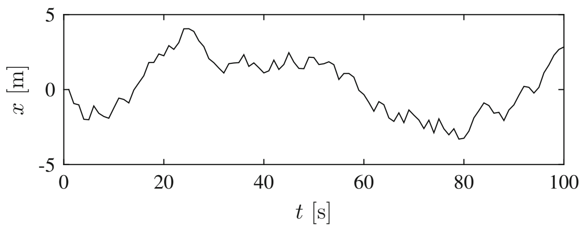

Fig. 4.12 Random motion of a grain of dust

t (s)	0	1	2	3	4	5	6	7	8	9
y (m)	0	15	60	135	240	375	540	735	960	1215

(**a**) Draw the motion diagram and the displacements for this motion.
(**b**) Use the motion diagram to find the average velocity as a function of time after lift-off.
(**c**) Use the motion diagram to find the average acceleration as a function of time after lift-off.

4.12 Capturing the motion of a falling ball. We use an ultra-sonic motion detector to measure the vertical position of a small ball. We throw the ball upwards, and measure the position until it hits the ground. You find the measured data in the file ballmotion.d.[7] Each line in the file consists of a time, t_i, measured in seconds, and a distance, x_i, measured in meters.
(**a**) Plot the position as a function of time for the ball.
(**b**) How long time does it take until the ball hits the ground?
(**c**) Plot the average velocity as a function of time for the ball.
(**d**) What is the maximum and minimum velocity of the ball?
(**e**) What is the initial velocity: The velocity of the ball at the start of the motion?
(**f**) Plot the average acceleration as a function of time for the ball.
(**g**) When is the maximum and minimum accelerations? Does this correspond with your physical intuition?

4.13 Motion graphs. A car is driving along a straight road. Sketch the position and velocity as a function of time for the car if:
(**a**) The car drives with constant velocity.
(**b**) The car accelerates with a constant acceleration.
(**c**) The car brakes with a constant acceleration.

4.14 Random walker. Figure 4.12 shows the motion of a tiny grain of dust bouncing randomly around in an air chamber.
(**a**)When is the grain to the left of the origin?
(**b**)When is the grain to the right of the origin?
(**c**) Is the grain ever exactly at the origin?

[7] http://folk.uio.no/malthe/mechbook/ballmotion.d.

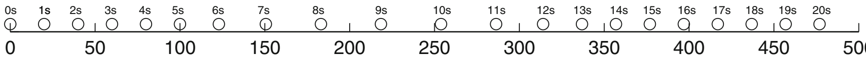

Fig. 4.13 Motion diagram for a car

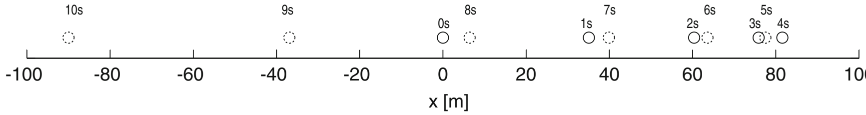

Fig. 4.14 Can you describe the motion?

4.15 Motion diagram for a car. Figure 4.13 shows the motion diagram for a car driving along a straight road.
(**a**) Describe the motion of the car.
(**b**) Sketch the position as a function of time.
(**c**) Estimate the velocity of the car throughout the motion.
(**d**) Estimate the acceleration of the car throughout the motion.

4.16 Discover the motion. Figure 4.14 shows the motion diagram for a motion.
(**a**) Describe the motion qualitatively.
(**b**) Suggest a process that leads to this motion diagram.

4.17 The fastest indian. In the film "The World's Fastest Indian" Anthony Hopkins plays Burt Munro who reaches a velocity of 201 mph in his 1920 Indian motorcycle.
(**a**) At this velocity, how far does the Indian travel in 10 s?
(**b**) How long time does the Indian need to travel 1 km?

4.18 Meeting trains. A freight train travels from Oslo to Drammen at a velocity of 50 km/h. An express train travels from Drammen to Oslo at 200 km/h. Assume that the trains leave at the same time. The distance from Oslo to Drammen along the railway track is 50 km. You can assume the motion to a long a line.
(**a**) When do the trains meet?
(**b**) How far from Oslo do the trains meet?

4.19 Catching up. Your roommate sets off early to school, walking leisurely at 0.5 m/s. Thirty minutes after she left, you realize that she forgot her lecture notes. You decide to run after her to give her the notes. You run at a healthy 3 m/s.
(**a**) What is her position when you start running?
(**b**) What is your position when $t < t_1$?
(**c**) Sketch the position of you and your roommate as functions of time and indicate in the figure where you catch up with her.
(**d**) How long time does it take until you catch up with her?
(**e**) How far has she come when you catch up with her?

Now you have developed a strategy to solve such a problem, let us make the problem more complicated and see if you still can use your strategy. First, let us

assume that you start off at $v_0 = 5$ m/s, but then you tire gradually, so that your speed drops off with distance, x, reducing your speed by 1 m/s for every hundred meters you run, until you reach a speed of $v_1 = 2$ m/s, which you can keep for a long time.
(**f**) Show that your velocity as a function of position can be written as:

$$v(x) = \begin{cases} v_0 - b\,x & \text{when } v < v_1 = 2\,\text{m/s} \\ v_1 & \text{otherwise} \end{cases} \tag{4.80}$$

where $b = 1$ m/s/100 m.
(**g**) Plot or sketch $v(x)$.
(**h**) If you know your position and velocity at a time t, how can you find the position and velocity at $t + \Delta t$, a small time-step later?
(**i**) Write a program to find your position as a function of time. (Remember that you first start running at the time $t = t_1 = 1800, texts$. Before this you are standing still.)
(**j**) Validate your program by setting $b = 0$ and comparing the calculated $x(t)$ with the exact result, $x_e(t) = v_0(t - t_1)$ when $t > t_1$.
(**k**) How can you use this result to find where you catch up with your roommate?
(**l**) Where do you catch up with your roommate?
(**m**) What parts of your solution strategy are general, that is, what parts of your strategy do not change if we change how either person moves?

4.20 Electron in electric field. An electron is shot through a box containing a constant electric field, getting accelerated in the process. The acceleration inside the box is $a = 2000$ m/s^2. The width of the box is 1 m and the electron enters the box with a velocity of 100 m/s.
(**a**) What is the velocity of the electron when it exits the box?

4.21 Archery. As an expert archer you are able to fire off an arrow with a maximum velocity of 50 m/s when you pull the string a length of 70 cm.
(**a**) If you assume that the acceleration of the arrow is constant from you release the arrow until it leaves the bow, what is the acceleration of the arrow?

4.22 Collision. A car travelling at 36 km/h crashes into a mountainside. The crunch-zone of the car deforms in the collision, so that the car effectively stops over a distance of 1 m.
(**a**) Let us assume that the acceleration is constant during the collision, what is the acceleration of the car during the collision?
(**b**) Compare with the acceleration of gravity, which is $g = 9.8$ m/s^2.

4.23 Braking distance. When you brake your car with your brand new tyres, your acceleration is 5 m/s^2.
(**a**) Find an expression for the distance you need to stop the car as a function of the starting velocity.

With your old tires, the acceleration is only two thirds of the acceleration with the new tyres.
(**b**) How does this affect the braking distance?
(**c**) Your reaction time is 0.5 s. If a child jumps into the street 30 m ahead of you when you are driving 50 km/h, are you able to stop with your new tires? What would happen if you did not change tyres?

4.24 Motion with constant acceleration. An object starts at $x = x_0$ with a velocity $v = v_0$ at the time $t = t_0$ and moves with a constant acceleration a_0. Show that the velocity v when the object has moved to a position x is $v^2 - v_0^2 = 2a_0(x - x_0)$.

4.25 Position plots. The position $x(t)$ of a particle moving along the x-axis is given in Fig. 4.15.
(**a**) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(**b**) Indicate in the figure where the velocity is maximal and minimal.
(**c**) Indicate in the figure where the acceleration is positive, negative, and zero?

4.26 Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in Fig. 4.16.
(**a**) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(**b**) Indicate in the figure where the velocity is maximal and minimal.
(**c**) Indicate in the figure where the acceleration is positive, negative, and zero?
(**d**) Indicate in the figure where the acceleration is maximal and minimal.

4.27 Velocity plots. The velocity $v(t)$ of a particle moving along the x-axis is given in Fig. 4.17.

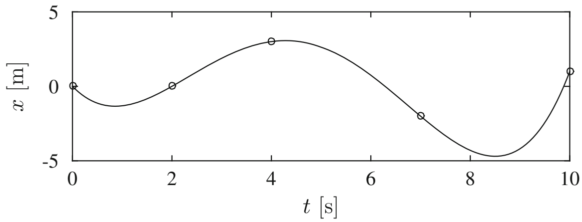

Fig. 4.15 The position of a particle moving along the x-axis.

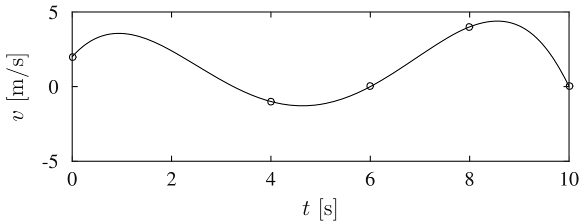

Fig. 4.16 The velocity of a particle moving along the x-axis.

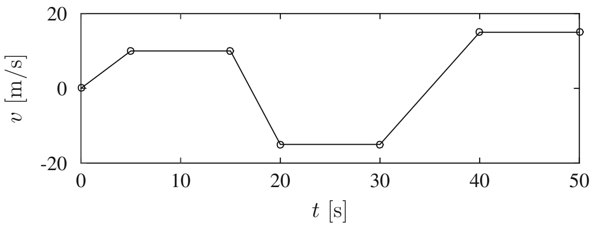

Fig. 4.17 The velocity of a particle moving along the x-axis

(**a**) Indicate in the figure where the velocity of the particle is positive, negative, and zero?
(**b**) Indicate in the figure where the particle speeds up and slows down.
(**c**) Indicate in the figure where the particle is stationary—that is, where it does not move.
(**d**) Indicate in the figure where acceleration is the largest and the smallest.
(**e**) Sketch the position as a function of time, $x(t)$.

4.28 A swimming bacterium. When the heliobacter bacteria swims, it is driven by the rotational motion of its tiny tail. It swims almost at a constant velocity, with small fluctuations due to variations in the rotational motion. As a simple model for the motion, we assume that the bacteria starts with the velocity $v = 10\,\mu$ m/s at the time $t = 0$ s, and is then subject to the acceleration, $a(t) = a_0 \sin(2\pi t/T)$, where $a_0 = 1\,\mu$ m/s^2, and $T = 1$ ms.
(**a**) Find the velocity of the bacterium as a function of time.
(**b**) Find the position of the bacterium as a function of time.
(**c**) Find the average velocity of the bacterium after a time $t = 10T$.

4.29 Resistance. (This problem requires some knowledge of statistics). An electron is moving with a constant acceleration, a_0, through a conductor. However, there are many small irregularities in the conductor—called scattering centers. If the electron hits a scattering center it stops, that is, its velocity immediately becomes zero. The scatting centers have a constant density. The probability for the electron to hit a scattering center when it moves a distance Δx is $P = \Delta x/b$, where b is a length describing the typical length between two scattering centers. Assume that the electron starts from rest. (For simplicity, we measure lengths in nm and time in ns, and you can assume that $b = 1$ nm and that $a_0 = 1$ nm/ns^2). First, we address the case without scattering.
(**a**) Write a program to find the motion of the electron using Euler-Cromer's method to find the velocity and position from the acceleration. Plot the position, $x(t)$, and velocity, $v(t)$, of the electron as functions of time and compare with the exact result.
(**b**) During the time interval Δt, the electron moves from $x(t)$ to $x(t + \Delta t)$. The probability for the electron to stop during this interval is $P = (x(t + \Delta t) - x(t))/b$. Explain why the following method models a collision:

```
dx = x[i+1]-x[i]
p = dx/b
if (random.uniform(0,1)<p):
   v[i+1] = 0
```

where `random.uniform(0,1)` produces a random number uniformly distributed between 0 and 1.
(**c**) Rewrite your program to include the effect of collisions using the algorithm described above. Plot the position, $x(t)$, and the velocity, $v(t)$, as functions of time. What do you see? Comment
(**d**) Find the average velocity v_{avg} for the electron.
(**e**) How does v_{avg} depend on a_0 and b? Can you make a theory that gives the value of v_{avg}?
(**f**) (Requires knowledge of statistics). What is the probability density for the distance, X, between two collisions?

4.30 Ball on vibrating surface. A ball is falling vertically through air over a vibrating surface. The position of the surface is $x_w(t) = A \cos \omega t$, where $A = 1$ cm and ω is called the angular frequency of the vibrations. The ball starts from a position $x = 10$ cm at $t = 0$ s. The acceleration of the ball is given as:

$$a(x, v, t) = \begin{cases} -g & x > x_w \\ -g - C(x - x_w) & x \le x_w \end{cases} . \tag{4.81}$$

where $g = 9.81\ \mathrm{m/s^2}$ and $C = 10000.0\ \mathrm{s^{-2}}$.
(**a**) Write down the equation you need to solve to find the motion of the ball. Include initial conditions for the ball.
(**b**) Write down the algorithm to find the position and velocity at $t_{i+1} = t_i + \Delta t$ given the position and velocity at t_i. Use Euler-Cromer's scheme.
(**c**) Write a program to find the position and velocity of the ball as a function of time.
(**d**) Check your program by comparing the initial motion of the ball with the exact solution when the acceleration is constant. Plot the results.
(**e**) Check your program by first studing the behavior when the vibrating surface is stationary, that is, when $A = 0$ m and $x_w = 0$ m. Plot the resulting behavior. Ensure that your timestep is small enough, $\Delta t = 10^{-5}$ s. What happens if you increase the timestep to $\Delta t = 0.02$ s?
(**f**) Finally, use your program to model the motion of the ball when the surface is vibrating. Use $A = 0.01$ m, $\omega = 10\ \mathrm{s^{-1}}$, and simulate 5 s of motion. Plot the results. What is happening?
(**g**) What happens if you increase the vibrational frequency to $\omega = 30\ \mathrm{s^{-1}}$?. Plot the results. Can you explain the difference from $\omega = 10\ \mathrm{s^{-1}}$?

Projects

4.31 Sliding on snow. In this project we address the motion of an object sliding on a slippery surface—such as a ski sliding in a snowy track. You will learn how to find the equation of motion for sliding systems both analytically and numerically, and to interpret the results.

We start by studying a simplified situation called frictional motion: A block is sliding on a surface. moving with a velocity v in the positive x-direction. The forces from the interactions with the surface results in an acceleration:

$$a = \begin{cases} -\mu(|v|)g & v > 0 \\ 0 & v = 0 \\ \mu(|v|)g & v < 0 \end{cases} , \tag{4.82}$$

where $g = 9.8\,\mathrm{m/s^2}$ is the acceleration of gravity. Let us first assume that $\mu(v) = \mu = 0.1$ for the surface. That is, we assume that the coefficient of friction does not depend on the velocity of the block. We give the block a push and release it with a velocity of 5 m/s. (**a**) Find the velocity, $v(t)$, of the block.
(**b**) How long time does it take until the block stops?
(**c**) Write a program where you find $v(t)$ numerically using Euler's or Euler-Cromer's method. (Hint: You can find a program example in the textbook.) Use the program to plot $v(t)$ and compare with your analytical solution. Use a timestep of $\Delta t = 0.01$.

The description of friction provided above is too simplified. The coefficient of friction is generally not independent of velocity. For dry friction, the coefficient of friction can in some cases be approximated by the following formula:

$$\mu(v) = \mu_d + \frac{\mu_s - \mu_d}{1 + v/v^*} , \tag{4.83}$$

where $\mu_d = 0.1$ often is called the dynamic coefficient of friction, $\mu_s = 0.2$ is called the static coefficient of friction, and $v^* = 0.5$ m/s is a characteristic velocity for the contact between the block and the surface.
(**d**) Show that the acceleration of the block is:

$$a(v) = -\mu_d g - g\frac{\mu_s - \mu_d}{1 + v/v^*} , \tag{4.84}$$

for $v > 0$.
(**e**) Use your program to find $v(t)$ for the more realistic model, with the same starting velocity, and compare with your previous results. Are your results reasonable? Explain.

The model we have presented so far is only relevant at small velocities. At higher velocities the snow or ice melts, and the coefficient of friction displays a different dependency on velocity:

$$\mu(v) = \mu_m \left(\frac{v}{v_m} \right)^{-\frac{1}{2}} \text{ when } v > v_m \ , \tag{4.85}$$

where v_m is the velocity where melting becomes important. For lower velocities the model presented above with static and dynamic friction is still valid.
(**f**) Show that

$$\mu_m = \mu_d + \frac{\mu_s - \mu_d}{1 + v_m/v^*} \ , \tag{4.86}$$

in order for the coefficient of friction to be continuous at $v = v_m$.
(**g**) Modify your program to find the time development of v for the block when $v_m = 1.5$ m/s. Compare with the two other models above: The model without velocity dependence and the model for dry friction. Comment on the results.
(**h**) The process may be clearer if you plot the acceleration for all the three models in the same plot. Modify your program to plot $a(t)$, plot the results, and comment on the results. What would happen if the initial velocity was much higher or much lower than 5 m/s?

Chapter 5
Forces in One Dimension

What determines how far a bungee-jumper falls before he starts moving upward? In this chapter you acquire the tools to answer this, sometimes critical, question.

We have introduced a structured approach to find the motion of a object from its acceleration and the initial conditions (see Fig. 5.1). But how do we find the acceleration? We could measure it directly, as we did with an accelerometer, but this is not satisfactory. Physics is not only about describing what is happening, but rather about explaining and predicting motion. In order to determine the motion, we need to be able to *predict* the acceleration of an object.

In this chapter we will show you that the acceleration of an object is related to the forces acting on the object. In order to predict the motion, we need to:

- Find what forces are acting on an object.
- Introduce quantitative models for the forces—we need numbers for the forces in order to have numbers for the acceleration.
- Determine the acceleration from the forces using Newton's second law of motion.
- "Solve" the motion from the differential equations of motion and the initial conditions.

We will address these points in detail: First we show how to identify the forces acting on an object. Then we introduce Newton's second law that relates forces to acceleration. Finally, we introduce models for some of the most common forces in the macroscopic world.

5.1 What Is a Force?

We all have an intuitive notion of a force. Imagine you give one end of a soft rubber band to a friend (see Fig. 5.2). As you pull the rubber band, your friend will experience a pull in the rubber band. She feels the force acting on her. As you pull harder, she

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_5

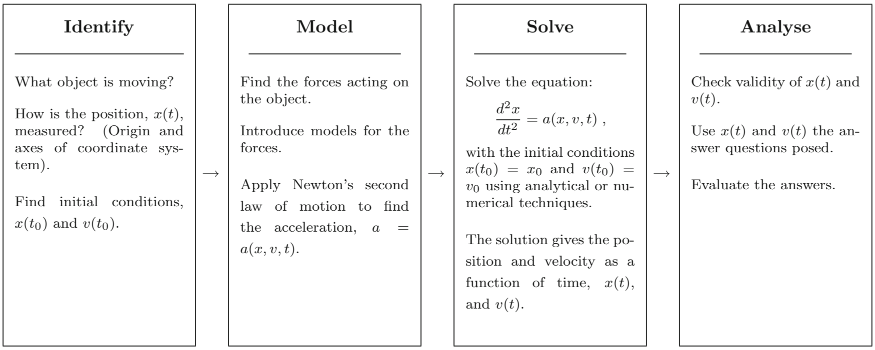


Fig. 5.1 The structured problem solving approach

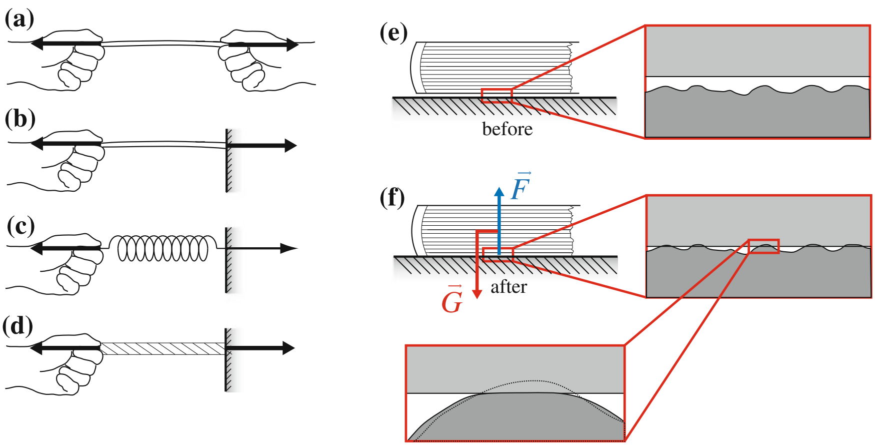


Fig. 5.2 Illustration of **a** two hands pulling on a rubber band, **b** a rubber band attached to a wall, **c** a spring attached to a wall, **d** a rope attached to a wall, **e** a book above a table, **f** a book on a table, **g** deformation of surface bump, **h** magnification of bump

will feel that the pull becomes stronger—the force acting on her becomes larger. In addition, the rubber band stretches. The harder you pull, the longer the rubber band becomes.

If you instead tie the rubber band to the wall, the rubber band will again elongate as you pull. But now it is not a person experiencing the pull, it is the wall. A force may indeed act on the wall as well as on a person. If we pull harder, the rubber band elongates further, and we expect the force on the wall to become larger. This suggests that the elongation of the band is a reasonable way to measure the magnitude of the force, and this is indeed the usual way to define a force: by prescribing how we can measure it. We can measure forces by how they deform rubber bands.

Now, there is nothing special about a rubber band. We could replace the rubber band by a spring or any other material. As you pull on the spring, the spring elongates.

If the spring is stiff, it elongates less than the rubber band, but it still elongates somewhat. A rope may be even stiffer, and would deform even less, but a careful measurement would show that also a rope elongates when pulled.

We are nearing a definition of a force. We could define a force as an interaction—a pull or a push on an object—that can be measured by the deformation of a spring. In this case the magnitude of the force increases with the deformation of the spring. This definition is not altogether satisfactory, but it illustrates a particular type of force—what we call a *contact force*. Contact forces occur where an object is in contact with other objects.

What about a book lying on a table, are there any forces acting on the book? The book is not moving, so we may be tempted to say no. Unfortunately, this would be wrong. When we pulled on the wall with the spring, the wall was not moving, but there was still a force acting on it. What about the book—where are the forces acting on the book? First, there is one force we have not discussed so far, the force of gravity. This is one of the fundamental forces in nature: There are gravitation forces between any two objects pulling the objects toward each other. There is a gravitational force from the Earth on the book, which pulls the book downward.

What is stopping the book from moving? The table! But how? We cannot see any deformation as we could for the rubber band. This is only because you do not look carefully enough. If you zoomed in on the contact between the book and the table using a microscope, you would see that the surface of the table and the surface of the book are not flat, but rough. Small surface irregularities can be seen along the surfaces. When the book is placed on the table, these small irregularities deform (see Fig. 5.2). Each irregularity acts as a small spring, and when the irregularities are deformed, that deformation is related to the contact force between the two objects. The sum of the forces from all of these small springs is the force from the table on the book.

If we zoom further in on the contact between one surface irregularity and the table, we realize that the contact force really is a sum of electromagnetic forces between the atoms on the surface of the book and the atoms on the surface of the table. The atoms are never in actual contact, but as the book and the table are pressed toward each other, electromagnetic forces will act from the table on the book. The electromagnetic force has been shown to be part of the electromagnetic and the weak nuclear force, which is one of three fundamental forces. The other two are gravity and the strong nuclear force, which is responsible for the interactions between subatomic particles and for the interactions in the nucleus. These are the three main forces in nature, and all forces are reducible to these forces.

In most cases, we will study objects that consist of many atoms. In practice, we cannot find the sum of the forces from all the individual atoms to find the magnitude of the force, but we will instead develop simplified models for the macroscopic forces we encounter. We will call such models **force models**, and they will be our main tools for determining forces on macroscopic objects.

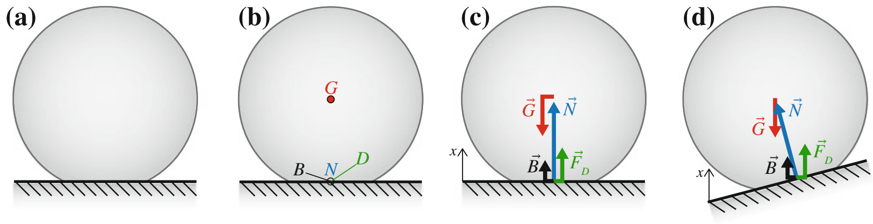


Fig. 5.3 Illustration of a ball bouncing off the floor

5.2 Identifying Forces

The first step in the "Model" box in our structured problem solving approach is to find the forces acting on the object. We therefore need a systematic way to find and identify the forces acting on an object, and we will do this by studying a specific example: A ball bouncing on the ground, as illustrated in Fig. 5.3. In the process of studying this example, we develop a general procedure for analyzing forces.

First, we need to discern between the *object*, also called the *system*, and the *environment*, which is everything else. In this case, the system is the ball, and the environment is everything else, such as the floor, the air surrounding the ball, and the Earth.[1] We have now found the first step in our procedure:

1. Divide the problem into *system* and *environment*.

In order to find the forces acting, we must realize a fundamental characteristic of a force:

> All forces acting on the *system* must have a source—an identifiable cause in the *environment*.

A force acting on the ball must be related to an interaction with something in the environment. This also means that we do not consider internal forces—forces between one part of the object and another part—we only consider external forces.

We have claimed that there are only three types of forces: gravity, the electromagnetic and weak nuclear force, and the strong nuclear force. However, this is not very helpful for our analysis of a macroscopic object such as the ball. Instead, we will divide forces into two main types:

> Forces are either *contact forces* or *long-range forces*.

[1] You will see that we need to include the Earth, since the gravitational force on the ball comes from the interaction with the whole Earth, and not just with the floor.

Contact forces, as evident from the name, are forces that occur at the contact between the system and the environment. We find contact forces by examining our drawing of the ball in Fig. 5.3. From the drawing, we see that the ball is in contact with the floor.

We generalize the procedure to find the contact points in the following steps:

2. Draw a figure of the object and everything in contact with the object.
3. Draw a closed curve around the system.
4. Find contact points—these are the points where contact forces may act.

What is the contact force at this contact? The force on the ball is from the floor, and we call this force the normal force. It is similar to the force acting on the book lying on the table, and may be considered as a sum of many forces acting along the interface between the ball and the floor. We introduce the symbol N for this force. The symbol N represents a number (with a notation), giving the strength of the force. (As we will see later, forces are measured in Newton, and N is therefore measured in Newtons.) This step in the general procedure can be summarized as:

5. Give names and symbols to all the contact forces.

The direction of the normal force from the floor on the ball depends on the direction of the floor as illustrated in Fig. 5.3d. In order to show both the direction and the magnitude of the force, we realize that a force must be a vector, and we introduce the symbol $\mathbf{N}$ for the normal force. To illustrate the normal force acting on the ball, we draw a vector starting in the contact point, acting in the direction of the normal force, and with a length related to its magnitude, as illustrated in Fig. 5.3c.

So far we have only discussed *one* of the contact forces between the ball and the environment. What other contact forces are there? The ball is also in contact with the air. The contact with the air results in several forces. Everywhere along the surface of the ball there will be small drag forces because of the difference in velocity between the surface of the ball and the air. Again, we simplify by assuming that all these small forces sum to a single force, the air resistance, $\mathbf{F}_D$, which is drawn as acting in a single point on the surface of the ball, as illustrated in Fig. 5.3c.

Similarly, there are differences in the pressure in the air, which would give rise to a buoyancy force, $\mathbf{B}$, which we again assume to be acting on the surface of the ball.

Finally, we must also look for the *long-range forces* affecting the ball. The only long-range force is the gravitational force acting from the Earth on the ball. We call this force, $\mathbf{G}$, and draw it as acting in the center of the ball, in the direction toward the center of the Earth.

This sums up the final step of our procedure:

6. Identify the long-range forces.

Free-Body Diagram

We have now defined a general procedure to find what we call the *free-body diagram* for the system. This is a diagram that identifies all the forces acting on the object, where each force is drawn as a vector starting from the point where the force is acting. The construction of the free-body diagram is central to mechanics—and it will typically be one of the first tasks you will do whenever you are solving a mechanics problem. We will therefore provide you with a detailed prescription for how to draw the free-body diagram.

Drawing a free-body diagram:
Follow these steps to find and identify all the forces acting on an object and then to draw the free-body diagram for the system.

- Divide the problem into *system* and *environment*.
- Draw a figure of the object and everything in contact with the object.
- Draw a closed curve around the system.
- Find contact points—these are the points where contact forces may act.
- Give names and symbols to all the contact forces.
- Identify the long-range forces.
- Make a drawing of the *object*. Draw the forces as arrows, vectors, starting from where the force is acting. The direction of the vector indicates the (positive) direction of the force. Try to make the length of the arrow indicate the relative magnitude of the forces.
- Draw in the axes of the coordinate system. It is often convenient to make one axis parallel to the direction of motion. When you choose direction of the axis you also choose the positive direction for the axis.

Note that when you draw a force, you indicate the positive direction for this force. If you later calculate the force and find that it is negative, it simply means that the force is acting in the opposite direction of what you thought or defined as the positive direction when you made the drawing.[2]

5.3 Newton's Second Law of Motion

We are now able to find and identify the forces acting on an object. However, we still need a connection between the forces and the motion of an object. This connection can be found through Newton's second law of motion, which relates the acceleration of an object to the forces acting on the object:

[2] You should, however, be aware that in some cases, this may mean that you have made an error in your assumptions, because some forces, such as the normal force due to a contact, cannot be negative unless the objects are glued together.

Newton's second law of motion: The force $\mathbf{F}$ on an object of inertial mass m is related to the acceleration $\mathbf{a}$ of the object through $\mathbf{F} = m\mathbf{a}$.

Newton's laws of motion are laws of nature that have been found by experimental investigations and have been shown to hold up to continued experimental investigations. Newton's laws are valid over a wide range of length- and time-scales. We use Newton's laws of motion to describe everything from the motion of atoms to the motion of galaxies.

Aspects of Newton's Second Law

Vector equation: Newton's second law is a vector equation: The acceleration is in the direction of the force, and the acceleration is proportional to the force. In this chapter, we will only study forces and motion in one dimension. We will therefore write $\mathbf{F} = F_x \, \mathbf{i}$, where $\mathbf{i}$ is the unit vector along the x-axis. The one-dimensional version of Newton's second law is then:

$$F_x = ma_x = m\frac{d^2x}{dt^2} \; . \tag{5.1}$$

Intertial mass: Newton's second law introduces a new property of an object—*the inertial mass*, m. We determine the inertial mass of an object by measuring the acceleration for a given applied force. The inertial mass is measured in Grams, with the notation g. Experimental studies show that inertial masses are additive: If we add two objects of masses m_A and m_B together, their total inertial mass is:

$$m = m_A + m_B \; . \tag{5.2}$$

Unit: Forces are measured in *Newton*, with the notation N. The definition of one Newton is that it is the force that gives an object with (inertial) mass of 1 kg an acceleration of 1 m/s^2. That is:

$$1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2} \; . \tag{5.3}$$

Net external force: The force, $\mathbf{F}$, in Newton's second law is the *net external force* acting on the object. By *external* we mean that the force has a cause outside the system, as we insisted when we drew a free-body diagram of an object. By *net force* we mean that if there are several forces acting on an object, it is the sum of all the external forces that causes the acceleration. We call this sum the net force:

$$\mathbf{F}_{\mathrm{net}} = \sum_j \mathbf{F}_j = m\mathbf{a} \; . \tag{5.4}$$

Here we have written a sum over various forces, where each force is identified by a subindex j. This is a typical way of writing the net force in shorthand. In practice, we replace the sum, $\sum_j$, by a sum of each of the external forces found in the free-body diagram.

For example, for the ball bouncing off the floor studied above, the net force on the ball is:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{N} + \mathbf{F}_D + \mathbf{B} \ . \tag{5.5}$$

Superposition: Forces are additive. We say that they obey the *superposition principle*. The acceleration due to many forces, $\mathbf{F}_i$, is the same as the acceleration of one force equal to the sum of all the small forces.

Point particles: Newton's second law applies to a *point particle*, an object located in a single point. The acceleration $\mathbf{a}$ is the acceleration of this point. While the concept of a point particle is mathematically useful, it is not that useful in a world of macroscopic objects that have spatial extent, such as any object we typically describe in mechanics.

Can we still use Newton's second law for extended objects? Yes! Newton's second law is valid for the motion of any macroscopic object—even objects that deform or change during the motions, such as bouncing football. But for a macroscopic object we need to be very precise in how we describe the position of the object, because the position is a single point whereas the object is located in many points. What point to choose? It turns out that Newton's second law is valid if we choose a particular point called the center of mass of the object (or any point on the object that does not move relative to the center of mass). We will introduce the center of mass formally later, but for now we can use the (geometric) center of the object or any other point that does not move relative to the center.

Because we can use Newton's second law for both point particles and extended objects, we do not need to discern between point particles and extended objects for now. Usually, we find it most convenient to work with real, extended physical objects, and we draw forces onto the extended objects, as shown in Fig. 5.3.

5.3.1 Example: Acceleration and Forces on a Lunar Lander

This example demonstrates how we can find forces from the acceleration, both in the case where the net force is zero and where the acceleration is measured.

You are leading a team that is building the return module for the next lunar expedition. You have designed a module that breaches if exposed to air resistance forces above 10^6 N for more than 5 s. The mass of the module is 5000 kg. To test your design you are using a numerical model that models the entry of the lander into the Earth's atmosphere. The result of such a simulation is in the form of the

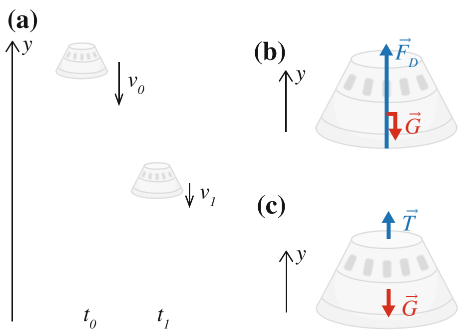


Fig. 5.4 **a** Sketch of descent of the reentry module, **b** free-body diagram of the module during reentry, and **c** during weighing

accelerations, $a(t_i)$, at a sequence of discrete times, t_i, in the file reentry.d.[3] Can the hull sustain this entry?

Free-body diagram: We illustrate the descent of the module in a sketch, as shown in Fig. 5.4. Our system is the module, and we describe its vertical position by $y(t)$.

We draw the system alone in a separate figure in order to draw the free-body diagram. The module is in contact with the surrounding air, giving rise to the air drag force, $\mathbf{F}_D$, which acts upward when the module is moving downward. This is the only contact force. In addition, it is affected by a long-range force—the gravitational force from the Earth, $\mathbf{G}$, which acts downward toward the Earth.

Newton's second law: The net force on the module is:

$$\mathbf{F}_{\text{net}} = \mathbf{F}_D + \mathbf{G} , \tag{5.6}$$

where $\mathbf{F}_D$ acts upward when the module is moving downward, hence $\mathbf{F}_D = F_D\mathbf{j}$, and $\mathbf{G}$ acts downward, that is, in the negative y-direction, $\mathbf{G} = -G\mathbf{j}$:

$$\mathbf{F}_{\text{net}} = (F_D - G)\,\mathbf{i} . \tag{5.7}$$

We remove the vector notation since we are looking at motion along the y-axis, getting the net force along the y-axis:

$$F_{\text{net}} = F_D - G , \tag{5.8}$$

which is the equation you will usually start from—we do not usually include the whole vector notation derivation when discussing a one-dimensional problem.

Newton's second law for motion along the y-axis gives:

$$F_{\text{net}} = F_D - G = ma_y , \tag{5.9}$$

[3] http://folk.uio.no/malthe/mechbook.

since we know $a_y(t_i)$ at given times t_i, we may use this relation to find $F_D(t_i)$ at the same time intervals, if we only knew $G(t_i)$.

Measuring the gravitational force: To determine $G(t_i)$, we need a force model—a model that gives us a number value (and unit) for the gravitational force. You probably already know this law, $G = mg$, and we will introduce it later on—but what could you do if you did not know it? We could device a experiment to measure the gravitational force G on the module. We hang the module in a wire, and measure the force in the wire when the module is at rest. This is the principle behind a weight. Figure 5.4c illustrates the free-body diagram for this experiment. Since the module is not moving the only contact force is $\mathbf{T}$, the force from the wire on the module. Newton's second law in the y-direction becomes:

$$F_{\text{net}} = T - G = ma_y \; . \tag{5.10}$$

However, we have designed this experiment so that $a_y = 0\,\text{m/s}^2$, since the module is not moving, therefore

$$T - G = 0 \;\Rightarrow\; T = G \; . \tag{5.11}$$

We have found that we can measure G by measuring T, which gives $T = G = 49{,}000\,\text{N}$.

You will find that this use of Newton's second law is very common, and we will return to it many times. Problems where there is no motion—or motion with constant velocity—are often called static problems.

Calculating air resistance force: Since we now know that $G = 49{,}000\,\text{N}$ (and we assume this is a constant throughout the motion), we can now use the data we have for $a_y(t_i)$ for the module to find the air resistance force on the module. From Newton's second law we have:

$$F_D - G = ma_y \;\Rightarrow\; F_D = G + ma_y \; , \tag{5.12}$$

Since the acceleration is changing throughout the motion, the air resistance force F_D is also a function of time. We read the accelerations from the file reentry.d[4] using the following program.

```
t,a = loadtxt('reentry.d',unpack=True)
W = 49000.0   # N
m = 5000.0     # kg
D = W + m*a plot(t,D)
xlabel('t [s]')
ylabel('D [N]')
```

Analysis: We see from the plot of $F_D(t)$ in Fig. 5.5 that while F_D is decreasing, it is larger than the limit for more than 5 s. With this entry, the hull will breach!

[4] http://folk.uio.no/malthe/mechbook/reentry.d.

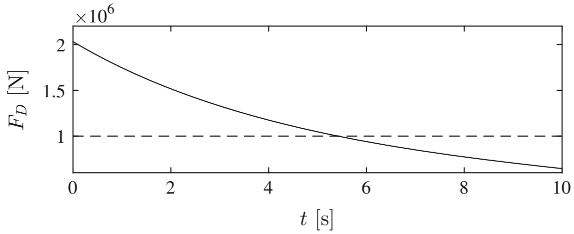


Fig. 5.5 Plot of the air resistance force, F_D, as a function of time

Additional material: We can find the time when the air resistance force becomes less than $F_D^C = 10^6$ N, by first finding the smallest i where $F_D(t_i)$ is less than F_D^C, and then finding the corresponding t_i. This is done by:

```
>> i = min(find(FD<1e6))
>> ti = t[i]

 ti =      5.4246
```

This shows that the air resistance force falls to F_D^C after 5.42 s. The module needs to be redesigned. You may get ideas as to how when you learn about air resistance later in this chapter.

5.4 Force Models

In order to use Newton's second law to determine the acceleration of an object, we need to find out how large a force is—we need to determine its magnitude and direction. For this, we need theories that provide numerical values for the forces. We call such models "force models". The force models may be based on direct, experimental measurements. We often call such models phenomenological or experimental force models. The force models can also be based on a more fundamental model or a model based on a microscopic view of the interactions.

In the following we introduce models for some of the most common types of forces acting between macro- and microscopic objects. These models will be your toolbox for addressing physical processes—you need to continually build on this toolbox, as this will be your reservoir of physical knowledge. If you want to describe a ball falling through air, you need mathematical expressions for the forces on the ball: both the force due to gravity and the force due to air resistance. If you want to describe the motion of a nanometer sized particle in water close to a charged surface you need to introduce (probably sophisticated) models for the forces between the particle and the individual water molecules and between the particle and the surface.

The range of problems you can solve depends on your knowledge of interactions—forces—and on your ability to simplify a complicated situation to a model that only contains forces you know how to address.

5.5 Force Model: Gravitational Force

Another of Newton's great accomplishments is his discovery of the law of gravity.

According to **Newton's law of gravity**, there are attractive, gravitational forces between all objects. The gravitational force on object A from object B is:

$$\mathbf{F}_{\text{from B on A}} = \gamma \frac{m \cdot M}{r_{AB}^3} \mathbf{r}_{AB} \ , \tag{5.13}$$

Here $\mathbf{r}_{AB}$ is a vector pointing from the center of object A to the center of object B, and r_{AB} is the length of this vector, corresponding to the distance between the centers of objects A and B (see Fig. 5.6). The quantities m and M are the gravitational masses of objects A and B respectively, and γ is the gravitational constant.

All experimental evidence shows that the gravitational masses are the same as the inertial masses. We will therefore use the same symbol, m for both the inertial mass and the gravitational mass of an object.

Constant Gravity

The general gravitational law becomes simpler for an object near the surface of the Earth or another planet. In this case, the distance r_{AB} from the object to the center of the Earth is approximately constant and equal to the radius R of the Earth. The gravitational force can therefore be approximated by:

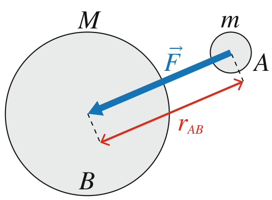


Fig. 5.6 The gravitational force **F** from object B on object A

The gravitational force near the surface of a planet is approximately constant, and equal to:

$$\mathbf{G} = -m \underbrace{\frac{\gamma M}{R^2}}_{=g} \mathbf{j} = -mg\,\mathbf{j} \,, \tag{5.14}$$

where the unit vector $\mathbf{j}$ points in the upward direction, and g is called the acceleration of gravity .

The constant g only depends on the radius and mass of the planet.

The Acceleration of Gravity

Why do we call g the acceleration is gravity? Because this is the acceleration of an object that is subject only to a gravitational force, which is easily seen from Newton's second law applied to an object only affected by gravity:

$$m\mathbf{a} = \mathbf{G} = -mg\,\mathbf{j} \,, \tag{5.15}$$

$$\mathbf{a} = -g\,\mathbf{j} \,, \tag{5.16}$$

For an object on the surface of the Earth, the acceleration of gravity is approximately $g = 9.81\ \text{m/s}^2$, whereas for an object on the surface of the Moon, the acceleration of gravity is $g_m = 0.17\,g$. You can find a table of the acceleration of gravity on the surface of various objects in the solar system in Table 5.1.

Table 5.1 The acceleration of gravity on the surface of various objects in the Solar system

Body	Mass (kg)	Radius (km)	g (m/s^2)	g/g_e
Sun	1.99×10^{30}	6.96×10^5	274.13	27.95
Mercury	3.18×10^{23}	2.43×10^3	3.59	0.37
Venus	4.88×10^{24}	6.06×10^3	8.87	0.90
Earth	5.98×10^{24}	6.38×10^3	9.81	1.00
Moon	7.36×10^{22}	1.74×10^3	1.62	0.17
Mars	6.42×10^{23}	3.37×10^3	3.77	0.38
Jupiter	1.90×10^{27}	6.99×10^4	25.95	2.65
Saturn	5.68×10^{26}	5.85×10^4	11.08	1.13
Uranus	8.68×10^{25}	2.33×10^4	10.67	1.09
Neptune	1.03×10^{26}	2.21×10^4	14.07	1.43
Pluto	1.40×10^{22}	1.50×10^3	0.42	0.04

There are local variations of g along the Earth's surface due to deviations of the spherical shape of the Earth, due to topographical variations, and due to differences in density in the Earth's crust. In addition, there are variations in the effective g due to the rotation of the Earth. Surveys of variations in g due to density differences in the crust are used as a remote sensing technique that gives important information about the properties of rocks present in the Earth's crust. This technique is routinely used for example for petroleum exploration.

5.6 Force Model: Viscous Force

You will often encounter objects that are in contact with a surrounding fluid such as a air or water. We therefore need a force model for the interaction between fluids and solid objects. Unfortunately, there is no fundamental law of nature for such an interaction. Instead, we must determine the force model from experiments or calculations based on an underlying model for fluid flow, and use this result as our model. Fortunately, experiments and calculations show that the force from the fluid has a simple form—it depends on the velocity of the object.

Drag Force at Low Velocities

If you pull a sphere through water, you expect a contact force F_D from the water on the sphere counteracting the motion of the sphere relative to water because water is forced to flow around the sphere. Both experiments and theoretical models show that for low velocities the fluid flows smoothly around the object (see Fig. 5.7), and the force, F_D, from the fluid is proportional to the velocity of the object relative to the fluid.

The **viscous force** on an object moving at a velocity v relative to a fluid is:

$$F_D = -k_v v \ , \tag{5.17}$$

where k_v is a constant that depends on the objects size, shape and surface, as well as on the (dynamic) viscosity of the fluid. Stokes showed that at low velocities

$$k_v = 6\pi\eta R \ , \tag{5.18}$$

where R is the radius of the sphere and η is the viscosity of the fluid. The viscosity of air is $\eta = 1.82 \times 10^{-5}\,\mathrm{Nsm^{-2}}$ (at room temperature), and the viscosity of water is $\eta = 1.00 \times 10^{-3}\,\mathrm{Nsm^{-2}}$ (at room temperature).

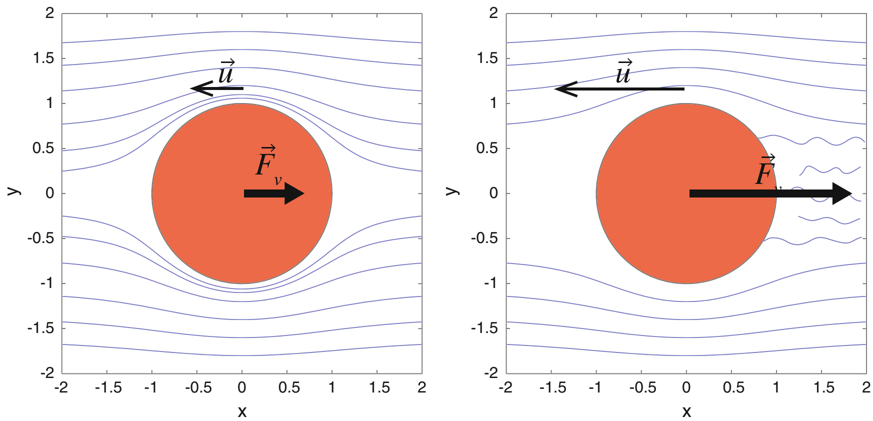


Fig. 5.7 Illustration of a drag force on an object due to the motion of the object relative to the surrounding fluid. At low velocities the fluid flow around the object is smooth, and the force is approximately proportional to the velocity. At higher velocities, the fluid flow becomes turbulent near the surface, the flow becomes irregular, and the force is approximately proportional to the square of the velocity

We call a force that is proportional to the velocity (of the object) a *viscous force*. If the force is a contact force from a fluid we also often use the term *drag force* to describe the interaction.

Drag Force at High Velocities

At larger velocities the fluid flow around the object becomes more irregular (see Fig. 5.7), and the drag force is not mainly related to the forces required to drag the fluid along the surface of the object, but instead depends on the under-pressure generated behind the object. In this case, the force is proportional to the square of the velocity, and we call this law the square law of air resistance.

The **drag force** on an object moving at a velocity v relative to a fluid is:

$$F_D = -Dv^2 , \tag{5.19}$$

acting in the direction opposite the velocity.

The minus sign shows that the force acts in the direction opposite the velocity. The prefactor D is a constant that depends on the objects size, shape and surface, and the density of the fluid. Experimental data gives an approximative value for D for a spherical object:

$$D \simeq 12.0\,\rho\,R^2 \; . \tag{5.20}$$

where ρ is the density of the fluid, and R is the radius of the sphere.

General Model for Fluid Drag

The behaviors for low and high velocities are special cases of a more general model for the drag force. Experiments show that the drag force can be written in the general form:

$$F_D = \frac{1}{2}\rho\,S\,C_D(v, \eta, \rho, d)v^2 \; , \tag{5.21}$$

where ρ is the density of the fluid. For air $\rho = 1.293\,\mathrm{kg/m^3}$ at normal pressures and temperatures. S is the cross-sectional area of the object—that means the area of the object's projection on a plane normal to the direction of motion. For a sphere with radius r the cross-sectional area is $S = \pi r^2$. The speed of the object relative to the fluid is v. The viscosity of the fluid is η, and d is a characteristic length scale, such as the diameter of a sphere. The coefficient $C_D(\ldots)$ is called the drag coefficient and describes the details of the air resistance—the physics of drag is hidden in this function. For a given object type—such as a sphere of a given material—experiments show a remarkable feature: The drag coefficient only depends on one number, Re, called the Reynold's number:

$$C_D(v, \eta, \rho, d) = C_D(Re) \; , \;\; Re = \frac{\rho d}{\eta} v \; . \tag{5.22}$$

This is a surprisingly compact description with many interesting implications. For example, if you increase the velocity by a factor 10 you get the same drag coefficient if you either reduce the radius by a factor 10 or increase the viscosity by a factor 10. The general function for the drag coefficient, $C_D(Re)$, for a smooth sphere is shown in Fig. 5.8. (You can find the data-set for part (a) in cdreynolds.d[5]).

For small velocities, we expect to recover Stokes' general result: $F_D = 6\pi\eta R v$, which implies that $C_D(Re) \propto Re^{-1}$ for small v (and Re). This is indeed what we observe in Fig. 5.8, where Re^{-1} is shown as a straight line. For large velocities, we see that the drag coefficient becomes approximately constant, and the force is therefore proportional to the square of the velocity. Our simplified models for small and large velocities are therefore consistent with the experimental results, and we

[5] http://folk.uio.no/malthe/mechbook/cdreynolds.

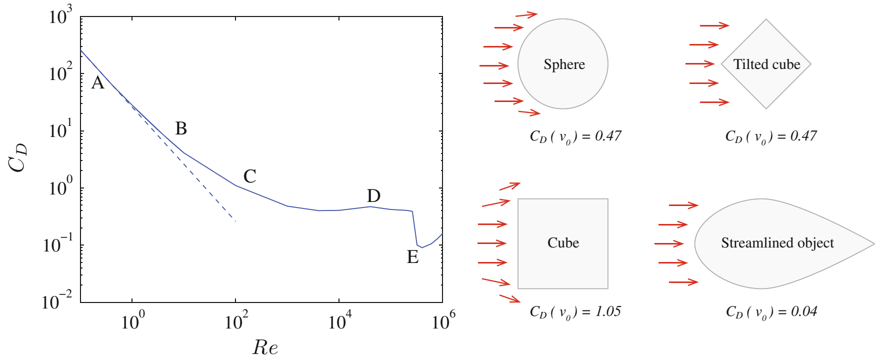


Fig. 5.8 **a** The drag coefficient C_D as a function of Reynold's number, Re $= \rho d v/\eta$ based on experimental data. **b** The drag coefficient C_D at the same velocity for object with the same cross-sectional area, but with different shapes

now also have a better understanding of what small and large means: Small means $Re \ll 1$ and large means $Re \gg 1000$.

Something strange happens when the Reynold's number reaches Re $= 3.2 \times 10^5$: The drag coefficient drops significantly! A careful calculation (which you can program yourself) shows that not only the drag coefficient but also the fluid drag force falls. How can the drag force decrease when the velocity increases? This effect is due to boundary-layer turbulence (which we will not explain here). The transition point where this effect kicks in depends on surface properties of the object. For a rough surface, such as that of a golf ball, the transition occurs for a lower Reynolds number than for a smooth ball. This is the reason why golf balls have a rough surface: The air drag force for large velocities is reduced by this design.

What happened to aerodynamic design? This is also hidden in the drag coefficient. The value of the drag coefficient depends on the shape of the object, and more aerodynamic designs have lower drag coefficients for approximately the same cross-sectional area, as illustrated in Fig. 5.8.

5.6.1 Example: Falling Raindrops

This example demonstrates how we can find the motion of an object subject to a constant force (gravity), and to a velocity-dependent force.

Raindrops are often as small as $d = 1$mm in diameter as they start falling. Here, we apply the structured problem-solving approach to find their velocity, first without air resistance, and then with a model for viscous drag.

Sketch and Identify: We always start addressing a problem through a sketch which should include *the system*, a raindrop, *the environment*, and *a coordinate system* (see Fig. 5.9). We describe the position of the raindrop by its vertical position $y(t)$ as a function of time t.

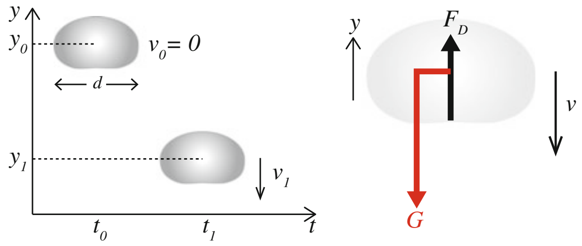


Fig. 5.9 *Left* Illustration of a raindrop falling down. *Right* Free-body diagram for the drop. (Notice that a real raindrop is neither perfectly spherical nor "drop-shaped", it is pushed flat by the air resistance force)

Model: The motion of the raindrop is determined by the forces acting on it. We draw the forces in a free-body diagram, as illustrated in Fig. 5.9. The sketch shows that the raindrop is only in contact with the surrounding air, which gives rise to an air resistance force F_D. In addition, the raindrop is affected by gravity, G, from the Earth.

We have force models for each of these forces. We know that gravity is $\mathbf{G} = -mg\mathbf{j}$, where m is the mass of the raindrop. For the air resistance F_D we assume that we can use the viscous law:

$$\mathbf{F}_D = -k_v v(t)\,\mathbf{j}\ , \tag{5.23}$$

where $v(t)$ is the velocity of the drop. You should check with yourself that this force indeed has the correct sign. Remember that when the drop falls downward its velocity is negative.

Newton's second law: We apply Newton's second law to find the acceleration of the drop:

$$\mathbf{F}^{\text{net}} = \mathbf{F}_D + \mathbf{G} = -mg\,\mathbf{j} - k_v v(t)\,\mathbf{j} = ma\,\mathbf{j}\ , \tag{5.24}$$

which corresponds to

$$-mg - k_v v(t) = ma\ . \tag{5.25}$$

We lack two of the numbers in this equation: m and k_v. We can find the mass of the drop by assuming that it is spherical and made of water. The volume of a sphere is $V = (4\pi/3)r^3$, where $r = d/2$ is the radius of the sphere, and the mass density of water is $\rho = 1000.0\,\text{kg/m}^3$. The mass of the drop is therefore

$$m = \rho\, V = \rho\,\frac{4\pi}{3}r^3 = \rho\,\frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = 5.24 \times 10^{-7}\,\text{kg}\ . \tag{5.26}$$

We find k_v from Stokes' formula in (5.18): $k_v = 6\pi R\eta$. The radius of the raindrop is $r = 0.5 \times 10^{-3}$ m, and the viscosity of the air is $\eta = 1.82 \times 10^{-5}\,\text{Nsm}^{-2}$. This gives $k_v = 1.85 \times 10^{-7}\,\text{Nsm}^{-1}$.

Finally, to calculate the motion of the drop, we need to know its initial conditions: It starts from $y(0) = h$ at $t = 0$ s at rest, that is, with $v(0) = 0$ m/s.

Simplified model: No air resistance: First, what happens in the simplified case when we have no air resistance? In that case, $k_v = 0$, and the acceleration is a constant

$$a = -g \ . \tag{5.27}$$

We find the velocity as function of time by direct integration:

$$v(t) - v(0) = \int_0^t a \, dt = -gt \ . \tag{5.28}$$

This corresponds to a free fall, as we have seen previously. We expect this to only be a good approximation as long as the air resistance term is small, that is as long as $k_v v(t)$ is much smaller than mg, that is when

$$k_v v \ll mg \ \Rightarrow \ v \ll \frac{mg}{k_v} = \frac{5.24 \times 10^{-7} \,\mathrm{kg}\, 9.8 \,\mathrm{m/s^2}}{1.85 \times 10^{-7} \,\mathrm{Nsm^{-1}}} \simeq 27.8 \,\mathrm{m/s}. \tag{5.29}$$

We will check how good this approximation is further on.

Simplified model: Constant velocity: What will happen as the drop starts to fall? It starts from zero velocity, hence the initial acceleration will be $a = -g - (k_v/m)v = -g$. As the drop falls, the velocity becomes a negative number, but with increasing magnitude. The acceleration, $a = -g - (k_v/m)v$ will therefore approach zero. However, if the acceleration becomes zero, the velocity will no longer change, and the drop will have reached a stationary velocity—a velocity that does not change with time. This occurs when

$$a = -g - \frac{k_v}{m} v = 0 \ \Rightarrow \ -g = \frac{k_v}{m} v \ \Rightarrow \ v = -\frac{mg}{k_v} \ . \tag{5.30}$$

We call this velocity the *terminal velocity*, v_T:

$$v_T = \frac{mg}{k_v} \ . \tag{5.31}$$

We therefore expect the drop to approach the velocity $v = -v_T$ as time increases.

Full model: Numerical solution: We now know both the initial behavior, $a = -g$, and the asymptotic behavior, $a \to 0 \,\mathrm{m/s^2}$, $v \to -v_T$. We can find the velocity by solving

$$a = \frac{dv}{dt} = -g - \frac{k_v}{m} v \ , \tag{5.32}$$

with initial conditions $v(t_0) = 0$ m/s using Euler's method:

$$v(t_i + \Delta t) = v(t_i) + \Delta t \cdot a(t_i, v(t_i)) \ . \tag{5.33}$$

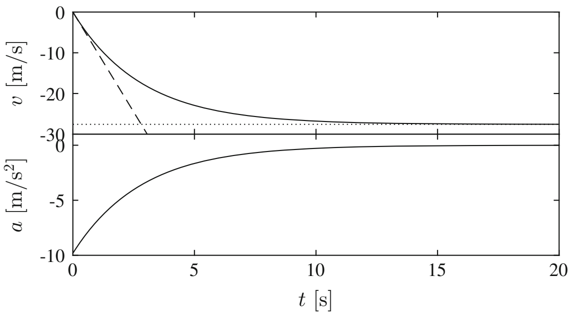
This method is implemented in the following program:

```
from pylab import *
#
# Physical variables
g = 9.81
kv = 1.85e-7 # Nsm^-2
m = 5.2e-7   # kg
time = 20.0
dt = 0.001
# Initial conditions
v0 = 0.0
# Numerical initialization
n = int(round(time/dt))
v = zeros(n,float)
a = zeros(n,float)
t = zeros(n,float)
# Set initial values
v[0] = v0
# Integration loop
for i in range(n-1):
  a[i] = -g - (kv/m)*v[i]
  v[i+1] = v[i] + a[i]*dt
  t[i+1] = t[i] + dt
```

Analysis of numerical solution: The resulting plot of $v(t)$ and $a(t)$ is shown in Fig. 5.10. We see that both $v(t)$ and $a(t)$ behaves as we expected. The drop starts with zero velocity and an initial acceleration of $-g$. The acceleration reduces as the velocity increases and the drop reaches a stationary state where it moves with constant velocity. We compare with the two simplified models. The behavior without air resistance is plotted as a dashed line, and it is indeed a reasonable approximation for small velocities, that is when $|v| \ll v_T = 27.5$ m/s. For long times the velocity approaches $v \to -v_T$, which is illustrated by the dotted line in the plot. The simplified solutions are therefore useful to check if our numerical solution is correct.

Full model: Analytical solution: Now that we have both found the simplified and the numerical solution, we are ready to attempt an analytical soultion. In this case

Fig. 5.10 Plot of $v(t)$ and $a(t)$ for the drop



we are fortunate, since the particular equation in (5.32) can be solved analytically using separation of variables. We simplify the equation by writing:

$$\frac{dv}{dt} = -g - \frac{k_v}{m}v = -g - \frac{g}{v_T}v = -g\left(1 + \frac{v}{v_T}\right) , \tag{5.34}$$

with initial condition $v(0) = 0$ m/s. We separate v and t on each side of the equation:

$$\frac{dv}{1 + v/v_T} = -gdt , \tag{5.35}$$

and integrate each side from $t_0 = 0$ to t:

$$\int_{v_0}^{v(t)} \frac{dv}{1 + v/v_T} = -\int_0^t gdt . \tag{5.36}$$

We introduce $u = 1 + v/v_T$, $du = dv/v_T$, $dv = v_T du$:

$$\int_1^{1+v(t)/v_T} v_T \frac{du}{u} = -g(t - 0) \Rightarrow v_T \ln\left(1 + \frac{v(t)}{v_T}\right) = -gt , \tag{5.37}$$

$$\ln\left(1 + \frac{v(t)}{v_T}\right) = -\frac{gt}{v_T} \Rightarrow, 1 + \frac{v(t)}{v_T} = e^{-gt/v_T} \Rightarrow v(t) = v_T\left(e^{-gt/v_T} - 1\right) . \tag{5.38}$$

Full model: Symbolic solution: You can solve (5.34) using the symbolic solver in Python. First, we define the variables `g`, `u` (corresponding to V_T) and `v(t)`

```
>> from sympy import *
>> v = Function('v')
>> from sympy.abc import t
>> from sympy.abc import g
>> from sympy.abc import u
```

Python can then solve the equation with the initial condition by

```
>> dsolve(Derivative(v(t),t)+g+g/u*v(t),v(t))
v(t) == C1*exp(-g*t/u) - u
```

This corresponds to the solution

$$v(t) = v_T\left(e^{-gt/v_T} - 1\right) . \tag{5.39}$$

In most cases, machines are much better than humans at integration. In your career as a physicist it is therefore more important to be able to formulate problems so that they can be solved numerically or symbolically than to be able to solve them analytically yourself.

Test your understanding: What would happen if the drop started with a velocity $v_0 = -2v_T$?

5.7 Force Model: Spring Force

The two most common contact forces for a macroscopic object are due to

- contact with a fluid or
- contact with another solid.

We have seen that we can use a velocity-dependent force to model fluid-solid contacts. What about solid-solid contacts? The contact forces between two solid objects come from the deformation of the objects (and from surface forces such as adhesion and friction, but we will address such effects later). How can we model contact forces due to deformation?

Spring Force

First, we can *measure* the force due to deformation directly. Figure 5.11 illustrates an experiment where we pull a rubber band and measure the force needed to extend the band a distance ΔL. Figure 5.11 shows that the force, F increases linearly with ΔL:

The force F required to extend an object by ΔL is the **spring force**:

$$F = k\Delta L \ , \tag{5.40}$$

This relationship is valid for small deformations ΔL for practically all materials.

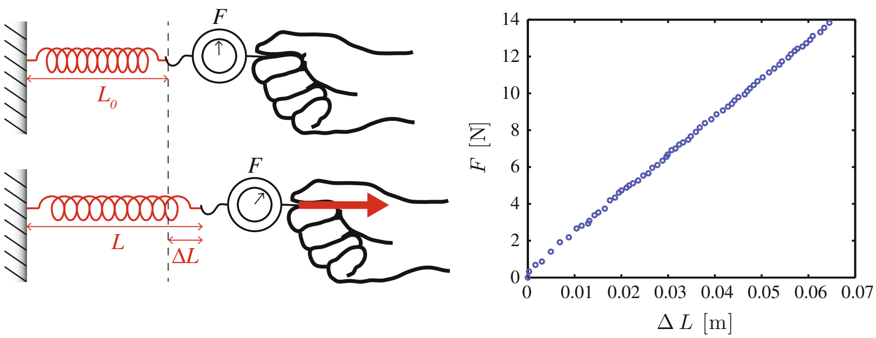


Fig. 5.11 Illustration of an experiment to measure the force needed to extend a spring a distance ΔL, and a plot of the force, $F(\Delta L)$

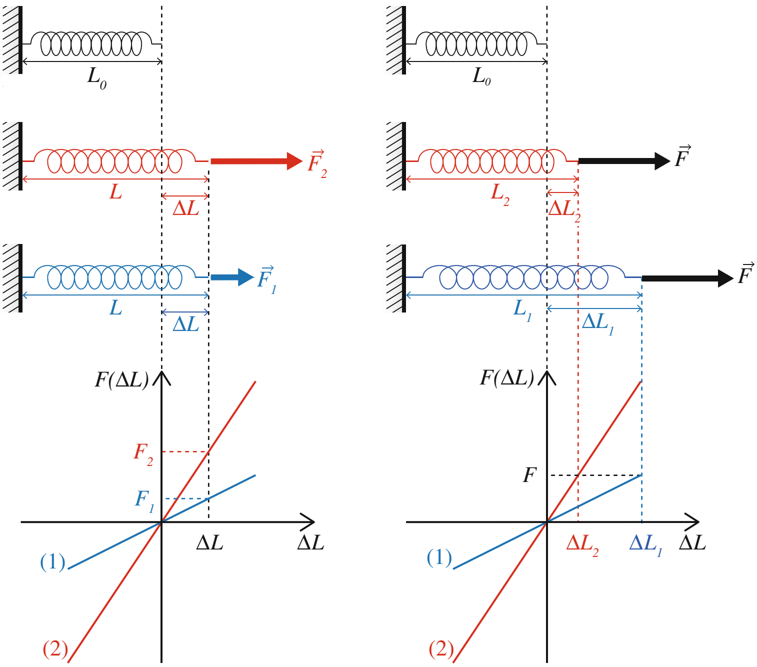


Fig. 5.12 Illustration of a stiff spring (*1*) and a weak spring (*2*) extended by the same length ΔL (*on the left*), and affected by the same force F (*on the right*)

Why do we call this force a *spring force*? Because a coiled spring is constructed so that the deformation force is proportional to the extension ΔL also for large extensions. Coiled springs are commonly used in experimental demonstrations and in many mechanical devices. Here, we usually base our discussions on the behavior of a spring, but keep in mind that the results are valid for the deformation of any object as long as the deformation is small compared to the size of the object.

Spring constant, k : The constant k is called the *spring constant*, which is the slope of the $F(\Delta L)$ curve. Figure 5.12 shows the behavior for a weak spring (1) and for a stiff spring (2). The stiff spring has a larger spring constant than the weak spring, $k_2 > k_1$.

If we want both springs to extend the same length ΔL, we need to apply a larger force to the stiffer spring than to the weaker spring: $F_2 > F_1$ (see left side in figure).

If we apply the same force to both springs, the weak spring extends further than the stiff spring: $\Delta L_1 > \Delta L_2$ (see right side in figure). This is consistent with our intuition: If we pull with the same force on a rubber band and on a stiff rope, the rubber band deforms more than the rope. Hence the rubber band has a smaller spring constant than the rope.

The spring constant characterizes a particular object. If we redo the experiment with the same spring, we find the same spring constant every time.

Elongation, ΔL: A physical object such as a spring has a non-zero length when it is not stretched. We call this length the *equilibrium length*, L_0, of the spring. The elongation of the spring is the difference between the length of the spring, L, and the equilibrium length L_0:

$$\Delta L = L - L_0 \ . \tag{5.41}$$

Various elongations are illustrated in Fig. 5.12. If the spring is stretched, ΔL is positive. We usually assume that a spring can be also compressed, even if this may not be physically realistic. (You cannot really compress a rope, it will buckle instead). For a compressed spring, the length of the spring is smaller than the equilibrium length, and ΔL is negative.

Sign: You have to determine the sign in front of $k\Delta L$ in (5.40) using your physical intuition. It is not that difficult: All you have to remember is that in order to extend the spring, you need to pull at the spring in the direction you extend it.

Spring-Block Models

We now know that the force required to extend a spring a length ΔL is $F = k\Delta L$. How can we use this to determine the motion of an object *attached to a spring*? In Fig. 5.13, we have illustrated two general cases: A block attached to a spring, and a ball in contact with a spring. Both objects slide on a frictionless surface, so that there are no other horizontal forces: The only horizontal force acting on the block (ball) is from the spring.

A Block Attached to a Spring
The force, F, on the block from the spring is the same as the force acting on the spring from the block, but it has the opposite direction. (Later, when we introduce Newton's third law, we see that these forces are action-reaction pairs.)

$$F(\Delta L) = -k\Delta L \ . \tag{5.42}$$

The sign is chosen so that when the spring is extended—it is longer than in equilibrium—the force is negative, that it, the force acts in the negative x-direction.

We describe the position of the block by the position, x, of the left-hand side of the block, where it is attached to the spring. The other end of the spring is attached to a wall at $x = 0$. The extension of the spring depends on the position of the block:

$$\Delta L = L - L_0 \ , \tag{5.43}$$

where $L = (x - 0) = x$ is the length of the spring.

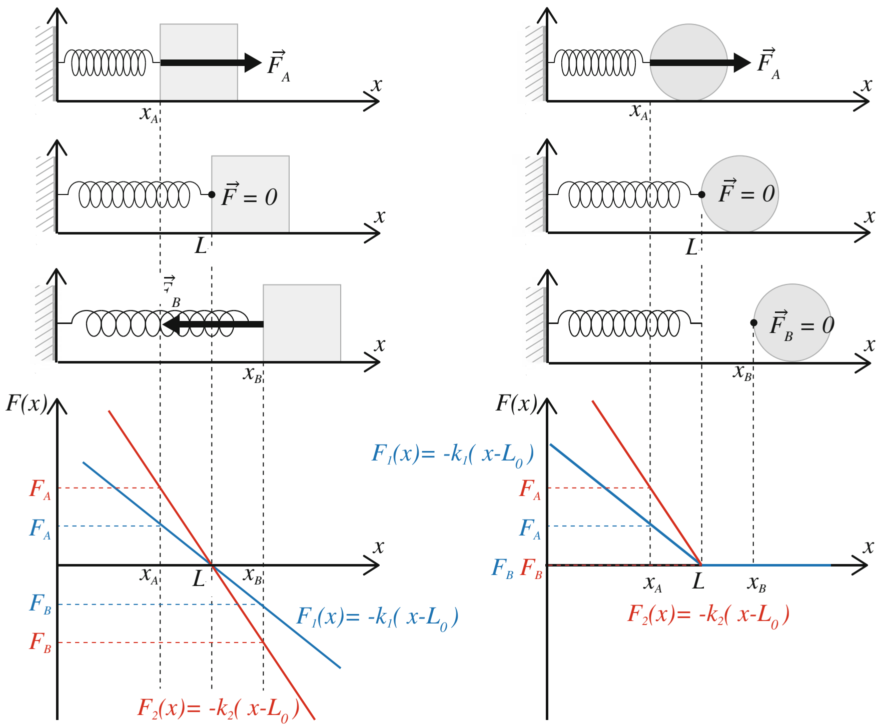


Fig. 5.13 Illustration of the force on a block attached to a spring (*left*), and on a ball colliding with (but not attached to) a spring (*right*)

The force F on the block consequently depends on the position of the block:

$$F = F(x) = -k(x - L_0) . \tag{5.44}$$

The spring force acting on an object attached to the spring is therefore an example of a **position-dependent force**.

Why is it important if a force is position-dependent? Because this means that the position occurs on both sides of the equation of motion following from Newton's second law:

$$F = ma \Rightarrow a = \frac{d^2x}{dt^2} = -\frac{k}{m}(x - L_0) . \tag{5.45}$$

We need to solve a differential equation to determine the motion of the block.

This also illustrates another important lesson: If the force model includes variables that change during motion, we need to rewrite the force model to include the position of the object. A common mistake is to stop at (5.42), where the force, $F(L)$, depends on the length L, and not realize that L is a function of the position, $L = L(x)$, and that the equation must be solved as a differential equation.

A Ball Colliding with a Spring

The ball in Fig. 5.13 is not attached to the spring—it is not glued to the spring as the block was. The ball is therefore only affected by the spring force as long as the spring is compressed, that is, when $\Delta L < 0$. We must rewrite the force model to reflect this:

$$F(\Delta L) = \begin{cases} -k\Delta L \ , & \Delta L < 0 \\ 0 \ , & \Delta L \geq 0 \end{cases} \tag{5.46}$$

The rest of the analysis is exactly the same as for the ball: The spring force is position dependent, and can be written as:

$$F(\Delta L) = \begin{cases} -k\,(x - L_0) \ , & x < L_0 \\ 0 \ , & x \geq L_0 \end{cases} \tag{5.47}$$

Notice that this force model is non-linear and the problem therefore requires special consideration when you solve it: You must remember that the force is zero as soon as the ball loses contact with the spring.

Notice also that the ball-spring contact is a lot like the normal force. This is not a coincidence. The spring-ball model is actually our simplest model for the normal force, as we discuss in the following section, and the spring-block model is the simplest model for a contact force between two attached objects. These are probably the models you will use the most during your physics studies.

Contact Forces

The spring force was introduced as an approximative model for the force due to deformation. It is based on experimental evidence: We find the law by measuring the force as a function of the deformation. And the law is surprisingly versatile: We can use it as a model for almost any contact force between solid objects. Let us see exactly how we map a complicated, realistic deformation problem onto the simplified spring model description.

Figure 5.14 shows a computer simulation of a collision between a soft, round disk and a solid wall that does not deform. The top figures show the deformation of the ball, where the colors indicate the magnitude of the forces inside the disk. During the collision, the ball is deformed, but how do we measure the extent of deformation? The simplest approach is to look at how much the ball has changed from its original,

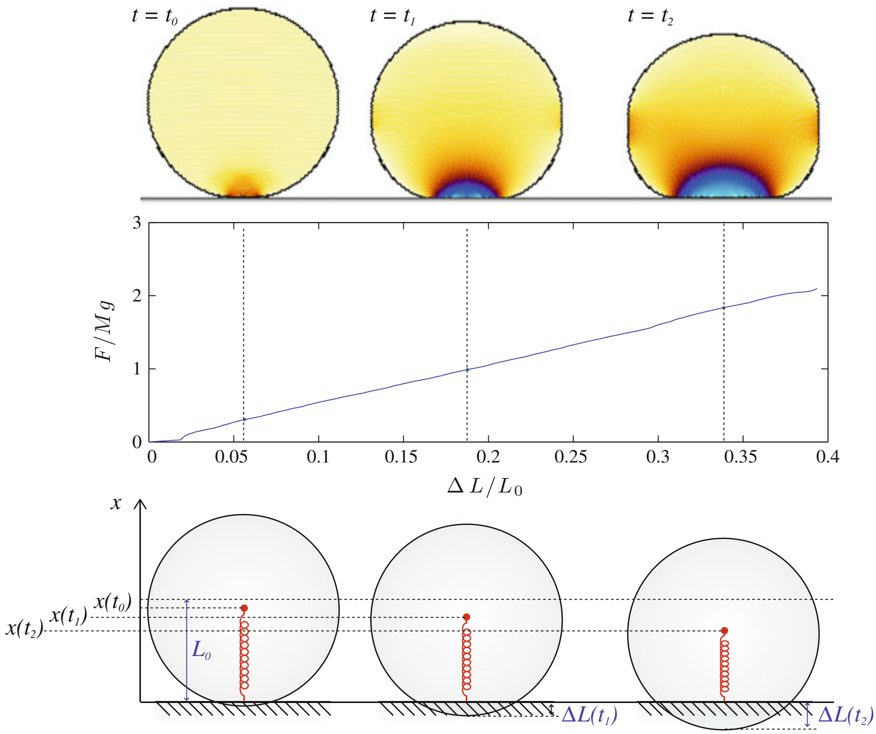


Fig. 5.14 *Top* Computer model of a soft ball colliding with a solid wall. *Middle* Plot of the force from the wall on the ball as a function of displacement. *Bottom* Illustration of how the displacement ΔL is measured, and an illustration of the spring-block model used to represent the soft ball

round shape. This measurement of ΔL is illustrated in the bottom figures, where we see that the original shape overlaps with the wall, and we define ΔL as the length of this overlap. The middle figure shows the force from the wall on the ball as a function of deformation, ΔL, and it is indeed a linear function, $F = k\Delta L$, as we found for a spring!

We can therefore use the spring model as a model for the deformation of a ball, or as a first approximation for the deformation of any deforming object. In this case we say that we use the spring model as an approximative model for the contact force. In the bottom of Fig. 5.14 we illustrate that we model the contact between the disk and the floor by representing the disk as a mass located at the center of the sphere connected to a massless spring of length $L_0 = R$ equal to the radius of the sphere.

We may also express the force in terms of the position x of the center of the ball. We notice that the spring is compressed, $\Delta L = L - L_0 = x - R < 0$, during the collision, and we expect the force on the ball to act upward while the ball is compressed, hence:

$$F = -k\Delta L = -k\,(x - R)\ , \tag{5.48}$$

is the force on the ball from the wall.

Generally, we do not know what the spring constant will be for such a model. We need either to measure the spring constant or to find the spring constant from a theoretical consideration based on for example elasticity theory. You must also ensure that you use a reasonable version of the spring model. For example, for the collision between a ball and a wall in Fig. 5.14, where the ball does not adhere to the wall, we must use a spring-ball model without attachment.

We can use the spring model both when the object itself is deformed, as illustrated in Fig. 5.14, as well as when the wall deforms while the ball remains practically undeformed—a steel ball bouncing on a mattress, or when both the ball and the wall deforms.

Test your understanding: How would you represent the force between an undeformable ball and a deformable surface? Make a drawing, and show how you introduce the spring that models the deformation.

Normal and Contact Forces

The force from the wall on the ball is the *normal force* on the ball. However, a normal force only acts as long as the objects are in contact.

We can **model a normal force** between two objects using a spring model:

$$F = \begin{cases} k\Delta L & \text{while in contact} \\ 0 & \text{otherwise} \end{cases} , \tag{5.49}$$

where ΔL describes the deformation of the two objects.

For the ball in Fig. 5.14 the normal force F is:

$$F = \begin{cases} -k(x-R) \ , & x < R \\ 0 \ , & x \geq R \end{cases} . \tag{5.50}$$

We can use this force model to find the motion of a bouncing ball if we also include the effect of gravity, $G = mg$. (We assume air resistance is negligible). The acceleration is then found from:

$$a = \frac{1}{m} F^{\text{net}} = F - G \ , \tag{5.51}$$

and the resulting motion is illustrated in Fig. 5.15.

The normal force only acts when two objects are pressed together. On the other hand, if the objects are *attached* to each other, the contact force acts both when the objects are pressed together and when they are pulled apart. Examples of attached objects are: two objects that are glued or welded together or attached by surface forces such as adhesion, two parts of one larger solid body such as the two lobes on a dumbbell, or two atoms in a diatomic molecule.

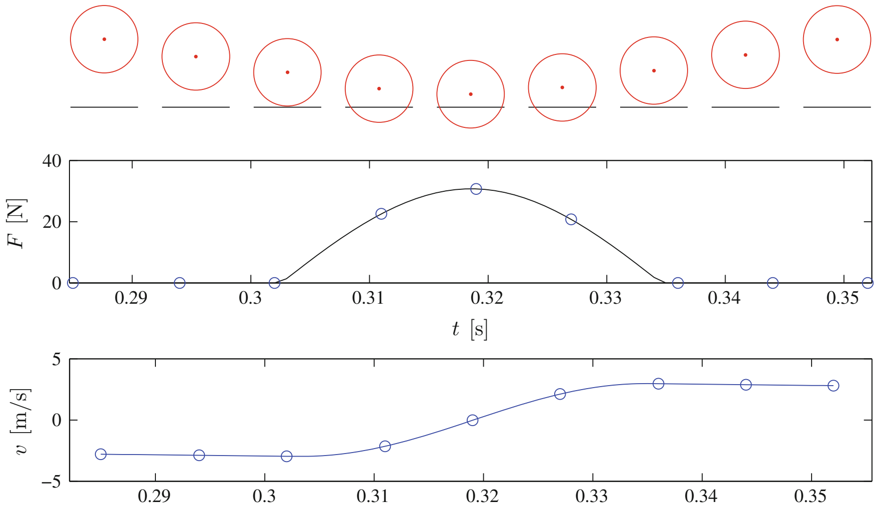


Fig. 5.15 Illustration of a ball bouncing on a floor, the normal force $F(t)$, and the velocity $v(t)$ before, during, and after the bounce

We can **model an attachment force** between two objects using a spring model:

$$F = k\Delta L \ , \tag{5.52}$$

where ΔL represents the elongation of the contact, which is typically equal to the change in distance between the centers of the two objects in contact.

General Position-Dependent Force

The spring model is not only a model for the deformation of springs. It is an extremely versatile model that provides a good description of the deformation of almost any object, spanning length scales from the deformation of the Earth's surface to the deformation of objects only a few atom diameters in size. It describes the deformation of rubber bands, strings, wires, wheels, bars, and the interactions between molecules and atoms. However, the spring model is typically only valid for small deformations: the deformation should be small compared to the size of the system.

The spring model is probably the most powerful and common model for interactions that you will encounter in physics. Why is that? Because many types of forces depend on the position of an object relative to another object: Contact forces, gravitational forces, electromagnetic forces, and inter-atomic forces all depend on the position, x, of the object they act on: $F = F(x)$. For motion near a point $x = b$, we can approximate the force on the object with Taylor's formula for $F(x)$:

$$F(x) = F(b) + F'(b)(x - b) + \mathscr{O}(x - b)^2 \; . \tag{5.53}$$

The force on the object at a position x near b, that is when $(x - b)$ is small, can be approximated by the first order term:

$$F(x) \simeq F(b) + F'(b)(x - b) \; . \tag{5.54}$$

This force model has the same form as the contact force model in (5.50) except for the constant $F(b)$. However, in most cases we measure the displacement from an equilibrium position where no force is acting and therefore $F(b) = 0$.

The spring model corresponds to a Taylor expansion of a position-dependent force $F(x)$ around an equilibrium point, b:

$$F(x) \simeq F'(b)\,(x - b) = k\,(x - b) \tag{5.55}$$

where $k = F'(b)$.

For example, for the ball in contact with the wall in Fig. 5.14, the deformation of the ball is small, and we can use the spring model as an approximation to the real deformation force.

5.7.1 Example: Motion of a Hanging Block

This example demonstrates how we can find the motion of an object affected by a spring force, using both numerical and analytical methods. The methods introduced can be applied to find the behavior of an object subject to any type of position-dependent force.

A block of mass $m = 1$ kg is hanging from a spring with spring constant $k = 100$ N/m. The other end of the spring is attached to the ceiling. We apply the structured problem-solving approach to find the motion of the block after it is released, and then make the model more realistic by adding air resistance.

Sketch and Identify: The *system* is the block, and the environment consists of the spring, the ceiling, the ground, and the surrounding air. The motion of the block is illustrated in Fig. 5.16, where $x(t)$ is used to describe the position of the block.

Model: We use the free-body diagram in Fig. 5.16 to identify the forces acting on the block: It is affected by a contact force, $\mathbf{F}$, from the spring, air resistance, $\mathbf{F}_D$, and gravity $\mathbf{G} = -mg\mathbf{j}$. We use a spring force model for the force from the spring:

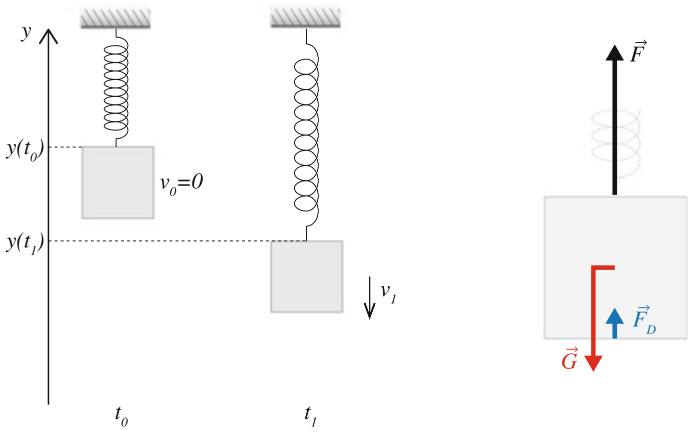


Fig. 5.16 Sketch of a block hanging in a spring (*left*), and the free-body diagram of the system (*right*)

$$F = \pm k \Delta L \ , \tag{5.56}$$

where the elongation ΔL depends on the position, y, of the block. For simplicity, we place the coordinate system so that the spring is in equilibrium when $y = 0\,\mathrm{m}$, so that $y = \Delta L$ as illustrated in Fig. 5.16. If the block is pulled down in the negative y-direction, the spring will act to pull the block up in the positive direction, therefore the correct sign of the spring force is:

$$\mathbf{F}(y) = -ky\,\mathbf{j} \ . \tag{5.57}$$

Newton's second law: Newton's second law along the y-axis gives

$$\mathbf{F}^{\mathrm{net}} = \mathbf{F} + \mathbf{G} + \mathbf{F}_D = F\,\mathbf{j} + G\,\mathbf{j} + F_D\,\mathbf{j} = ma\,\mathbf{j} \ , \tag{5.58}$$

where we remove the vectors to get:

$$F^{\mathrm{net}} = F + F_D + G = -ky + F_D - mg = ma \ . \tag{5.59}$$

Equilibrium model: First, let us consider the equilibrium situation—where the block does not move when released. Since the block does not move, air resistance is zero, $F_D = 0$, and the acceleration is also zero, which gives:

$$-ky + F_D - mg = -ky - mg = ma = 0 \ , \tag{5.60}$$

which gives

$$y_{eq} = -\frac{mg}{k} , \tag{5.61}$$

for the equilibrium position of the block.

What would you expect to happen if you instead released the block from a position above this equilibrium position? We would expect the block to oscillate, but the oscillations would grow smaller, and the block would eventually stop at the equilibrum position. Does our model system reproduce this behavior?

Simplified model—No air resistance: We start from a simplified model, where we assume that air resistance is negligible, so that $F_D = 0$. From (5.59) we see that the acceleration is

$$a = \frac{d^2x}{dt^2} = -g - \frac{k}{m}y , \tag{5.62}$$

and the block starts at rest, $v(t_0) = 0$ m/s, at $y(t_0) = 0$ m, where $t_0 = 0$ s.

Simplified model—Numerical solution: We can find the motion of the block using an Euler-Cromer scheme. Generally, we advice you to use a fourth-order Runge-Kutta method for oscillator problems, but we use Euler-Cromer here to make the programming transparent. (Notice that the direct Euler scheme is not stable for this equation). The Euler-Cromer scheme for (5.62) reads:

$$v(t_i + \Delta t) = v(t_i) - \frac{k}{m}x(t_i) - g\,\Delta t \tag{5.63}$$
$$x(t_i + \Delta t) = x(t_i) + v(t_i + \Delta t)\,\Delta t$$

which is implemented in the following program:

```
from pylab import *
# Initialize
m = 1.0     # kg
k = 100.0   # N/m
v0 = 1.0    # in m/s
time = 2.0 # s
g = 9.8     # m/s^2
# Numerical setup
dt = 0.0001 # s
n = int(round(time/dt))
t = zeros(n,float)
y = zeros(n,float)
v = zeros(n,float)
# Initial values
y[0] = 0.0
v[0] = v0
# Simulation loop for i in range(n-1):
    F = -k*y[i] - m*g
    a = F/m
    v[i+1] = v[i] + a*dt
    y[i+1] = y[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
```

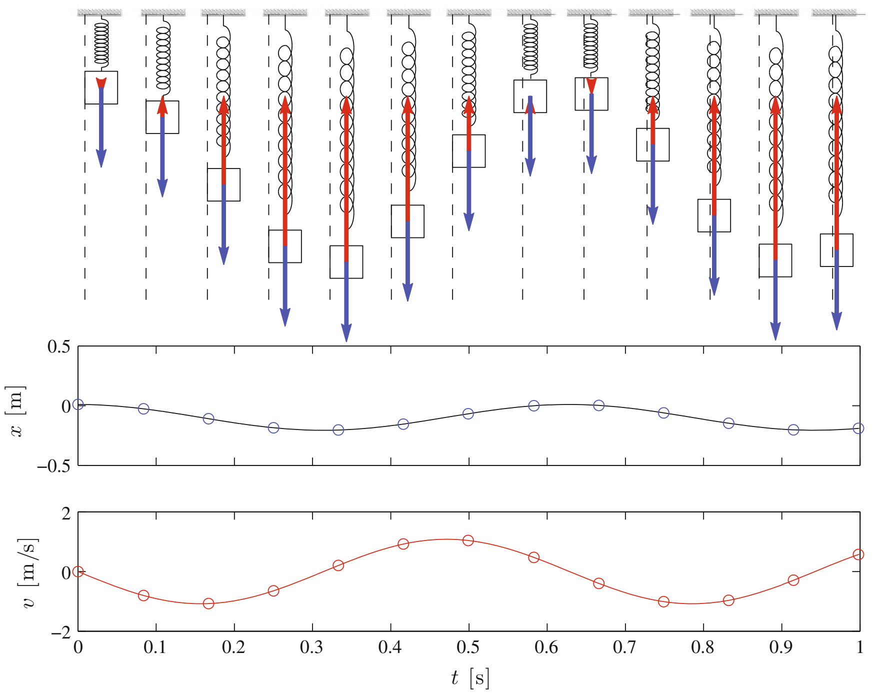


Fig. 5.17 Illustration of block position and plot of $y(t)$ and $v(t)$ from a numerical solution. The *top* figures illustrates the position of the block, and the forces acting on the block. *Red arrow* shows the spring force and *blue arrow* shows gravity

Here we chose a particular value for the time step, Δt, but how was this value chosen? Generally, we try to choose Δt as small as is practically possible: Small enough to ensure that the error is small, but not so small that the calculation takes too long time. In this case, the time step must be much smaller than time it takes for the block to swing back a forth one time, otherwise the results will not make any sense. However, it is a good rule to check your results by reducing your time-step by a factor 10 and observing if your solution is stable to such a change.

While the resulting motion is illustrated in the plots in Fig. 5.17, the motion becomes clearer by visualizing the dynamics using the simplest possible tool: the plotting function. We use a sequence of `plot` commands to give an impression of the dynamics by adding the following program lines

```
for ii in range(n-1):
    subplot(2,1,1)
    plot(t,y,'-b',t[ii],y[ii],'ob');
    xlabel('t [s]')
    ylabel('y [m]')
    subplot(2,1,2)
    plot(t,v,'-b',t[ii],v[ii],'ob');
    xlabel('t [s]')
    ylabel('v [m/s]')
```

Notice that we are not showing every time step, but jump in steps of 10 using `range(0,n−1,10)` in the `for-loop`. You should tune this to a number that gives a suitable dynamics on your computer. Using this tool, you can build your intuition of the impact of various variables. For example, you can check what happens if you change the initial velocity, v_0, or the spring constant k, for example, you could try changing the initial velocity to $v_0 = -0.1$ m/s, or you could set $y_0 = 0.1$ m and $v_0 = 0.0$ m/s.

If you want to make a more dynamic visualization of the motion of the block you can use the `VPython` tools, which allows you to make real-time graphical visualizations of the motion. We provide a simple program example to show how you can visualize and control the motion of an object using `VPython`:

```
from numpy import *
from visual import *
ipos = 0 dt = 0.001 # in s
m = 0.1    # in kg
k = 20.0   # in N/m
d = 0.1    # in m
x = 0.1
v = 0.1
# Initialize display
L = 0.02, H = 0.02, W = 0.02
scene = display(x=0,y=0,width=800,height=400,
                center=(0.1,H/2,0.0),range=(0.1,0.1,0.1));
ground = box(pos=(0.1,-0.001,0.0),size=(0.2,0.002,d));
block = box(pos=(x,H/2,W/2),size=(L,H,W),color=color.blue);
spring = helix(pos=(0,H/2,W/2),axis=(x-L/2,0,0),radius=H/4);
while 1:
    rate(100);
    a = -k/m*(x-d);
    v = v + a*dt;
    x = x + v*dt;
    block.x = x;
    spring.axis = (x - L/2,0.0,0.0);
    if scene.mouse.clicked:
        mc = scene.mouse.getclick();
        x = mc.pos.x;
        print 'x0 = ',x,', v0 = ',v
```

The main program structure is the same as before, but we are now not storing the values of the positions and velocities of the block—we are only visualizing their instantaneous values. The first part consists of the commands necessary to set up the window, the ground, the block and the spring. In the loop we use the function `rate(100)` to ensure that this loop can only run 100 times a second. We also read the mouse position to set new values of the position—which corresponds to setting new initial values. You should try the program—see if you can get the block to stand almost completely still. You can see how the program looks on screen in Fig. 5.18.

Simplified model—Analytical solution: *(You may skip this section without loss of continuity—solving differential equations often involves tricks that require experience, often are non-intuitive and not simple to follow.)*

For this particular problem, we can also find the exact solution, which means finding a function $y(t)$ that satisfies (5.62) and the initial conditions. The problem is simplified by rewriting the equation using the equilibrium position $y_{eq} = -mg/k$

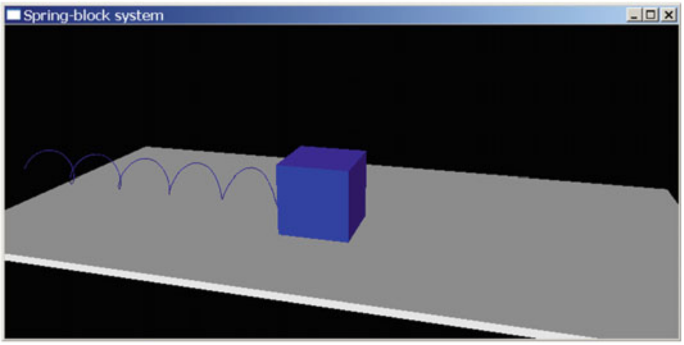


Fig. 5.18 Visualization of the system using `VPython`

$$a = -g - (k/m)\, y = -(k/m)\left(y - y_{eq}\right) , \tag{5.64}$$

where we introduce a new variable, $u = y - y_{eq}$. Since y_{eq} is a constant, this does not change the second derivative with respect to time:

$$\frac{d^2u}{dt^2} = \frac{d^2y}{dt^2} = -(k/m)\, u , \tag{5.65}$$

where the initial conditions now are $u(0) = y(0) - y_{eq} = -y_{eq}$ and $du/dt(0) = v(0)$.

Finding an analytical solution means finding a function $y(t)$ that satisfies (5.65) and the initial conditions. If we have found one such function, we can be sure this is *the* solution, because there is a uniqueness theorem in the mathematics of ordinary differential equations.

What functions become minus themselves after being derived twice? You may know (if you have already learned this trick), that this is true for the trigonometric functions sin and cos. The general solution to (5.65) is:

$$u(t) = A\cos(\omega t) + B\sin(\omega t) . \tag{5.66}$$

If we insert (5.66) in (5.65) we find that $\omega = \sqrt{k/m}$. The two prefactors A and B must be determined from the initial conditions:

$$u(0) = A\cos(0) = A = -y_{eq} = mg/k , \tag{5.67}$$

and

$$du/dt(0) = B\omega\cos(0) = 0 \;\Rightarrow\; B = 0 \tag{5.68}$$

So the complete solution is

$$y(t) = u(t) + y_{eq} = (mg/k)\cos\omega t - (mg/k) \ . \tag{5.69}$$

(Please do not be discouraged if you did not understand how we found the solution in this case. You will solve this equation many times in your career, and each time you will learn to know it better. Eventually, it will become a natural part of your knowledge base.)

Simplified model—Symbolic solution: You can solve (5.62) using the symbolic package in Python. First, you need to define all the relavant variables in the problem, where we introduce $q = k/m$ for simplicity:

```
>> from sympy import *
>> y = Function('y')
>> from sympy.abc import t
>> from sympy.abc import g
>> from sympy.abc import q
```

Then, we solve the differential equation, which includes the second derivative of $y(t)$, which is written as `Derivative(y(t),t,2)`. In addition, we need to provide the initial conditions at both $y(0) = y_0$ and $dy/dt(0) = 0$. To solve the equation, we need to rewrite it so that the right hand side consists of a zero:

$$\frac{d^2y}{dt^2} = -g - q\,y \ \Rightarrow\ \frac{d^2y}{dt^2} + g + q\,y = 0\ . \tag{5.70}$$

We can then use `dsolve` to find the analytical solution:

```
>> dsolve(Derivative(y(t),t,2)+g+q*y(t),y(t))
y(t) == -g/q + (C1*sin(t*Abs((re(q)**2 + im(q)**2)**(1/4)*....
```

Ooops! This is the correct result, but it is in a form which you may not recognize, unless you have some experience with complex numbers. The key lies in the sign of q. In our case, q is a positive number, but we have not told the symbolic solver this. We can redefine `q` and try again:

```
>> q = Symbol('q', real=True, positive=True)
>> dsolve(Derivative(y(t),t,2)+g+q*y(t),y(t))
y(t) == C1*sin(sqrt(q)*t) + C2*cos(sqrt(q)*t) - g/q
```

This result was much simpler to interpret! This is indeed the answer we found previously, we only need to substitute the correct initial conditions, just like we did above.

Simplified model—Analysis: The solution we have found so far show an everlasting, oscillating motion:

$$y(t) = (mg/k)\cos(\omega t) - (mg/k) \ . \tag{5.71}$$

The block oscillates up and down with a period $T = 2\pi/\omega$. However, this is not what we would expect in a realistic situation, where we expect the motion to be damped and that the block eventually comes to rest. We can study one way this may happen by introducing air resistance.

Full model—Numerical solution: Now, we assume that the block also is affected by a non-negligible air resistance force, F_D described by an advanced model for air resistance: At large velocities the air resistance force is described by a quadratic law with drag coefficient $D = 0.15\,\mathrm{m}^{-1}$, and at low velocities it is described by a viscous drag force with drag constant k_v. We assume that the transition occurs at $v_t = 0.01$ m/s:

$$F_D(v) = \begin{cases} -Dv(t)|v(t)| \ , & v > v_t \\ -k_v v(t) \ , & v < v_t \end{cases} \tag{5.72}$$

A continuous force requires the two behaviors to be the same at $v = v_t$, that is:

$$-Dv_t^2 = -k_v v_t \ \Rightarrow \ k_v = Dv_t \ . \tag{5.73}$$

(You are not yet expected to come up with a law like this, but you should be able to solve problems if you are given such a law).

This gives us a force model for F_D and we can find the motion of the block using a numerical scheme such as Euler-Cromer, as implemented in the following program. Notice the use of the `if`-statement in order to test if the ball is experiencing the high-velocity or the low-velocity air resistance force:

```
from pylab import *
# Initialize
m = 1.0       # kg
k = 100.0     # N/m
v0 = 0.0      # in m/s
time = 15.0 #
s g = 9.8     # m/s^2
D = 2.5       # m^-1
vt = 0.2     # m/s
kv = D*vt
# Numerical setup
dt = 0.0001 # s
n = int(round(time/dt))
t = zeros(n,float)
y = zeros(n,float)
v = zeros(n,float)
# Initial values
y[0] = 0.0
v[0] = v0
# Simulation loop
for i in range(n-1):
    if (v[i]<vt)
        FD = -kv*v[i]
    else
        FD = -D*v[i]*abs(v[i])
    F = -k*y[i] - m*g + FD
    a = F/m
    v[i+1] = v[i] + a*dt
    y[i+1] = y[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
```

The resulting behavior is shown in Fig. 5.19. The dashed line shows the equilibrium solution, $y = y_{eq}$, and the numerical solution does indeed converge towards this. The dotted line shows the solution without air resistance, which demonstrates that the air resistance does not affect the oscillation period significantly.

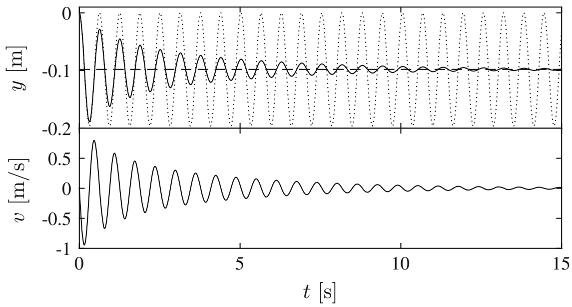


Fig. 5.19 Plot of the position $y(t)$ and velocity $v(t)$ of the block when air resistance forces are included

5.8 Newton's First Law

What happens if the net external force on a body is zero? Applying Newton's second law, we find:

$$\mathbf{F}_{\text{net}} = 0 = m\mathbf{a} \Rightarrow \mathbf{a} = \frac{d\mathbf{v}}{dt} = 0 \,. \tag{5.74}$$

The acceleration is zero, which means that the velocity of the object is constant. This is often referred to as Newton's first law:

Newton's first law: An object in a state of uniform motion tends to remain in that state unless an external force changes its state of motion.

Why do we need a separate law for this? Is it not simply a special case of Newton's second law? Yes, Newton's first law can be deduced from the second law as we have illustrated. However, the first law is often used for a different purpose: Newton's First Law tells us about the limit of applicability of Newton's Second law. Newton's Second law can only be used in reference systems where the First law is obeyed. But is not the First law always valid? No! The First law is only valid in reference systems that are not accelerated. If you observe the motion of a ball from an accelerating car, the ball will appear to accelerate even if there are no forces acting on it. We call systems that are not accelerating *inertial systems*, and Newton's first law is often called the law of inertia. Newton's first and second laws of motion are only valid in inertial systems. We will discuss references systems and inertial systems in detail when we discuss motion in two and three dimensions in Chap. 6.

A system is an inertial system if it is not accelerated—that is, the reference system must not be accelerating linearly or rotating. Unfortunately, this means that most systems we know are not really inertial systems. For example, the surface of the Earth is clearly not an inertial system, because the Earth is rotating. The Earth is also not an inertial system, because it is moving in a curved path around the Sun. However, even if the surface of the Earth is not strictly an inertial system, it may be considered to be approximately an inertial system for many laboratory-size experiments.

5.9 Newton's Third Law

So far, we have studied the motion of a single object by considering the interactions between the object (the system) and the environment (everything else). But most problems we deal with include an interplay between several objects. How can we address such systems?

We already started to address systems with several components when we addressed forces between a system and its environment. When you press your finger toward the table, you experience a force, and the table experiences a force. How are these forces related?

First, we realize that they act on different objects. There is a force from the finger on the table, and a force from the table on the finger. If the finger is the system, there is a contact force on the finger from the table, which is in the environment. On the other hand, if the table is the system, there is a contact force on the table from the finger, which is the environment in this case.

Could we get around this by introducing a more precise description of the interaction between the table and the finger? We argued that the top of the table is really not flat but rather rough on a microscopic scale—its surface consists of microscopic bumps. When I press my finger on the table, I press onto these small bumps, so that the bumps are deformed, and the bumps push down on the table. However, when my finger pushes toward the bumps, there is still a force from the finger on the bumps and a force from the bumps on my finger. Similarly, the bumps press down on the rest of the table, and the rest of the table pushes up on the bumps.

We realize that all forces, all interactions between objects, come in pairs. If there is a force from object A on object B, there is also a force from object B on object A. This fundamental principle of interactions is called Newton's third law. We do not know of any force that do not obey this law: All forces appear in pairs. Newton's third law is usually formulated as:

Newton's third law of motion: For every action there is an equal and opposite reaction.

This is a classical formulation of Newton's third law. The words "action" and "reaction" here means force and counter-force. If you push with your finger (F) on the table (T), there is a force, an action, $\mathbf{F}_{\text{from F on T}}$, from the finger on the table. Typically we write this by subindicies:

$$\mathbf{F}_{\text{from F on T}} \,, \tag{5.75}$$

for the force from F on T.

Newton's third law then states that there is an equal and oppositely directed force from the table on the finger. That is:

$$\mathbf{F}_{\text{from T on F}} = -\mathbf{F}_{\text{from F on T}} \,, \tag{5.76}$$

It is important to realize that the two forces in the force-pair *act on different objects*.

Application of Newton's Third Law

Let us introduce and apply Newton's third law through a simple example. We address a two crates lying on top of each other on the ground, as illustrated in Fig. 5.20a. The system consists of three bodies: the top crate (A), the bottom crate (B), and the ground (E). The ground is really the whole Earth, and we therefore use the symbol (E). We use the standard procedure to establish the free-body diagram for the compound system.

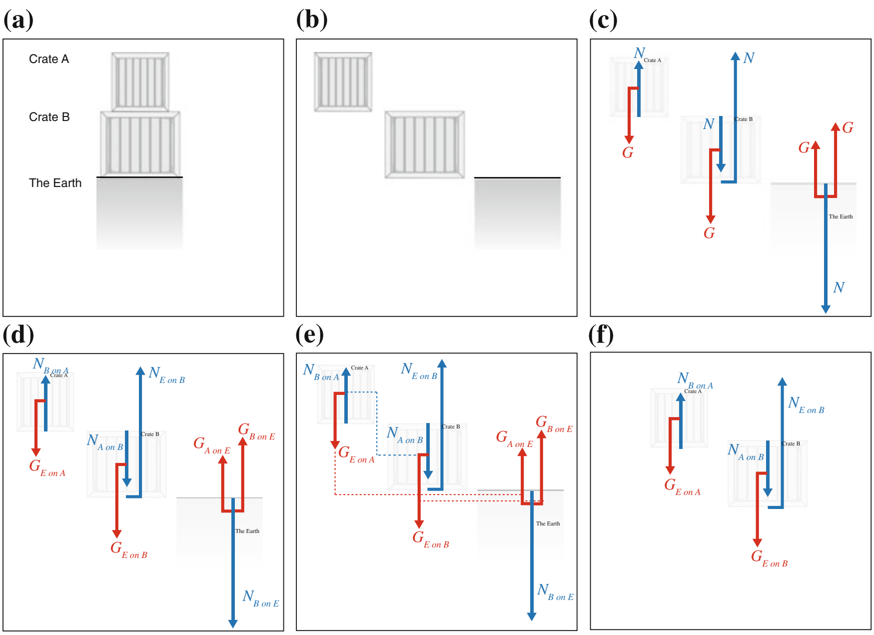


Fig. 5.20 Illustration of two crates lying on the ground. Various steps in the process of designing the free-body diagram are illustrated

Drawing a free-body diagram for a compound system:
Follow these steps to find and identify all the forces acting on an each object, and then to draw the free-body diagram for each of the objects the system.

- Draw each object as separate systems.
- Find all forces on all objects, and draw them as vectors.
- Express the forces as $F_{\text{A on B}}$.
- Identify action-reaction pairs.
- Check that every force has a unique reaction, and that they act on different objects. Draw in the axes of the coordinate system.

1. **Draw all objects as separate systems.** This is done in Fig. 5.20b. We draw the systems apart, so that we have room to fill in more details such as forces and coordinate systems for the various systems.
2. **Find all forces on all objects, and draw them as vectors**. This is done in Fig. 5.20c. For each object, we find where it is in contact with other objects, or with the environment, and draw in the contact forces. Finally, we add the long-range forces. Here, the only long-range force is gravity. Note that gravity acts between all the objects, but we have only included the gravitational forces between the Earth and each of the two crates, since the gravitational force between the two crates are negligible.
3. **Express the forces as $F_{\text{from A on B}}$**. For each of the forces we find what object it is acting on, and in what object the force has it origin. This is shown in Fig. 5.20d.
4. **Identify action-reaction pairs.** We draw a dotted line between each action-reaction pair. All force in our figure should have an origin either in one of the other objects, or in the environment. For clarity, we have placed the gravitational force from the crate on the Earth, in a point close to the surface of the Earth. In reality, these forces act in the center of the Earth. We have illustrated this process in Fig. 5.20e
5. Check that **every force has a unique reaction**, and that they act on different objects. Draw in the axes of the coordinate system.

The result of this analysis is a free-body diagram for each of the objects. The proposed method is elaborate. You may argue that it is too elaborate. As you become experienced, you no longer need to be this rigorous in your approach. An expert would draw the free-body diagram directly, but would still recognize action-reaction pairs in the drawing, even though he will probably not mark them.

When you become more experienced in recognizing action-reaction pairs, you may also exclude The Earth from the drawing in Fig. 5.20e. Instead, you will draw two objects, and consider everything outside the two objects to be the environment. You can do this, because you are typically not interested in the motion of the Earth, but only the motion and in the forces on the two crates. In this case, you will have two types of forces in your drawing of one of the objects: either a force has its origin in one of the other objects, in this case your free-body diagrams include the action-reaction pair, or a force has its origin in the environment. This is illustrated in Fig. 5.20f. Here, the two gravitational forces on the two crates, and the normal force on the lower crate, all have their origin in the environment, and their reactions are therefore not included in the figure.

One of the most common mistakes when applying Newton's third law is to assign an incorrect action-reaction pair. For example, it is common to assign the normal force $F_{\text{from B on A}}$ to be the reaction force to $G_{\text{from E on A}}$ in Fig. 5.20d. Such an error can be spotted in two ways: First, the action-reaction pair in this case acts on the same object. They must always act on different objects according to Newton's third law. Second, the two forces are due to quite different mechanisms and contacts. For contact forces, the action and reaction forces act in the same point. For long-range forces, the action-reaction pair is due to the same type of interaction between the two objects. In Fig. 5.20d the reaction force to the gravity from the Earth on crate A must therefore also be a gravitational force and also between the same two objects.

Structured Approach to Compound Problems

With the addition of Newton's third law, we now have sufficient tools to address the forces between objects in problems with several moving components. We call such systems *compound systems*. Solving problems with several components are not more complicated than solving problems with just a single object—we simply apply the structured problem-solving approach to each of the objects.

5.9.1 Example: Weight in an Elevator

In this example you learn to analyze problems with several bodies, using Newton's third law to relate force pairs, and apply Newton's second law to find the motion of each body in the system.

Problem: A person of mass m is standing in an elevator of mass M. The person is standing on spring-weight with negligible mass and spring constant k. Find the force shown by the weight and the force in the wire if both the elevator and the person is moving with a constant acceleration a. Discuss what really happens as the elevator starts moving from rest.

Our plan is to find the forces acting on each object and use Newton's second law to find the acceleration of each object. If all objects are moving with the same acceleration, we can use this to find the forces between the object. In the second part the objects do not necessarily have the same acceleration, and we need to introduce force models for the internal forces to relate the positions.

Sketch and identify: In this problem we are studying the motion of the elevator (E), the weight (W), and person (P) as illustrated in Fig. 5.21, described by the corresponding positions, $x_E(t)$, $x_W(t)$, and $x_P(t)$.

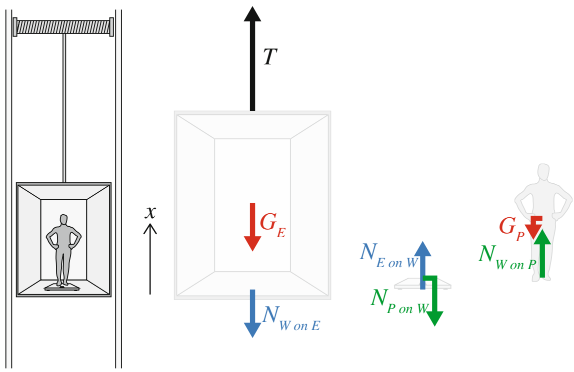


Fig. 5.21 Illustration of the elevator system and free-body diagrams for the elevator, the weight, and the person

Model: We find the forces acting on each object separately in three free-body diagrams (see Fig. 5.21). The *elevator* is affected by the external force T from the wire, the contact force $N_{\text{W on E}}$ from the weight, gravity, $G_E = m_E g$. The *weight* is affected by the contact force $N_{\text{E on W}}$ from the elevator and the contact force $N_{\text{P on W}}$ from the person. The gravitational force is negligible, $G_W = 0$. The *person* is affected by the contact force $N_{\text{W on P}}$ from the weight and by gravity, $G_P = m_P g$. We notice that the contact forces $N_{\text{E on W}}$ and $N_{\text{W on E}}$ are action-reaction pairs, and similarly for $N_{\text{P on W}}$ and $N_{\text{E on W}}$. We apply Newton's second law in the vertical direction for each object:

Elevator:

$$\sum_j F_{j,E} = T - N_{\text{W on E}} - G_E = m_E a_E \ . \tag{5.77}$$

Weight:

$$\sum_j F_{j,W} = N_{\text{E on W}} - N_{\text{P on W}} = \underbrace{m_W}_{=0} a_W = 0 \ . \tag{5.78}$$

Person:

$$\sum_j F_{j,P} = N_{\text{W on P}} - G_P = m_P a_P \ . \tag{5.79}$$

Notice that the directions of the forces are included in the choice of signs. This means that $N_{\text{W on P}}$ is the magnitude of this contact force. Newton's third law therefore gives that $N_{\text{W on P}} = N_{\text{P on W}} = N_P$, where we have written N_P for simplicity, and similarly $N_{\text{W on E}} = N_{\text{E on W}} = N_E$.

Simplified model—No motion: First, let us address the case where the elevator is not moving. In this case all the accelerations are zero, and we find from (5.79) that

$$N_{\text{W on P}} - G_P = m_P a_P = 0 \Rightarrow N_P = G_P \ . \tag{5.80}$$

Since what we read from the weight is the normal force applied to the weight, this means that we can read the gravitational force on the person from the weight, which is what we call the weight of the person!

The force, T, from the wire can be found from (5.77):

$$T - N_{\mathrm{W\ on\ E}} - G_E = m_E a_E = 0 \ , \tag{5.81}$$

which gives

$$T = N_E + G_E \ , \tag{5.82}$$

where (5.78) always gives $N_E = N_P$, which we combine with $N_P = G_P$, getting:

$$T = N_E + G_E = G_E + G_P = (m_E + m_P)\, g \ . \tag{5.83}$$

When the system is at rest, the force from the wire is equal to the sum of the gravitational forces.

Simplified model—No relative motion: Second, we assume that the elevator, the weight, and the person do not move relative to each other. They will therefore have the same acceleration, $a_E = a_W = a_P = a$. In this case, the weight still displays the contact force N_P, but this force is now given by (5.79), which gives:

$$N_P = G_P + m_P a_P = m_P g + m_P a = m_P\,(a + g) \ . \tag{5.84}$$

So the number you read on the weight is larger if the elevator is accelerating upward.

Full model—Force models: Finally, what happens if we open for the possibility that the objects move relative to each other? As the elevator starts to move, the weight starts to compress, increasing the force on the person, until the person starts moving, changing the compression in the spring in the weight, and so on. We must "Solve" to find the motion of each of the objects based on force models for the interactions. We can still use (5.77), since the mass of the weight is negligible, which means that $N_E = N_P = N$, and the forces acting on each end of the spring are the same!

How do we model the contact force N? It depends on the compression of the spring inside the weight. Let us describe the position of each side of the weight. If we describe the motion of the person by the position of her feet, they are in contact with the top of the weight, and the position of the top of the weight is therefore $x_P(t)$. Similarly, we can describe the position of the elevator by the position of the point where the weight is in contact with the elevator, so that the bottom of the weight is at $x_E(t)$. The change in the extension of the spring inside the weight is therefore

$$\Delta L = L - L_0 = x_P - x_E - L_0 \ , \tag{5.85}$$

where L_0 is the equilibrium distance between the top and bottom of the weight. The weight is compressed by the contact force, N, which is given as a spring force:

$$N = \pm k \Delta L \tag{5.86}$$

where we must choose the sign to ensure that a compressive force must be applied to compress the spring. The normal force must be positive if the spring is compressed. For compression $\Delta L < 0$. The sign must therefore be negative to ensure a positive contact force:

$$N = -k\Delta L = -k\left(x_P - x_E - L_0\right) \ , \tag{5.87}$$

Notice that the contact force cannot be negative, since the person is not glued to the weight. A weight can push, but not pull, on a person standing on it.

Full model—Newton's second law: Finally, we have a force model for the contact forces, and we can use Newton's second law to find equations for the positions of the person and the elevator. Newton's second law for the elevator from (5.77) gives:

$$T - N - G_E = m_E a_E \ , \tag{5.88}$$

and similarly Newton's second law for the person from (5.79) gives:

$$N - G_P = m_P a_P \ , \tag{5.89}$$

where N from (5.87) is a function of both x_E and x_P.

We can rewrite these equations as:

$$a_E = \frac{d^2 x_E}{dt^2} = \frac{1}{m_E}\left(T + k\left(x_P - x_E - L_0\right) - m_E g\right) \ , \tag{5.90}$$

$$a_P = \frac{d^2 x_P}{dt^2} = \frac{1}{m_P}\left(-k\left(x_P - x_E - L_0\right) - m_P g\right) \ . \tag{5.91}$$

While these differential equations may seem daunting, they are not that difficult to solve analytically (but we will not do that here), and they are simple to solve numerically (which we will do here).

Full model—Initial conditions: What are the initial conditions? We know that the person and elevator are starting from rest. Then an additional force F was added to the elevator. Since they are starting from rest, we know that the initial velocities are zero: $v_E(t_0) = 0\,\text{m/s}$ and $v_P(t_0) = 0\,\text{m/s}$. What about the initial positions? We know that initially the person is standing on the weight. The weight must therefore be compressed. We find the compression from Newton's second law for the person, in the case where the acceleration is zero:

$$a_P(t_0) = -k\left(x_P(t_0) - x_E(t_0) - L_0\right) - m_P g = 0 \ , \tag{5.92}$$

which gives

$$x_P(t_0) = x_E(t_0) + L_0 - \frac{m_P g}{k} \, , \tag{5.93}$$

where we are free to choose the coordinate system so that $x_E(t_0) = 0\,\mathrm{m}$.

Notice that the elevator was started by adding a force F to the tension T already in the wire when the system is at rest. We find the initial value for T from Newton's second law for the elevator in the case where the acceleration is zero:

$$a_E(t_0) = T + k\,(x_P - x_E - L_0) - m_E g = 0 \, , \tag{5.94}$$

where (5.92) gives $k\,(x_P - x_E - L_0) = m_P g$, which inserted in (5.94) gives:

$$T = m_E g + m_P g = (m_E + m_P) g \, . \tag{5.95}$$

This is not surprising: At rest, the force T from the wire must balance the total gravitational force on the elevator with the person, $T = G_P + G_E$, just as we found above. And the result would be the same if we assumed the elevator, weight, and person to be a single object with mass $M = m_E + m_P$.

Full model—Numerical solution: While this problem can be solved analytically, the numerical solution and discussion is simpler, and provides the most essential insights into the process. We therefore apply Euler-Cromer's method to find the positions and velocities of the person and the elevator starting from $t = t_0 = 0.0\,\mathrm{s}$. The method is implemented in the following program:

```
from numpy import *
```

The positions x_E and x_P are compared with the positions found using the acceleration of the whole system found previously:

$$a = \frac{T - (m_E + m_P) g}{m_E + m_P} \, . \tag{5.96}$$

The resulting behavior is shown in Fig. 5.22. We see that the position of the person on the weight oscillates around the stationary solution. The position of the elevator also oscillates, but the oscillations are much smaller and are therefore not visible on this plot.

Analysis: What we read off the weight is the spring force, which is:

$$N = -k\,(x_P - x_E - L_0) \tag{5.97}$$

The spring force is plotted as a function of time in Fig. 5.23. The spring force, which is the force read off the weight, oscillates around the expected value, $N = m_P(g+a) = 823.2\,\mathrm{N}$, indicated by the straight line in the plot.

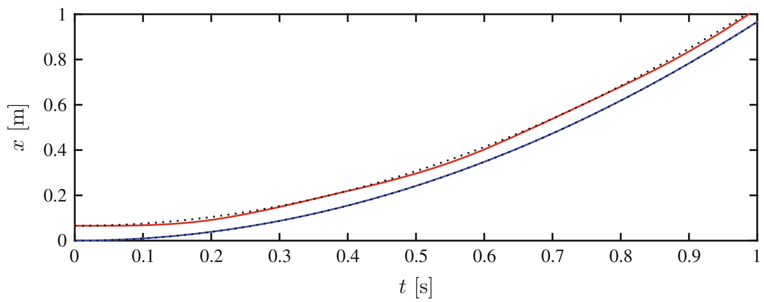


Fig. 5.22 Plot of the positions x_E (*blue*) and x_P (*red*) as a function of time

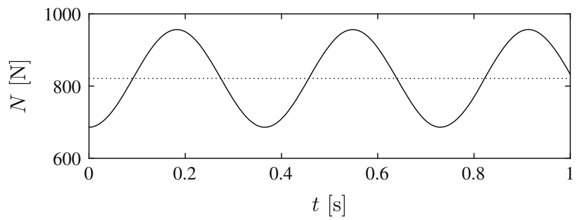


Fig. 5.23 Plot of the force N recorded by the weight as a function of time

Final comment: Here, we have found an alternative behavior of the weight-spring system: The weight is oscillating! Such a behavior is to be expected in a real system, but there will also be additional forces that will tend to reduce the oscillations, so that after a few oscillations, the weight will stop at the expected value, $F = 823.2\,\mathrm{N}$

Summary

Newton's second law:

- Newton's second law relates the acceleration of an object to the net forces acting on it: $\sum_j \mathbf{F}_j = m\mathbf{a}$, where the sum is over all forces acting on the object, and m is the inertial mass.
- All forces acting on *a system* has a source in *the environment*.
- Forces can be *contact forces* acting on the boundary between the system and the environment.
- Forces can be *long range forces* from an object in the environment.
- Forces are drawn as vectors starting at the point where the force is acting, pointing in the direction of the force, and with a length indicating the length of the force.
- The force may be a given quantity, $\mathbf{F}$.

- The gravitational force acts between all objects. On the surface of the Earth the gravitational force on an object is $\mathbf{W} = -mg\,\mathbf{j}$, where $\mathbf{j}$ is a unit vector pointing upward, g is the acceleration of gravity, and m is the gravitational mass. The gravitational mass is equal to the inertial mass.
- The contact force from a fluid on a moving object depends on the velocity of the object relative to the fluid. The simplest force model is the viscous force, $\mathbf{D} = -k_v\mathbf{v}$, where the constant k_v depends on the viscosity of the fluid and the size of the object.
- The contact force from a solid depends on the distance between the object and the solid. The simplest force model is the spring force, $\mathbf{F} = -k\Delta L\,\mathbf{i}$. Where ΔL is the elongation, and k is the spring constant which gives the stiffness of the contact. The spring model is one of the most fundamental force models because it is the first order Taylor expansion of any position-dependent force.

Problem-solving approach:

- We **identify** the object and its initial conditions.
- We **model** the behavior by find the forces acting on the object, introducing force models for all the forces, and applying Newton's second law to find an equation for the acceleration of the object.
- We **solve** the problem by finding the position and velocity from the acceleration and the initial conditions using numerical or analytical techniques.
- We **analyze** the solution to validate it, and use the solution to answer the original question posed.

Exercises

Discussion Questions

5.1 Single force. Can an object affected only by a single force have zero acceleration?

5.2 Zero velocity. If you throw a ball vertically it has zero velocity at its maximum point. Does it also have zero acceleration at this point?

5.3 Acceleration of gravity. You measure the acceleration of gravity in an elevator moving at a velocity of 9.8 m/s downwards. What will you measure?

5.4 Hammerhead. The head of your hammer is loose. How would you hit the shaft in order to fasten the hammerhead? Does this work if you are an astronaut working in space?

5.5 In the army. You are told by a friend in the army that the force you feel when you fire a gun is the same as the force felt by a sandbag hit by the bullet because the two forces are action-reactions pairs. Is this true?

5.6 Car with trailer. A car pulls on a trailer with a given force, but the trailed pulls back at the car with the same force. So why does not the trailer remain at rest, independently of how hard the car pulls?

5.7 Wet dog. When a dog is wet it shakes its body violently to get dry. Explain how this works.

5.8 Whiplash. Explain why a properly adjusted headrest will reduce the chance of whiplash injury if your car is hit from behind.

5.9 Seat belts. Explain why seat belts reduce the risk of injury if you are involved in a car accident. How would you improve seatbelt design?

5.10 Parachute. If you jump from a plane you quickly reach the terminal velocity. Why do you die if you hit the ground at terminal velocity, but not if you open your parachute at the same velocity?

5.11 Tug-of-war. Two persons are pulling at each end of a long rope. If the rope is effectively massless, the tension in the rope is the same all along the rope, and the force on each end of the rope must therefore be of equal magnitude. How can then one of them win in a tug-of-war?

5.12 Bouncing ball. You drop a ball onto a weight, and it bounces back up. Does the value displayed by the weight change during the bounce, and if it does, how does it change? Explain.

5.13 Air resistance. You throw a ball straight up and measure the velocity as it passes you on its way down. Will the velocity be larger, the same, or smaller if you did the same experiment in vacuum?

Problems

5.14 Parachuter. A person jumps from an airplane, falling freely for several seconds before he pulls the cord of his parachute and the parachute unfolds.
(**a**) Identify the forces acting on the parachuter and draw a free-body diagram of the parachuter before he has pulled the chord.
(**b**) Identify the forces acting on the parachuter and draw a free-body diagram of the parachuter after he has pulled the chord.
(**c**) Sketch the net force acting on the parachuter as a function of time, $F(t)$.

5.15 Forces on a drop of water. A drop of water is hanging from a faucet.
(**a**) Identify the forces acting on the drop and draw a free-body diagram of the drop.
The drop finally falls down towards the sink.
(**b**) Identify the forces acting on the drop and draw a free-body diagram of the drop.

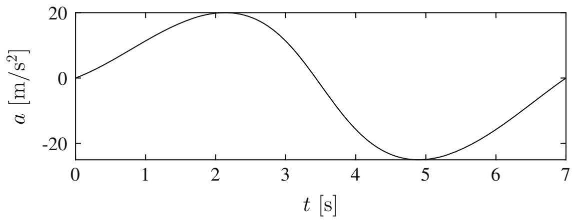


Fig. 5.24 Acceleration of a car

5.16 Forces on an anchor. Susan is standing on the floor in a boat, pulling a rope attached to an achor.
(**a**) Identify the forces acting on the anchor and draw a free-body diagram of the anchor.
(**b**) Identify the forces acting on Susan and draw a free-body diagram of Susan.
(**c**) Identify the forces acting of the boat and draw a free-body diagram of the boat.

5.17 Force on a car. A car driver needs to make a rapid maneuver. You have access the an accelerometer fitted in the car. The acceleration is shown in Fig. 5.24. You can assume that the only horizontal force on the car is from the ground on the wheels of the car. The mass of the car is $m = 1000$ kg.
(**a**) Identify in what time interval the car is speeding up and when it is slowing down.
(**b**) Draw the force on the car from the ground as a function of time.
(**c**) What is the maximum and minimum force on the car?
(**d**) Make of drawing of the car, and draw the force on the car from the ground when the car is speeding up and when the car is slowing down.

Alan is sitting on his seat in the car. He does not move relative to the seat throughout the maneuver. His mass is $m = 70$ kg.
(**e**) Draw a free-body diagram for Alan during the maneuver, including only horizontal forces. Can you use the same free-body diagram throughout the whole maneuver, also when the force on the car is negative?
(**f**) What is the maximum and minimum force acting on Alan during the maneuver?

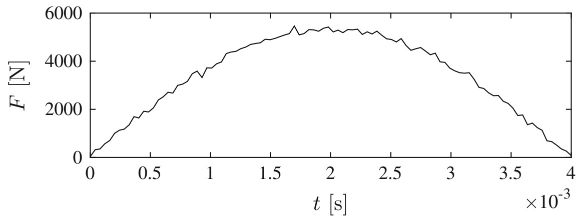
5.18 Pulling a train. A locomotive exerts a force $F = 20{,}000$ N on a train cart loaded with automobiles. The mass of the cart, including its load, is 10,000 kg.
(**a**) What is the acceleration of the train cart?

Another car with mass 2000 kg is added to the load on the train cart.
(**b**) What is the acceleration of the train cart now?

5.19 Firing a bullet. A bullet of mass 0.1 kg is fired through a 1 m rifle barrel. Assume a constant force $F = 1000$ N is applied on the bullet while in the barrel. What is the velocity of the bullet as it leaves the barrel?

5.20 Jumping into snow. Albert jumps from the roof of his house into a deep pile of snow. He starts 5 m above the snow, and stops 1 m into the snow. What force was exerted on Albert? Albert's mass is $m = 70$ kg. You can ignore air resistance, and you can assume that the force on Albert from the snow is constant.

Fig. 5.25 Force on baseball



5.21 Force sensor reading. A small force sensor has been fitted on the surface of a baseball, measuring the force from the bat on the ball. The recorded force is shown in Fig. 5.25, and you can find the data in the file baseballforce.d.[6] The mass of the baseball is $m = 0.145$ kg.
(**a**) Draw a free-body diagram of the baseball.
(**b**) Find the acceleration of the baseball as a function of time.
(**c**) Find the velocity of the baseball as a function of time.
(**d**) What is the velocity of the baseball as it leaves the bat?

5.22 Vertical throw. You stand at the top of a bridge, throwing a rock directly downwards towards the water underneath. You give the rock an initial velocity, v_0, and the rock starts at a height h_0 above the surface. Air resistance is negligible.
(**a**) Draw a free-body diagram of the baseball.
(**b**) Find the acceleration of the baseball as a function of time.
(**c**) Find the velocity of the baseball as a function of time.
(**d**) What is the velocity of the baseball as it leaves the bat?

5.23 Reaction time. Your reaction time can be measured with the help of a fried using a ruler. Your friend holds the ruler vertically between your thumb and index finger. When he releases the ruler, you grab it as soon as you can. If the ruler is placed with the 0 cm mark initially between your fingers, how can you use how far the ruler has fallen to find your reaction time? You can assume that you use negligible time to actually grab the ruler as soon as you start moving your finger.
(**a**) Draw a free-body diagram for the ruler when it is falling.
(**b**) Find the position of the ruler as a function of time.
(**c**) Find your reaction time, if the ruler fell a vertical distance h before you grabbed it.
(**d**) If you are driving in your car at 80 km/h, how far do you travel during your reaction time?

5.24 Terminal velocity of heavy and large objects. You drop two spheres from a high tower. First, assume that the spheres have the same diameter, d, and surface properties, so that they have the same air resistance, but they have different masses, m_A and m_B. The air resistance is described using a quadratic law with the coefficient D for both spheres.

---

[6] http://folk.uio.no/malthe/mechbook/baseballforce.d.

(**a**) Draw a free-body diagram for a sphere as it is falling.
(**b**) Find an expression for the acceleration of either sphere.
(**c**) Which object has the largest acceleration—the object with the largest or with the smallest mass?

Now, let us modify the experiment. We now drop two spheres of different diameter, d_1 and d_2, but the spheres are solid and made of the same materials, for example steel. They will therefore have different masses, m_1 and m_2. Still, air resistance for both spheres are described using a quadratic law, but the coefficient D depends on cross-sectional area of the sphere, and therefore on the diameter: $D = C_0 d^2$, where C_0 is a constant.
(**d**) Find an expression for the acceleration of such a sphere as a function of the diameter of the sphere.
(**e**) Which object has the largest acceleration—the object with the largest or with the smallest diameter?

5.25 Space shuttle with air resistance. During lift-off of the space shuttle the engines provide a force of 35 million Newtons. The mass of the shuttle is approximately 2 million kg.
(**a**) Draw a free-body diagram of the space shuttle immediately after lift-off.
(**b**) Find an expression for the acceleration of the space shuttle immediately after lift-off.

Let us assume that the force from the engines is constant, and that the mass of the space shuttle does not change significantly over the first 20 s.
(**c**) Find the velocity and position of the space shuttle after 20 s if you ignore air resistance.

Let us assume that we can describe the air resistance force on the space shuttle with a square law, $F = -Dv|v|$, where $D \simeq 388$ kg/m.
(**d**) Develop a program to find the velocity and position of the space shuttle using numerical methods.
(**e**) Find the velocity and position of the space shuttle after 20 s if you include air resistance.
(**f**) Plot the velocity and position and compare with the results without air resistance. Comment on the results.

Notice that D depends on the density of the surrounding air, and the density falls when as the shuttle ascends, hence D actually depends on the height of the shuttle.

5.26 Experiments in Pisa. On a visit to Pisa, you decide to redo Galileo's original experiment based on your knowledge of physics. You bring to steel spheres of the same size to the top of the tower. One sphere is hollow and the other is solid.
(**a**) Draw a free-body diagram for one of the spheres.
(**b**) How would you describe air resistance for each of the spheres?
(**c**) Find an expression for the acceleration of the sphere as a function of its mass.
(**d**) Which of the two spheres have the largest acceleration?

5.27 Stretching an aluminum wire. A thin aluminum wire is stretched 1 mm when a 10 kg weight is suspended from it. Assume the wire can be modelled as a spring, what is the spring constant for the wire?

5.28 Two masses and a spring. Two particles of $m = 0.1$ kg are attached with a spring with spring constant $k = 100$ N/m and equilibrium length $b = 0.01$ m. Both particles start at rest and the spring is at equilibrium. An external force $F = 1000$ N acts during 1 s on one of the particles in the direction of the other particle. Find the position of both particles as a function of time from the time $t = 0$ s when the external force starts acting. (You may solve this problem analytically or numerically).

Projects

5.29 Modeling a 100 m race. In this project we will develop an advanced model for the motion of a sprinter during a 100 m race. We will build the model gradually, adding complications one at a time to develop a realistic model for the race.
(**a**) A sprinter is accelerating along the track. Draw a free-body diagram of the sprinter, including only horizontal forces. Try to make the length of the vectors correspond to the relative magnitudes of the forces.

Let us assume that the sprinter is accelerated by a constant horizontal driving force, $F = 400$ N, from the ground all the way from the start to the 100 m line (averaged over a few steps). The mass of the sprinter is $m = 80$ kg.
(**b**) Find the position, $x(t)$, of the sprinter as a function of time.
(**c**) Show that the sprinter uses $t = 6.3$ s to reach the 100 m line.

This is a bit fast compared with real races. However, real sprinters are limited by air resistance. Let us introduce a model for air resistance by assuming that the air resistance force is described by a square law:

$$D = (1/2)\rho C_D A(v - w)^2 \tag{5.98}$$

where ρ is the density of air, A is the cross-sectional area of the runner, C_D is the drag coefficient, v is the velocity of the runner, and w is the velocity of the air. At sea level $\rho = 1.293$ kg/m^3, and for the runner we can assume $A = 0.45$ m^2, and $C_D = 1.2$. You can initially assume that there is no wind: $w = 0$ m/s.

Assume that the runner is only affected by the constant driving force, F, and the air resistance force, D.
(**d**) Find an expression for the acceleration of the runner.
(**e**) Use Euler's method to find the velocity, $v(t)$, and position, $x(t)$ as a function of time for the runner. The runner starts from rest at the time $t = 0$s. Plot the position, velocity and acceleration of the runner as a function of time. How did you decide on the time-step Δt? (Your answer should include the program used to solve the problem and the resulting plots).
(**f**) Use the results to find the race time for the 100 m race.

(**g**) Show that the (theoretical) maximum velocity of a runner driven by these forces is:

$$v_T = \sqrt{2F/(\rho\, C_D\, A)}\ . \tag{5.99}$$

The runner may have to run more than 100 m to reach this velocity. (We often call this maximum velocity the terminal velocity—"terminal" because the velocity increases until it reaches the terminal velocity, where the acceleration becomes zero). Find the numerical value of the terminal velocity for the runner. Do you think this is realistic?

So far the model only includes a constant driving force and air resistance. This is clearly a too simplified model to be realistic. Let us make the model more realistic by adding a few features.

First, there is a physiological limit to how fast you can run. The driving force from the runner should therefore decrease with velocity, so that there is a maximum velocity at which the acceleration is zero even without air resistance. While we do not know the detailed physiological mechanisms for this effect, we can make a simplified force model to implement the effect by introducing a driving force, F_D, with two terms: a constant term, F, and a term that decreases with increasing velocity, F_V: $F_V = -f_V\, v$, so that the driving force is:

$$F_D = F + F_V = F - f_V v\ . \tag{5.100}$$

Reasonable values for the parameters are $F = 400$ N, and $f_v = 25.8$ sN/m. (These values are chosen to make the maximum velocity reasonable—they are not based on a physiological consideration).

(**h**) If you assume that the runner is subject only to these two driving forces, what is his maximum velocity? (You can ignore the drag term, D, in this calculation).

In addition, during the first few seconds the runner is crouched and accelerating rapidly. In this phase, his cross-sectional area is smaller because he is crouched, and the driving force exerted by the runner is larger than later. Let us also introduce these aspects into our model.

First, let us assume that the crouched phase lasts approximately for a time, t_c. We do not expect this phase to end abruptly at a specific time. Instead, we expect the driving force to decrease gradually (and the cross-sectional area to increase gradually) as the runner is going from a crouched to an upright running position. A common way to approximate such a change is through an exponential function that depends on the time and the characteristic time, t_c. For example, by introducing an initial driving force, F_C:

$$F_C = f_c \exp(-(t/t_c)^2)\ . \tag{5.101}$$

When $t = 0$, the force is equal to f_c, but as time increases, the force decreases rapidly. When the time has reached t_c, the force has dropped to $1/e \simeq 0.37$ of the value at $t = 0$, and after a time $4t_c$ this contribution to the driving force has dropped to less than 2 % of its initial value.

Notice that we do not have any experimental or theoretical reason to use this particular form for the time dependence. We have simply chosen a convenient form as a first approximation, and then we use this form and try to get reasonable results with it. A better approach would be to have experimental data on how the force varied during the first few seconds, but unfortunately we do not know this. Making rough estimates that you can subsequently improve by better measurements, calculations, or theory will be an important part of how you apply physics in practice.

The total *driving* force is then:

$$F_D = F + f_c \exp(-(t/t_c)^2) - f_v v \; . \tag{5.102}$$

where reasonable values for the parameters are $f_c = 488$ N and $t_c = 0.67$ s. (These values are chosen so that the total race-time becomes reasonable).

In addition, we need to modify the air resistance force because the runner is crouched in the initial phase, so that the cross-sectional area is reduced. We therefore need to replace the cross-sectional area A in the expression for D with a time-dependent expression, $A(t)$, with the properties that: (1) when time is zero, the area should be reduced to 75 % of the area during upright running (again, we guess reasonable values); and (2) after a time much larger than t_c, the runner is upright, and the cross-sectional area should be A. Again, we introduce a modification to the area that depends on the exponential factor used above:

$$A(t) = A - 0.25A \exp(-(t/t_c)^2) = A\left(1 - 0.25 \exp(-(t/t_c)^2)\right) \; . \tag{5.103}$$

The air resistance force therefore becomes:

$$D = \frac{1}{2} A(t) \rho C_D (v - w)^2 = \frac{1}{2} A \left(1 - 0.25 e^{-(t/t_c)^2}\right) \rho C_D (v - w)^2 \tag{5.104}$$

The total force on the runner is:

$$F_{\text{net}} = F + F_C - F_V - D = F + f_c e^{-(t/t_c)^2} - f_v v - D \; , \tag{5.105}$$

where $F = 400$ N is a constant driving force, and the other terms have been addressed above.

(**i**) Modify your numerical method to include these new forces. Find and plot $x(t)$, $v(t)$, and $a(t)$ for the motion.

(**j**) How fast does he run 100 m?

(**k**) Compare the magnitudes of the various forces acting on the runner by plotting F (which is constant), F_C, F_V and D as a function of time for a 100 m race. Discuss how important the various effects are.

(**l**) Use the model to test how the resulting time on 100 m would change if the runner had a hind wind with a wind speed of $w = 1$ m/s. What if he was running into a wind with a wind speed of $w = 1$ m/s?

5.30 Modelling Bungee Jumping Numerically. In this exercise we will study a person bungee jumping. The bungee cord acts as an ideal spring with a spring constant k when it is stretched, but it has no strength when pushed together. The cord's equilibrium length is d. There is also a form of dampening in the cord, which we will model as a force which is dependent on the speed of the cord's deformation. When the cord is stretched a length x, and is being stretched with the instantaneous speed v, the force from the spring is given as

$$F(x, v) = \begin{cases} -k(x-d) - c_v v & \text{when} \quad x > d \\ 0 & \text{when} \quad x \leq d \end{cases}$$

where c_v is a constant that describes the dampening in the cord, and k is the spring constant.

We set $x = 0$ to be where the bungee cord is attached and let the positive direction of the x-axis point downwards. A person with a mass m places the cord around the waist and jumps from the point where it is attached. The initial velocity is $v_0 = 0$. You can neglect air resistance and assume that the bungee cord is massless. The motion is solely vertical. The acceleration of gravity is g.

(**a**) Draw a free-body diagram of the person when the bungee cord is taut. Name all the forces.

(**b**) At what height is the person hanging when the motion has stopped?

(**c**) Write a numerical algorithm that finds the persons position and velocity at the time $t + \Delta t$ given the persons position and velocity at a time t. And implement this algorithm in a program that finds the motion of a person bungee jumping.

(**d**) Use your program to plot the height as a function of time, $x(t)$, for a person of mass $m = 70$ kg jumping with a bungee cord of equilibrium length $d = 20$ m and spring constant $k = 150$ N/m, for $T = 60$ s with a timestep of $dt = 0.01$ s. The acceleration of gravity is $g = 9.8\,\mathrm{m/s^2}$. What is a reasonable choice for c_v? Explain your choice.

(**e**) Is the system conservative during the whole motion, parts of the motion, or not at all? Explain.

(**f**) How would our model be different if we included air resistance?

Chapter 6
Motion in Two and Three Dimensions

You have now learned to use Newton's second law. This is the main tool you need to solve problems in mechanics. However, so far, we have only addressed motion and forces in one dimension. Fortunately, the description of motion and Newton's laws do not change when we go to higher dimensions. We can continue to apply the structured problem-solving approach in exactly the same way as we did before. But in order to address motion in two- and three dimensions, we need to extend our mathematical and numerical methods to address two- and three-dimensional motion. This will be done in two parts: In this chapter we introduce general two- and three-dimensional motion. Later we will introduce constrained motion—motion constrained to occur along a specific path in the same way a rollercoaster cart is constrained to follow the rollercoaster track, or in the way a part of a rotating body is following the rotation of the body.

6.1 Vectors

The use of vectors to describe positions in two- and three-dimensional systems is essential in order to describe general motion. If you are familiar with the use of vectors you can safely jump to the next section.

Scalars and Vectors

A scalar is a single number (with notation), such as a length, a mass, or a temperature. In order to describe physical quantities such as a displacement or a force, we need to describe both a magnitude and a direction: A displacement is described by a distance and a direction; A force is described by the magnitude of the force and the direction it acts in.

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_6

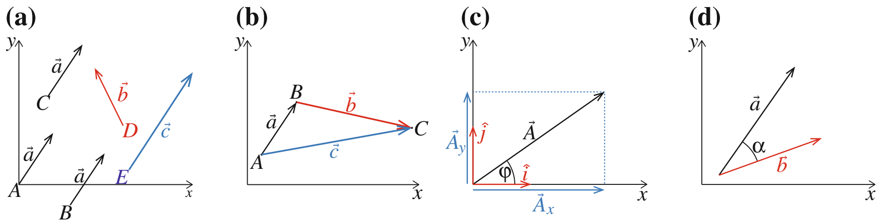


Fig. 6.1 **a** Illustration of vectors, **b** vector addition, **c** units vectors, decomposition and angle, **d** dot product

> A **vector** is a quantity with both direction and length. A vector can have physical units.

We indicate that a quantity is a vector by drawing a small arrow above it: **a**.[1] It will be important for you to use a notation that clearly separates vector and scalar quantities. Because one of the most common mistakes made by students is to mix up vectors and scalars in their calculations, with dire results, we strongly urge you to use the arrow notation for vectors and to stick with it.

Vector Addition

Vector addition is intuitive for the addition of displacements (see Fig. 6.1b): If you first move along the vector **a** from A to B, and then along the vector **b** from B to C, the net displacement is the vector:

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \ , \tag{6.1}$$

from point A to C.

This geometric definition of vector addition is general, and we use it also for vectors that are not displacements: We find the sum of two vectors **a** and **b** geometrically by placing the tail of vector **b** at the tip of vector **a**. The sum is called the **resultant** vector.

---

[1] Our definition of a vector is rather limited compared to the more general definition of vector spaces you may be used to in mathematics. It means we usually limit ourselves to vectors in one, two, and three Cartesian dimensions. Usually, we illustrate a vector by an arrow, as shown in Fig. 6.1a. The length of the arrow indicates the magnitude, and the direction shows the direction of the vector. Notice that it does not matter where we start drawing a vector. The vector **a** in Fig. 6.1 is the same even if it is drawn in position A, B, or C, but the vector **b** in position D is different, because it has a different direction than **a**, and the vector **c** in position E is different from both **a** and **b** since it has a different magnitude. Remember: The only thing that matters for a vector is its magnitude and direction—not where it starts.

From this definition, we realize that vector addition is **commutative**, the order of addition is arbitrary, and associative:

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}\ , \ \text{(commutative law)} \tag{6.2}$$

$$\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}\ , \ \text{(associative law)} \tag{6.3}$$

Scalar Multiplication

We can rescale the length of a vector by multiplying it with a scalar:

$$\mathbf{b}=2\mathbf{a}\ . \tag{6.4}$$

Vector **b** is twice as long as vector **a**, but still pointing in the same direction. By multiplying a vector with a scalar we change the magnitude, but not the direction of the vector.

If we multiply a vector by 0, we get a vector of zero length, called the **zero vector**:

$$0\mathbf{a}=\mathbf{0}\ . \tag{6.5}$$

If we multiply a vector by a negative number, we change the direction of the vector to point in the opposite direction. For example, if we multiply with a vector with -1, we get the negative vector—a vector of the same length, but which points in the opposite direction.

Vector Components

A coordinate system is a grid you choose to describe the world in numbers. You are free to choose any coordinate system you like: You may choose where to place the origin and how to orient the axis. When you have decided on this, we use the coordinate system to describe a vector by *decomposing* the vector in the given coordinate system.

Here, we use *Cartesian* coordinate systems, where the axes are orthogonal to each other. We describe the coordinate system by the position of the origin, O, and by unit vectors pointing along each axis: the x-, y-, and z-axis. The unit vectors are of unit length, of length 1, and do not have any unit. The unit vectors are orthogonal, they form 90° angles with each other, as illustrated in Fig. 6.1c. It is common to use the symbol **i**, **j**, and **k** for the unit vectors along the x, y, and z-axis respectively.

Any vector can be uniquely decomposed into a set of component vectors along each of the axes:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y \ , \tag{6.6}$$

as illustrated in Fig. 6.1c, where each of the component vectors can be written in terms of the unit vector along the axis:

$$\mathbf{A}_x = A_x \, \mathbf{i} \ , \ \mathbf{A}_y = A_y \, \mathbf{j} \ . \tag{6.7}$$

Here, the units of the vectors are in the scalar numbers A_x and A_y.

If you know the magnitude and direction of a vector, you can find the component of the vector from trigonometrical considerations. For example, the vector **A** shown in Fig. 6.1c, may be decomposed into its x- and y-components by:

$$\mathbf{A} = A_x \, \mathbf{i} + A_y \, \mathbf{j} = |\mathbf{A}| \cos\phi \, \mathbf{i} + |\mathbf{A}| \sin\phi \, \mathbf{j} \ . \tag{6.8}$$

We may write a vector by using the unit vectors or by writing the vector directly in coordinate form:

$$\mathbf{A} = A_x \, \mathbf{i} + A_y \, \mathbf{j} + A_z \, \mathbf{k} = \left(A_x, A_y, A_z\right) \ . \tag{6.9}$$

The Magnitude of a Vector Using Components

If the coordinate system is orthogonal, then all the axis are orthogonal to each other, and we can use Pythagoras' theorem to relate the magnitude to the vector to the components:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \ . \tag{6.10}$$

Addition, Subtraction and Scalar Multiplication Using Components

A particular advantage of the component form is that addition, subtraction, and scalar multiplications can be done for each component independently.

$$\underbrace{A_x \, \mathbf{i} + A_y \, \mathbf{j}}_{\mathbf{A}} + \underbrace{B_x \, \mathbf{i} + B_y \, \mathbf{j}}_{\mathbf{B}} = \underbrace{(A_x + B_x) \, \mathbf{i} + \left(A_y + B_y\right) \, \mathbf{j}}_{=\mathbf{A}+\mathbf{B}} \ . \tag{6.11}$$

$$c\mathbf{A} = c\left(A_x \, \mathbf{i} + A_y \, \mathbf{j}\right) = cA_x \, \mathbf{i} + cA_y \, \mathbf{j} \ . \tag{6.12}$$

The Dot Product

The **dot product** between two vectors **A** and **B** is defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\alpha \ , \tag{6.13}$$

where α is the angle between the two vectors, as illustrated in Fig. 6.1d.

The dot product is **linear**:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \ , \tag{6.14}$$

and **commutative**:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \ . \tag{6.15}$$

The dot product depends on the angle α, as illustrated in Fig. 6.1d. When two vectors are parallel and point in the same direction, the dot product is equal to the product of the magnitudes. A particular useful property of the dot product is that the dot product of two orthogonal vectors is zero. As a result the dot product is simple on component form

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \left(A_x\,\mathbf{i} + A_y\,\mathbf{j}\right) \cdot \left(B_x\,\mathbf{i} + B_y\,\mathbf{j}\right) \\ &= A_x B_x \underbrace{\mathbf{i} \cdot \mathbf{i}}_{=1} + A_x B_y \underbrace{\mathbf{i} \cdot \mathbf{j}}_{=0} + A_y B_x \underbrace{\mathbf{j} \cdot \mathbf{i}}_{=0} + A_y B_y \underbrace{\mathbf{j} \cdot \mathbf{j}}_{=1} \\ &= A_x B_x + A_y B_y \ . \end{aligned} \tag{6.16}$$

The value of the dot product is independent of the unit vectors used to decompose the vectors. We say that the dot product is invariant under a change of coordinate system.

What makes the dot product so useful, is that it can be used to decompose a vector onto a given set of unit vector—it can be used to find the components of a vector in any given coordinate system. A vector **A** can be written in component form as:

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k} \ . \tag{6.17}$$

How can we determine the components A_x, A_y, and A_z? We find them by using to dot product, and remembering that the unit vectors are orthogonal. We find component A_x from:

$$\mathbf{A} \cdot \mathbf{i} = A_x \underbrace{\mathbf{i} \cdot \mathbf{i}}_{=1} + A_y \underbrace{\mathbf{j} \cdot \mathbf{i}}_{=0} + A_z \underbrace{\mathbf{k} \cdot \mathbf{i}}_{=0} = A_x \ , \tag{6.18}$$

and similarly for the other components:

The component of a vector **A** can be found by dot-multiplication with the unit vectors **i**, **j**, and **k**:

$$A_x = \mathbf{A} \cdot \mathbf{i} \; , \; A_y = \mathbf{A} \cdot \mathbf{j} \; , \; A_z = \mathbf{A} \cdot \mathbf{k} \; , \tag{6.19}$$

so that

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{i}) \; \mathbf{i} + (\mathbf{A} \cdot \mathbf{j}) \; \mathbf{j} + (\mathbf{A} \cdot \mathbf{k}) \; \mathbf{k} \; . \tag{6.20}$$

Numerical Representation of Vectors

In Python a vector is represented by its component form. The vector **a**:

$$\mathbf{a} = 1\,\mathbf{i} + 2\,\mathbf{j} = (1, 2, 0) \;\; , \tag{6.21}$$

is generated by the following command:

```
a = array([1,2,0]);
```

Addition and Subtraction

All the ordinary mathematical operations can be applied to vectors just as you would apply them to scalars. For example, vector addition is achieved by:

```
b = array([2,-4,0])
c = a + b
print(c)
```

```
[ 3 -2 0]
```

You can decide if you want to use a vector in two- or three dimensions. For example, you could instead have defined the vector **a** as:

```
a = array([1,2]);
```

But notice that you cannot add two vectors that do not have the same number of components.

Scalar Multiplication

Scalar multiplication is similarly naturally implemented:

```
d = 3*a
print(d)
```

```
[3 6 0]
```

Componentwise Operations

Notice that there is room for error because of the way commands are interpreted. For example, if you add a scalar to a vector, this is interpreted as a componentwise addition: The scalar is added to each of the components:

```
e = a + 3
print(e)
```

```
[4 5 3]
```

Dot Product

The dot product is found by applying the function `dot`, which returns a scalar:

```
f = dot(a,b)
print(f)
```

```
-6
```

A common application of the dot product is to find the component of a vector **a** along the direction given by a vector **b**. In general, **b**, is not a unit vector. We therefore first need to find a unit vector in the direction of **b**:

$$\mathbf{u}_b = \frac{\mathbf{b}}{|\mathbf{b}|} , \tag{6.22}$$

and the component of **a** in this direction is given by the dot product:

$$a_b = \mathbf{a} \cdot \mathbf{u}_b = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} . \tag{6.23}$$

Numerically, this is done in exactly the same way:

```
ab = dot(a,b)/sqrt(dot(b,b));
print(ab)
```

Notice that we use the relation:

$$|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} , \tag{6.24}$$

for the magnitude of **b**.

Time Sequences of Vectors

We will often work with a sequence of vectors, corresponding to the time evolution of a vector. For example, we may be interested in the position, **r**, or the force, **F**, as a function of time, t:

$$\mathbf{r}(t) \text{ and } \mathbf{F}(t) \ . \tag{6.25}$$

Numerically, we will have a corresponding sequence of positions or forces at discrete times, t_i:

$$\mathbf{r}_i = \mathbf{r}(t_i) \text{ and } \mathbf{F}_i = \mathbf{F}(t_i) \ . \tag{6.26}$$

Fortunately, it is simple to both represent and apply mathematical operations to an element in a sequence.

We generate a sequence of n vectors $\mathbf{r}_i$ with x, y, and z coordinates by:

```
n = 10
r = zeros((n,3),float)
```

We can use mathematical vector operations directly on element in the sequence, as illustrated in the following example:

```
v = array([1.0,-2.0,2.0])
n = 10
r = zeros((n,3),float)
r[0] = array([0,0,0]);
dt = 0.1
r[1] = r[0] + v*dt
```

6.2 Description of Motion

The cheetah is the world fastest land animal. How can we characterize its motion as it chases its prey? How fast does it run and how fast does it turn?

Motion Diagram and Position Vector

Figure 6.2 shows a few frames from a movie of a cheetah chasing a Thompson gazelle. To quantify the motion we generate a motion diagram: We mark the position of the cheetah at regular time intervals and record the positions $\mathbf{r}(t_i)$ of the cheetah relative to the origin at time t_i.

We are free to choose the origin and the axes of the coordinate system. The origin determines where we measure the positions from. In Fig. 6.2 we have chosen a stationary point on the ground as the origin. In addition to the origin, a coordinate system consists of a set of axes that we use to decompose the position vector. The directions of the axes indicate the positive direction of the corresponding unit vector. The position can be decomposed along the x, y, and z-axes respectively (Table 6.1):

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \ , \tag{6.27}$$

where $x(t)$, $y(t)$, and $z(t)$ are lengths along the axes and hence have units of length. For example, the position at $t = 0.5$ s is:

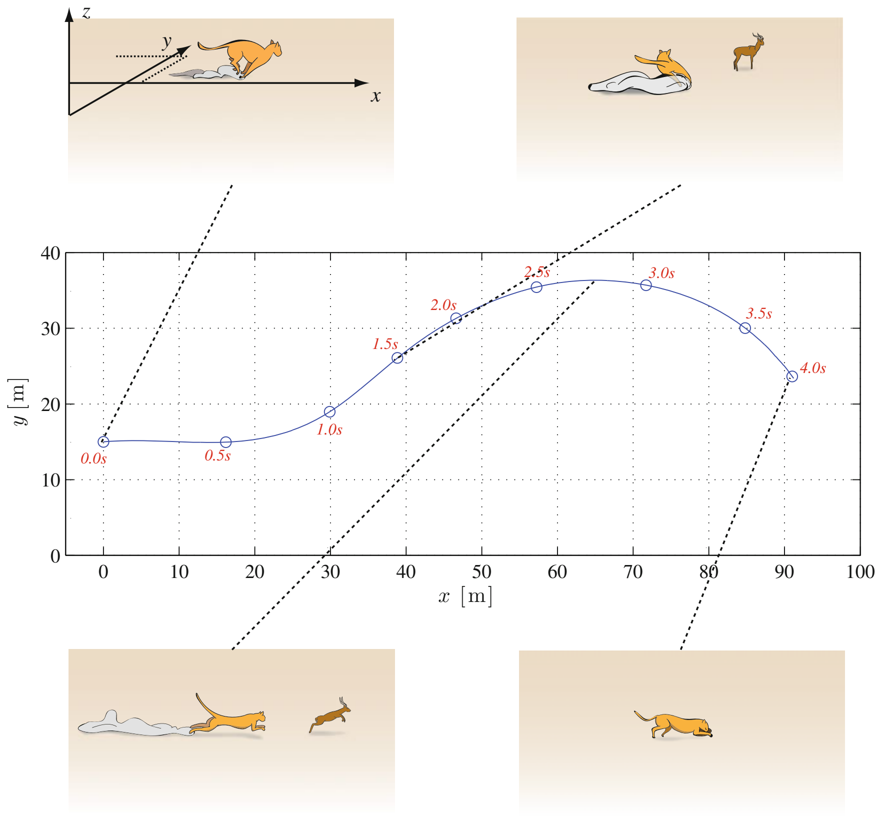


Fig. 6.2 Illustration of a cheetah chasing a Thompson gazelle, and an illustration of the two-dimensional motion and the two-dimensional motion diagram. (Illustration by S.B. Skattum)

Table 6.1 Positions of the running cheetah

t_i (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
x_i (m)	0.0	16.2	29.9	38.9	46.6	57.2	71.7	84.8
y_i (m)	15.0	14.95	19.0	26.1	31.3	35.4	35.7	30.0

$$\mathbf{r}(1.0\,\text{s}) = x(1.0\,\text{s})\,\mathbf{i} + y(1.0\,\text{s})\,\mathbf{j} = 29.9\,\text{m}\,\mathbf{i} + 19.0\,\text{m}\,\mathbf{j}\,, \tag{6.28}$$

where we have skipped the z-coordinate since all the motion is in the xy-plane. You find the complete dataset in the file cheetah.d.[2] Each line gives t_i, x_i, y_i, where t_i is measured in seconds, and x_i and y_i are measured in meters. We have tabulated the positions at $\Delta t = 0.5$ s intervals in the following table:

[2]http://folk.uio.no/malthe/mechbook/cheetah.d.

Velocity Vector

Figure 6.2 shows how the position changes over a time interval Δt. The change in position is also a vector and is called the **displacement**. The displacement from $t = 1.0\,\mathrm{s}$ to $t = 2.0\,\mathrm{s}$ is denoted $\Delta\mathbf{r}(1.0\,\mathrm{s})$:

$$\begin{aligned}\Delta\mathbf{r}(1.0\,\mathrm{s}) &= \mathbf{r}(2.0\,\mathrm{s}) - \mathbf{r}(1.0\,\mathrm{s}) = (46.6\,\mathrm{m}\,\mathbf{i} + 31.3\,\mathrm{m}\,\mathbf{j}) - (29.9\,\mathrm{m}\,\mathbf{i} + 19.0\,\mathrm{m}\,\mathbf{j}) \\ &= 16.7\,\mathrm{m}\,\mathbf{i} + 12.3\,\mathrm{m}\,\mathbf{j}\,.\end{aligned} \tag{6.29}$$

We can read the displacement directly from the motion diagram in Fig. 6.2 as the vector from $\mathbf{r}(1.0\,\mathrm{s})$ to $\mathbf{r}(2.0\,\mathrm{s})$. Because the displacement depends on a difference between two positions, it does not depend on the choice of origin.

We see from Fig. 6.2 that both the length and the direction of the displacement vectors are changing throughout the motion. The rate of change of the displacement, the velocity, must therefore also be a vector:

The **average velocity** from a time $t = t_0$ to a time $t = t_0 + \Delta t$ is defined as:

$$\bar{\mathbf{v}}(t_0) = \frac{\mathbf{r}(t_0 + \Delta t) - \mathbf{r}(t_0)}{\Delta t} = \frac{\Delta\mathbf{r}(t_0)}{\Delta t}\,. \tag{6.30}$$

It is measured in meters per second, m/s.

We find the average velocity at $t = 1.0\,\mathrm{s}$ using the data in the table above:

$$\bar{\mathbf{v}}(1.0\,\mathrm{s}) = \frac{\Delta\mathbf{r}(1.0\,\mathrm{s})}{1.0\,\mathrm{s}} = \frac{16.7\,\mathrm{m}}{1.0\,\mathrm{s}}\,\mathbf{i} + \frac{12.3\,\mathrm{m}}{1.0\,\mathrm{s}}\,\mathbf{j} = 16.7\,\mathrm{m/s}\,\mathbf{i} + 12.3\,\mathrm{m/s}\,\mathbf{j}\,. \tag{6.31}$$

If we instead use a time interval $\Delta t = 0.5\,\mathrm{s}$ to find the average velocity at $t = 1.0\,\mathrm{s}$ we find:

$$\begin{aligned}\Delta\mathbf{r}(1.0\,\mathrm{s}) &= \mathbf{r}(1.5\,\mathrm{s}) - \mathbf{r}(1.0\,\mathrm{s}) = (38.9\,\mathrm{m}\,\mathbf{i} + 26.1\,\mathrm{m}\,\mathbf{j}) - (29.9\,\mathrm{m}\,\mathbf{i} + 19.0\,\mathrm{m}\,\mathbf{j}) \\ &= 9.0\,\mathrm{m}\,\mathbf{i} + 7.1\,\mathrm{m}\,\mathbf{j}\,,\end{aligned} \tag{6.32}$$

$$\bar{\mathbf{v}}(1.0\,\mathrm{s}) = \frac{\Delta\mathbf{r}(1.0\,\mathrm{s})}{\Delta t} = \frac{1}{0.5\,\mathrm{s}}\,(9.0\,\mathrm{m}\,\mathbf{i} + 7.1\,\mathrm{m}\,\mathbf{j}) = 18.0\,\mathrm{m/s}\,\mathbf{i} + 14.2\,\mathrm{m/s}\,\mathbf{j}\,. \tag{6.33}$$

We see that the average velocity depends on the time interval Δt, just as we saw in the one-dimensional case. Again, we can understand this better by studying the displacement at $t = 1.0\,\mathrm{s}$ for smaller and smaller time intervals Δt, as shown in Fig. 6.3. We see that as Δt becomes smaller, the displacement also becomes smaller, but its direction approaches the tangent to the curve describing the motion around $t = 1.0\,\mathrm{s}$.

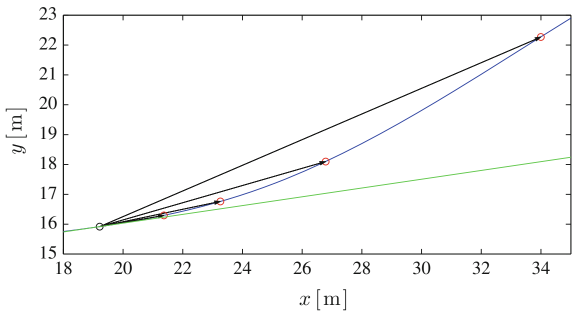


Fig. 6.3 Illustration of the average velocity, $\bar{\mathbf{r}}(2.0\mathrm{s})$ for decreasing time intervals Δt for the motion of the cheetah

The **instantaneous velocity** at the time t is defined as the limit of the average velocity when the time interval Δt goes to zero, that is, the time derivative of the position vector, $\mathbf{r}(t)$.

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt} \ . \tag{6.34}$$

Whenever we use the term velocity, we will mean the instantaneous velocity. The velocity vector is tangential to the trajectory.

Speed

The magnitude of the velocity vector is called the *speed*, v, defined as:

$$v(t) = |\mathbf{v}(t)| \ . \tag{6.35}$$

We use the word velocity for the velocity vector, and the word speed for the magnitude of the vector velocity.

Time Derivatives of Vector Functions

How do we find the derivative of a vector function such as $\mathbf{r}(t)$? The simplest approach is to write the vector in terms of the unit vectors for the coordinate system, and then take the derivative of each component:

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}\left(x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\right) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} \ , \tag{6.36}$$

where[3] we define the component-wise velocities as

$$v_x(t) = \frac{dx}{dt} \; , \; v_y(t) = \frac{dy}{dt} \; , \; v_z(t) = \frac{dz}{dt} \; . \tag{6.37}$$

The velocity vector can therefore also be written:

$$\mathbf{v}(t) = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j} + v_z(t)\,\mathbf{k} = \big(v_x(t), v_y(t), v_z(t)\big) \; . \tag{6.38}$$

Acceleration

It is clear from Fig. 6.2 that the average velocity of the cheetah is not constant throughout the motion—it is varying both in direction and magnitude. Just as we introduced velocity to characterize the change in the position vector, we introduce the acceleration vector to characterize the change in the velocity vector:

The **average acceleration vector** over a time interval Δt from t to $t + \Delta t$ is defined as:

$$\bar{\mathbf{a}} = \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} \; . \tag{6.39}$$

We define the **instantaneous acceleration vector**, or simply the instantaneous acceleration, as the limit of the average acceleration vector when the time interval approaches zero:

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} \; . \tag{6.40}$$

The acceleration vector is the time derivative of the velocity vector.

We find the acceleration in the vector component representation by componentwise derivation:

$$\begin{aligned} \mathbf{a}(t) &= \tfrac{d}{dt}\mathbf{v} = \tfrac{d}{dt}\big(v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}\big) \\ &= \tfrac{dv_x}{dt}\,\mathbf{i} + \tfrac{dv_y}{dt}\,\mathbf{j} + \tfrac{dv_z}{dt}\,\mathbf{k} = a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j} + a_z(t)\,\mathbf{k} \; , \end{aligned} \tag{6.41}$$

where we see that:

$$a_x(t) = \frac{dv_x}{dt} \; , \; a_y(t) = \frac{dv_y}{dt} \; , \; a_z(t) = \frac{dv_z}{dt} \; . \tag{6.42}$$

[3] Here we have implicitly assumed that the time derivatives of the unit vectors are zero. This is not necessarily the case: The unit vectors vary with time for rotating reference systems.

Since the velocity vector is the time derivative of the position vector

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) , \tag{6.43}$$

we can write the acceleration vector as the second time derivative of the position vector:

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v} = \frac{d}{dt}\frac{d}{dt}\mathbf{r} = \frac{d^2\mathbf{r}}{dt^2} , \tag{6.44}$$

which can be written on component form:

$$a_x(t) = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} , \; a_y(t) = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} , \; a_z(t) = \frac{dv_z}{dt} = \frac{d^2z}{dt^2} . \tag{6.45}$$

Notation for Time Derivatives

In physics, we use both the differential form, d/dt and the dot notation for time derivatives:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} , \tag{6.46}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} , \tag{6.47}$$

but we *do not* use the "marked" notation, $v_x = x'(t)$, often used in mathematics because we often use $x'(t)$ to mean the position x measured in another coordinate system. We therefore strongly recommend you to use the notations introduced here, either the d/dt notation or the dot-notation.

Interpretation of Motion Diagrams

It is often takes time to gain a good intuition for acceleration, in particular for two- and three-dimensional motions. Motion diagrams can help in developing that intuition by visualizing velocities and accelerations.

If the motion diagram is drawn using a constant time interval Δt, we can use the displacement vector as a visualization of the velocity, since the velocity is proportional to the displacement:

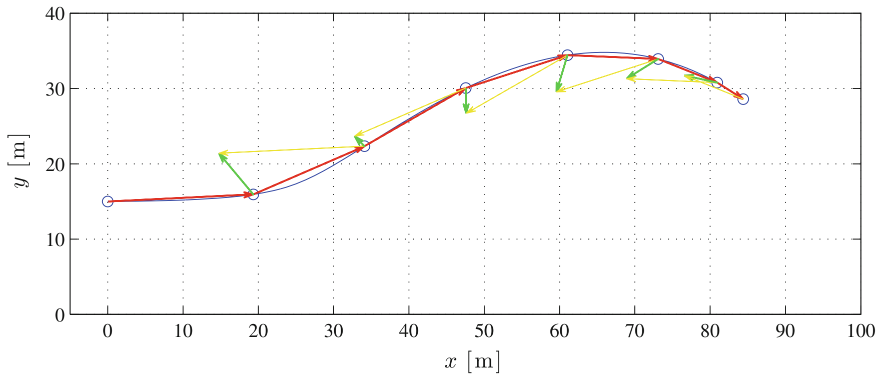


Fig. 6.4 Motion diagrams for the cheetah with $\Delta t = 0.5$ s illustrating both the displacements, interpreted as velocities, and the change in displacements, interpreted as accelerations. The constructions of the accelerations are illustrated

$$\bar{\mathbf{v}}(t_i) = \frac{1}{\Delta t}\Delta\mathbf{r}(t_i) \ . \tag{6.48}$$

The displacement vectors are illustrated by red vectors at intervals of 0.5 and 0.25 s in Fig. 6.4.

Notice that if we want to look at the *change* in velocity at the time $t = 1.0$ s, we would like to compare the velocity before the time $t = 1.0$ s with the velocity after the time $t = 1.0$ s. Now, the average velocity at the time $t = 1.0$ s is really the average velocity over the time interval from 1.0 to 1.5 s. And the average velocity at the time $t = 0.5$ s is the average velocity over the time interval from 0.5 to 1.0 s. Therefore, a reasonable way to characterize the change in velocity at $t = 1.0$ s is to characterize it as the change in velocity over the time interval from 0.5 t to 1.5 s:

$$\Delta\bar{\mathbf{v}}(1.0\,\mathrm{s}) = \bar{\mathbf{v}}(1.0\,\mathrm{s}) - \bar{\mathbf{v}}(0.5\,\mathrm{s}) \ . \tag{6.49}$$

We can interpret this as the average acceleration of the cheetah at $t = 1.0$ s, since it is (approximately) proportional to the average acceleration:

$$\bar{\mathbf{a}}(1.0\,\mathrm{s}) \simeq \frac{1}{\Delta t}\left(\bar{\mathbf{v}}(1.0\,\mathrm{s}) - \bar{\mathbf{v}}(0.5\,\mathrm{s})\right) \ . \tag{6.50}$$

This method therefore provides a way to use the motion diagram to find approximations for the acceleration vectors in each point on the motion diagram. Incidentally, this method is the same as the simplest numerical method to find the second order time derivative of the position vector.

1. We find the displacements, $\Delta \mathbf{r}$:

$$\Delta \mathbf{r}(t) = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \ , \tag{6.51}$$

   and interpret these as average velocity vectors for the motion.
2. We find the change in displacements, $\Delta\Delta \mathbf{r}$:

$$\Delta\Delta \mathbf{r}(t) = \Delta \mathbf{r}(t) - \Delta \mathbf{r}(t - \Delta t) \ , \tag{6.52}$$

   and interpret these as average acceleration vectors for the motion.

We have illustrated the motion diagram of the cheetah using this approach in Fig. 6.4. Notice that we can generate the average acceleration vectors by a graphical vector subtraction, as illustrated at $t = 1.0\,\mathrm{s}$ in Fig. 6.4.

6.2.1 Example: Mars Express

This example demonstrates how we can find the velocity and acceleration from both real data and mathematical representations of the position—based on real data provided by ESA.

The Mars Express probe was launched on June 2nd 2003 and reached Mars in December 2003 (see ESA's site for more information on Mars Express). We want to use the data provided by ESA to illustrate the motion of the Mars Express, analyze the velocity and acceleration of the module, and compare with the motion of the Earth and Mars (see Fig. 6.5).

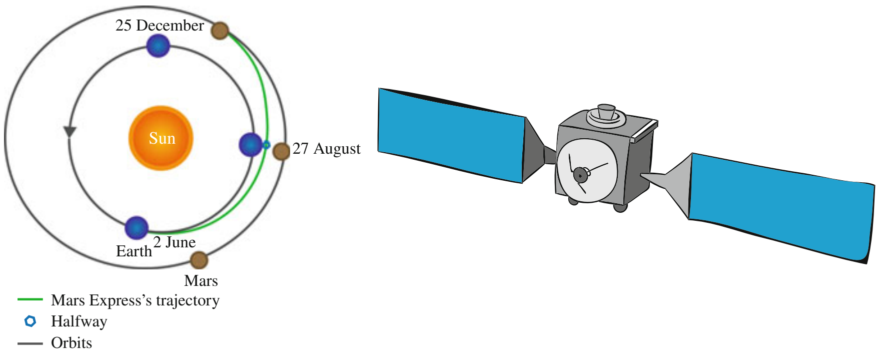


Fig. 6.5 The trajectory of the Mars Express spacecraft illustrated as a motion diagram and the Mars Express spacecraft. (Illustration by S.B. Skattum)

Reading and converting the data: The path of the Mars Express from Earth to Mars is approximately given by the data set marsexpresslr.d,[4] which has been reduced to be strictly two-dimensional. Each line in the data set contains the following information:

$$\begin{array}{cccc} t_1 & x_1 & y_1 & z_1 \\ t_2 & x_2 & y_2 & z_2 \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

The time is given in days, and the spatial coordinates are measured in kilometers. We read the data set into Python to examine the motion, using `loadtxt`:

```
from pylab import *
t,x,y=loadtxt('marsexpresslr.d',usecols=[0,1,2],unpack=True)
n = len(t)
dt = t[1] - t[0]
r = zeros((n,2),float)
r[:,0] = x
r[:,1] = y
```

This generates the arrays `t`, `x`, and `y`, which we combine to one array to form `r`. This provides us with a vector $\mathbf{r}(t_i)$, which contains the x- and y-components

$$\mathbf{r}(t_i) = x(t_i)\,\mathbf{i} + y(t_i)\,\mathbf{j} \; . \tag{6.53}$$

The vector representation in Python is useful and allows us to make operations on the whole vector in a very similar way to how we write the operations mathematically. We demonstrate this by converting the lengths to astronomical units. The x- and y-coordinates in the data-set are measured in kilometers, but we would like to measure lengths in Astronomical Units, where $1\text{AU} = 149{,}598{,}000$ km. An astronomical unit (1 AU) roughly corresponds to the average radius of the orbit of the Earth around the Sun—it is therefore a useful unit for describing planetary motion. We can convert the data by dividing the length by 1 AU. Mathematically, we would write this as:

$$\mathbf{r}'(t) = \frac{1}{1\text{ AU}}\,\mathbf{r}(t) \; . \tag{6.54}$$

Using the vector representation, the implementation in Python is almost identical:

```
AU = 149598000.0
r = r/AU
```

How does this work? The command r/a divides each element in the array by $a = 1$ AU—both the x and the y coordinate for all times i. This is a very intuitive and efficient way of coding, and you should learn to utilize the power of vectorization.

Plotting the trajectory—Low resolution data: The data-set provides the positions $\mathbf{r}_i$ at times t_i. We can visualize the trajectory by plotting all the positions of the module:

[4] http://folk.uio.no/malthe/mechbook/marsexpresslr.d.

```
plot(r[:,0],r[:,1])
axis('equal')
xlabel('x [au]')
ylabel('y [au]')
show()
```

where we have used `axis('equal')` to ensure that the scaling of the x- and y-axis are the same, so that a circle will appear as a circle and not as an ellipse. Notice that `r[:,0]` gives the x_i for all i, and similarly for the y_i.

The resulting trajectory is shown by a dotted line in Fig. 6.6a. Unfortunately, this data-set only contains the position of the module every 30 days. This only gives us a coarse illustration of the motion, but we are still able to see the main features.

Motion diagram: The plot of the trajectory itself does not provide much insight into the motion of the module. We can gain more insight through a motion diagram or by calculating the velocity and acceleration of the module along the trajectory. Since the data is so coarse, we start by illustrating the velocities and the accelerations in the diagrams.

The average velocity is proportional to the displacement:

$$\bar{\mathbf{v}}_i = \frac{1}{\Delta t} \Delta \mathbf{r}_i = \frac{1}{\Delta t} \left(\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i) \right) \ . \tag{6.55}$$

We can therefore illustrate the velocities by drawing the displacement vectors from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$. This is done by drawing an arrow from `r[i,:]` to `r[i+1,:]` using the `quiver` command, which draws an arrow from a point $\mathbf{r}_i$ and in a distance $\Delta \mathbf{r}_i = \mathbf{r}_{i+1} - \mathbf{r}_i$. Notice the additional arguments provided to the `quiver` command to ensure that the arrows have correct length and orientation. Also remember to replot the previous figure and to add `show()` at the end of the script.

```
for i in range(n-1):
    plot(r[i,0],r[i,1],'o')
    dr = r[i+1,:] - r[i,:]
    quiver(r[i,0],r[i,1],dr[0],dr[1],angles='xy',
           scale_units='xy',scale=1)
show()
```

The loop stops at $n-1$, since we cannot find the displacement from $i = n$ to $i = n+1$. The arrow illustrating the (average) velocities are shown in Fig. 6.6a.

Similarly, the acceleration is approximately given by the change in velocities:

$$\mathbf{a} \simeq \frac{1}{\Delta t} \left(\mathbf{v}(t_i) - \mathbf{v}(t_{i-1}) \right) \ , \tag{6.56}$$

We use the velocities in t_i and t_{i-1} because the velocity in t_i is calculated from the points t_i and t_{i+1} and the velocity in t_{i-1} is calculated from the points t_{i-1} and t_i. In this way, the acceleration is properly centered. This becomes clear by reviewing the motion diagram in Fig. 6.6a. To find the acceleration in point 1, we need to use the velocities (displacements) from point 0 and 1.

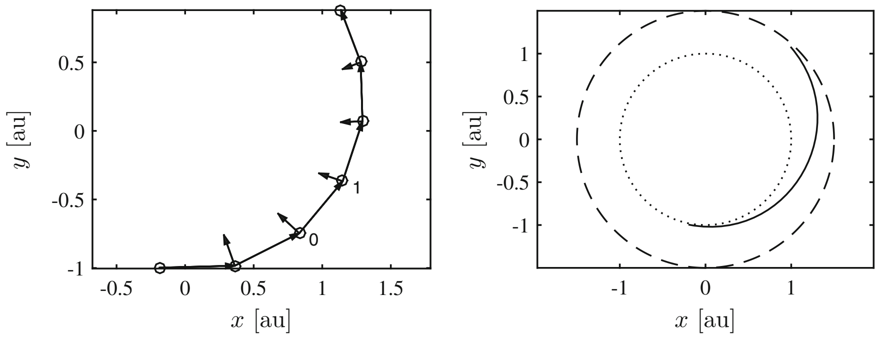


Fig. 6.6 The trajectory of the Mars Express spacecraft illustrated as a motion diagram. The *circles* shows the position of the module at 30 day intervals, the *arrows* indicate the velocities (displacements) and the accelerations (change in displacements)

We can therefore illustrate the accelerations by:

$$\mathbf{a} \simeq \frac{1}{\Delta t^2}\left((\mathbf{r}_{i+1} - \mathbf{r}_i) - (\mathbf{r}_i - \mathbf{r}_{i-1})\right) \ . \tag{6.57}$$

which is implemented as:

```
for i in range(1,n-1):
    plot(r[i,0],r[i,1],'o')
    dr = (r[i+1,:] - r[i,:])-(r[i,:] - r[i-1,:])
    quiver(r[i,0],r[i,1],dr[0],dr[1],angles='xy',
           scale_units='xy',scale=1)
show()
```

The accelerations are shown in Fig. 6.6a. Already from this very simple figure we gain insight into the motion: The acceleration is toward the center of the trajectory and the acceleration is decreasing in magnitude since the length of the arrows are decreasing.

Plotting the trajectory—High resolution data: Fortunately, we have access to data with much higher time resolution in the file marsexpresshr.d.[5] We read and plot the data using the same method as before:

```
from pylab import *
t,x,y=loadtxt('marsexpresshr.d',usecols=[0,1,2],unpack=True)
n = len(t)
r = zeros((n,2),float)
r[:,0] = x
r[:,1] = y
AU = 149598000.0
r = r/AU
plot(r[:,0],r[:,1]), axis('equal')
xlabel('x [au]')
ylabel('y [au]')
show()
```

[5] http://folk.uio.no/malthe/mechbook/marsexpresshr.d.

The resulting trajectory is shown in Fig. 6.6b. In this case, the density of points is so high that it does not make sense to plot the displacements—they are all very small.

Calculating velocity and acceleration: The velocity at t_i is the time derivative of $\mathbf{r}(t)$ at t_i, which we approximate by the numerical derivative over the time interval Δt, when Δt is small:

$$\mathbf{v}(t_i) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \simeq \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \ . \tag{6.58}$$

Notice that this is exactly the same form we used to calculate the displacement in the motion diagram. We can therefore calculate the velocities numerically using the same approach:

```
n = len(r)
dt = t[1]-t[0]
v = zeros((n,2),float)
for i in range (n-1):
    v[i,:] = (r[i+1,:]-r[i,:])/dt
```

The magnitude of the velocity, $v(t)$, is shown in Fig. 6.7.

Similarly, we approximate the acceleration using the numerical derivative of the velocity:

$$\mathbf{a}(t_i) \simeq \frac{\mathbf{v}(t_i) - \mathbf{v}(t_{i-1})}{\Delta t} \ , \tag{6.59}$$

which again is analogous to what we calculated in the motion diagrams:

```
a = zeros((n,2),float)
for i in range(2,n-1):
    a[i,:] = (v[i,:] - v[i-1,:])/dt
```

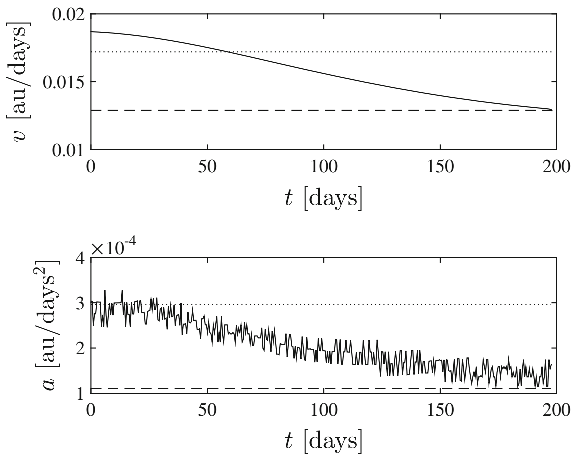


Fig. 6.7 The magnitude of the velocity and the acceleration of the module as function of time. (Notice the noise in the accelerations)

Figure 6.7 shows that the acceleration is decreasing in magnitude as the module moves from the Earth to Mars, but the data is noisier for the acceleration. The magnitudes of velocity and acceleration are calculated using the `norm`-function.

```
vv = zeros((n,1),float)
aa = zeros((n,1),float)
for i in range(n):
    vv[i] = norm(v[i,:])
    aa[i] = norm(a[i,:])
```

Mathematical models: Since we do not have any intuition for velocities or accelerations of space travel such as for the Mars Express, we need to compare with other relevant motions, such as the motion of the Earth and Mars around the Sun. For simplicity, we assume that both the Earth and Mars follow circular orbits with radi $R_E = 1$ au and $R_M = 1.5$ au, and periods (the time a complete revolution takes) of $T_E = 1$ year $= 365.25$ days and $T_M = 2$ years $= 730.5$ days. The trajectory for a circular orbit with radius R and period T can be described by

$$\mathbf{r}(t) = R\,(\mathbf{i}\,\cos\omega t + \mathbf{j}\,\sin\omega t)\ , \tag{6.60}$$

where $\omega = 2\pi/T$. This describes a circle with radius R and the trajectory is at $\mathbf{r} = R\mathbf{i}$ at $t = 0$ and at $t = T$, hence the period is T. If you are not familiar with this representation of circles, you can plot the trajectory by

```
>> R = 1.0
>> T = 365.25
>> t = linspace(0,TE,1000)
>> r = R*array([cos(2*pi*t/T) sin(2*pi*t/T)])
>> plot(r[0,:],rE[1,:]), axis('equal')
```

We are here primarily interested in the velocity and acceleration of a planet in the trajectory, and not in when the planet is where along the trajectory. Therefore we have not taken care to ensure that the initial positions at $t = 0$ for Mars or the Earth are correct relative to each other. (That is, we do not care where the planets are at $t = 0$).

We use this representation for both the Earth and Mars. The velocity of the planet in this circular orbit is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}R\,(\mathbf{i}\,\cos\omega t + \mathbf{j}\,\sin\omega t) = R\omega\,(-\mathbf{i}\,\sin\omega t + \mathbf{j}\,\cos\omega t)\ . \tag{6.61}$$

The magnitude of the velocity is therefore a constant:

$$|\mathbf{v}| = R\omega\,|-\mathbf{i}\,\sin\omega t + \mathbf{j}\,\cos\omega t| = R\omega = 2\pi R/T\ , \tag{6.62}$$

which corresponds to the distance traveled during one complete revolution divided by the time it takes to make a revolution. We therefore now have numbers we can use to compare with the results from our calculations. For the Earth (v_E) and for Mars (v_M) we find:

$$v_E = 2\,\pi\ 1.0\,\text{au}/\,(365.25\,\text{days}) = 0.017\,\text{au/days}\ , \tag{6.63}$$

$$v_M = 2\pi\ 1.5\,\text{au}/\,(2\,365.25\,\text{days}) = 0.013\,\text{au/days}\ . \tag{6.64}$$

These values are illustrated by a dotted and a dashed line in Fig. 6.7. Indeed, we see that the Mars Express holds approximately the same velocity as Mars when the module approaches Mars, and that the module starts with a velocity significantly larger than that of the Earth in its orbit.

We find the acceleration for these trajectories from the derivative of the velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} R\omega\,(-\mathbf{i}\ \sin\omega t + \mathbf{j}\ \cos\omega t) = -R\omega^2\,(\mathbf{i}\ \cos\omega t + \mathbf{j}\ \sin\omega t)\ , \tag{6.65}$$

which shows that the acceleration of the Earth (and Mars) always points in toward the center of the orbit. The magnitude of the acceleration is a constant, $a = R\omega^2$, which gives

$$a_E = \left(2^2\,\pi^2\,1.0\,\text{au}\right)/\,(365.25\,\text{days})^2 = 3.00\times 10^{-4}\,\text{au/days}^2\ , \tag{6.66}$$

$$a_M = \left(2^2\,\pi^2\,1.5\,\text{au}\right)/\,(2\,365.25\,\text{days})^2 = 1.11\times 10^{-4}\,\text{au/days}^2\ . \tag{6.67}$$

We have plotted these values as dotted (Earth) and dashed (Mars) lines in Fig. 6.7. We see that the acceleration of the module is the same as the acceleration of the Earth when it is near the Earth and Mars when it is near Mars.

We can also plot the trajectories of the Earth and Mars into the plot of the trajectory of the Mars Express. First, we calculate the trajectories and then plot them in the same plot:

```
hold('on')
RE = 1.0 TE = 365.25
tt = linspace(0,TE,1000)
omegaE = 2*pi/TE
rE = RE*transpose(array([cos(omegaE*tt) sin(omegaE*tt)]))
plot(rE[:,0],rE[:,1],':')
RM = 1.5
TM = 2*365.25
tt = linspace(0,TM,1000)
omegaE = 2*pi/TE
rM = RM*transpose(array([cos(omegaM*tt) sin(omegaM*tt)]))
plot(rM[:,0],rM[:,1],'--')
hold('off')
show()
```

Notice the compact way of calculating the trajectory. First, we generate an array `tt` of time from 0 days to the period T, and then we generate the x and y positions. Notice also the use of transpose (using the `transpose` function) to ensure that the vectors have the shape $(N, 2)$ and not $(2, N)$. The resulting trajectories are shown in Fig. 6.6.

Analysis of acceleration: We can use the data-set to find out how the magnitude of the acceleration depends on the distance, r, to the Sun. The distance, r, is the magnitude of the position vector, since all the positions are measured with the Sun in

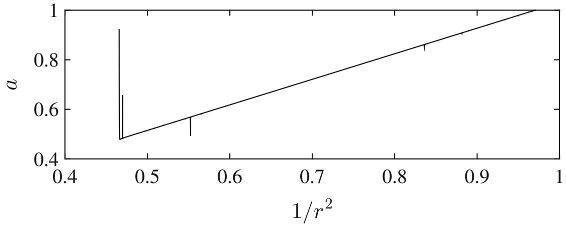


Fig. 6.8 A plot of the acceleration a as a function of $1/r^2$, where r is the distance from the Mars Express to the Sun

the origin of the coordinate system. The magnitude of the position vector is $r(t) = |\mathbf{r}|$. We have already found the magnitude of the velocity and the acceleration, and we can find the norm of the position in the same way:

```
rr = zeros((n,1),float)
for i in range(n):
    rr[i] = norm(r[i,:])
```

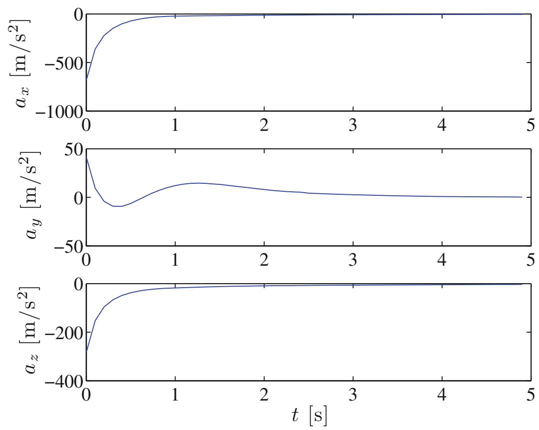
Let us now test the hypothesis that the acceleration is inversely proportional to the distance to the Sun, which is the essence of Newton's law of gravitation:

$$a(r) = \frac{c}{r^2} , \tag{6.68}$$

where c is a constant. We test this idea by plotting a as a function of $1/r^2$. If the resulting graph is a straight line, we have shown that the acceleration indeed is described by Newton's law of gravitation. The resulting plot is shown in Fig. 6.8. The plot shows that the data is consistent with Newton's law of gravitation, except for in the initial phase. However, in this case we expect the spacecraft to be affected by other effects, such as the effect of the engine driving it, and the gravitational force from the Earth.

6.3 Calculation of Motion

We are typically not given a description of the motion, instead we have measured the velocity or the acceleration or we have a theory for the acceleration, and we want to determine the motion. This requires us to integrate the equations of motion. Thus we need methods to determine the motion of an object in two and three dimensions given a set of measurements of the velocity or the acceleration, given a mathematical expression for the velocity or the acceleration, or given a differential equation for the velocity or acceleration. Here we equip you with the methods needed to actually solve mechanics problems.



t_i	$a_x(t_i)$	$a_y(t_i)$	$a_z(t_i)$
0.00s	-683.1m/s^2	41.1m/s^2	-283.0m/s^2
0.01s	-634.8m/s^2	36.6m/s^2	-263.3m/s^2
0.02s	-591.4m/s^2	32.3m/s^2	-245.6m/s^2
0.03s	-552.3m/s^2	28.4m/s^2	-229.9m/s^2
0.04s	-516.9m/s^2	25.0m/s^2	-215.6m/s^2
0.05s	-484.8m/s^2	21.7m/s^2	-202.4m/s^2
0.06s	-455.6m/s^2	18.8m/s^2	-190.5m/s^2
0.07s	-428.9m/s^2	15.9m/s^2	-179.9m/s^2
0.08s	-404.4m/s^2	13.7m/s^2	-169.9m/s^2
0.09s	-382.1m/s^2	11.4m/s^2	-160.9m/s^2

Fig. 6.9 Plot of the components of the accelerations recorded by the accelerometer in the probe, and a table listing the first 10 vales

Discrete Integration

In your line of work as a tornado chaser you have just developed a tornado probe, a small spherical ball you plan to shoot through a tornado to measure the wind velocities. The probe is fitted with a tiny accelerometer based on a MEMS-device (a micro-electromechanical system) that records the acceleration of the probe in the x, y, and z-directions every 0.01 s. After the flight, you recover the probe and read out the accelerations. How can you use these readings to find the velocity and position of the probe during its flight?

The acceleration was recorded at a sequence of times, t_i, with constant time intervals, $\Delta t = 0.01$ s, as shown in Fig. 6.9. We want to use this sequence of accelerations, $\mathbf{a}(t_i)$ to find both the sequence of velocities, $\mathbf{v}(t_i)$, and the sequence of positions, $\mathbf{r}(t_i)$ of the probe. Just as for one-dimensional motion, we use the expression for the numerical derivative of the velocity, the average acceleration, to determine the velocities. The average acceleration vector from the time t_i to $t_{i+1} = t_i + \Delta t$ is:

$$\bar{\mathbf{a}}(t_i) = \frac{\mathbf{v}(t_i + \Delta t) - \mathbf{v}(t_i)}{\Delta t} \; . \tag{6.69}$$

We solve for $\mathbf{v}(t_i + \Delta t)$ to step forward in time:

$$\begin{aligned} \mathbf{v}(t_i + \Delta t) - \mathbf{v}(t_i) &= \Delta t \cdot \bar{\mathbf{a}}(t_i) \\ \mathbf{v}(t_i + \Delta t) &= \mathbf{v}(t_i) + \Delta t \cdot \bar{\mathbf{a}}(t_i) \; . \end{aligned} \tag{6.70}$$

Even though the accelerometer does not provide the average acceleration during a time interval, but rather the instantaneous acceleration at the time, t_0, we get a reasonable approximation by assuming that the average acceleration is equal to the instantaneous acceleration:

$$\bar{\mathbf{a}}(t_0) \simeq \mathbf{a}(t_i) \; . \tag{6.71}$$

Indeed, this corresponds to the approximation used in the first order numerical derivative.

This produces a sequence of velocities, $\mathbf{v}(t_i)$. We can now use exactly the same procedure to find the positions from the velocities, since the velocity is the time derivative of the position. The velocity from time t_i to $t_{i+1} = t_i + \Delta t$ is approximately gives as the average velocity:

$$\mathbf{v}(t_i) \simeq \frac{\mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)}{\Delta t} \; , \tag{6.72}$$

which we again can solve for $\mathbf{r}(t_i + \Delta t)$ getting:

$$\begin{aligned} \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i) &\simeq \Delta t \cdot \mathbf{v}(t_i) \\ \mathbf{r}(t_i \Delta t) &\simeq \mathbf{r}(t_i) + \Delta t \cdot \mathbf{v}(t_i) \; . \end{aligned} \tag{6.73}$$

We can now find the position and velocities of the probe, starting at $t_0 = 0$s:

1. At $t = t_0 = 0\,\mathrm{s}$ the probe is launched from $\mathbf{r}_0 = -80.0\,\mathrm{m}\,\mathbf{i}$ with a velocity $\mathbf{v}(t_0) = \mathbf{v}_0 = 184.9\,\mathrm{m/s}\,\mathbf{i} - 18.5\,\mathrm{m/s}\,\mathbf{j} + 74.0\,\mathrm{m/s}\,\mathbf{k}$:

$$\mathbf{v}(0.0\,\mathrm{s}) = 184.9\,\mathrm{m/s}\,\mathbf{i} - 18.5\,\mathrm{m/s}\,\mathbf{j} + 74.0\,\mathrm{m/s}\,\mathbf{k} \; , \tag{6.74}$$

$$\mathbf{r}(0.0\,\mathrm{s}) = -80.0\,\mathrm{m}\,\mathbf{i} + 0.0\,\mathrm{m}\,\mathbf{j} + 0.0\,\mathrm{m}\,\mathbf{k} \; . \tag{6.75}$$

2. At $t_1 = t_0 + \Delta t = 0.01\,\mathrm{s}$, the velocity vector is:

$$\mathbf{v}(0.01\,\mathrm{s}) \simeq \mathbf{v}(0.0\,\mathrm{s}) + \Delta t\,\mathbf{a}(0.0\,\mathrm{s}) = 178.0\,\mathrm{m/s}\,\mathbf{i} - 18.1\,\mathrm{m/s}\,\mathbf{j} + 71.1\,textm/s\,\mathbf{k} \; , \tag{6.76}$$

where the acceleration $\mathbf{a}(0.0\,\mathrm{s}) = -683.1\,\mathrm{m/s}^2\,\mathbf{i} + 41.1\,\mathrm{m/s}^2\,\mathbf{j} - 283.0\,\mathrm{m/s}^2\,\mathbf{k}$ is listed in the table in Fig. 6.9. The position vector of the probe is:

$$\mathbf{r}(0.01\,\mathrm{s}) \simeq \mathbf{r}(0.0\,\mathrm{s}) + \Delta t\,\mathbf{v}(0.0\,\mathrm{s}) = -78.2\,\mathrm{m}\,\mathbf{i} - 0.18\,\mathrm{m}\,\mathbf{j} + 0.73\,\mathrm{m}\,\mathbf{k} \; . \tag{6.77}$$

This trajectory is illustration in Fig. 6.10, where we have also illustrated the velocity field of the tornado. We will return to this example with a more physical approach in the next chapter.

The method we have presented here is called Euler's method for integration. However, just as for one-dimensional motion, a small modification of the method makes it stronger: If we instead use the newly calculated velocity, $\mathbf{v}(t_i + \Delta t)$ when we calculate the new positions, we get Euler-Cromer's method , which usually has

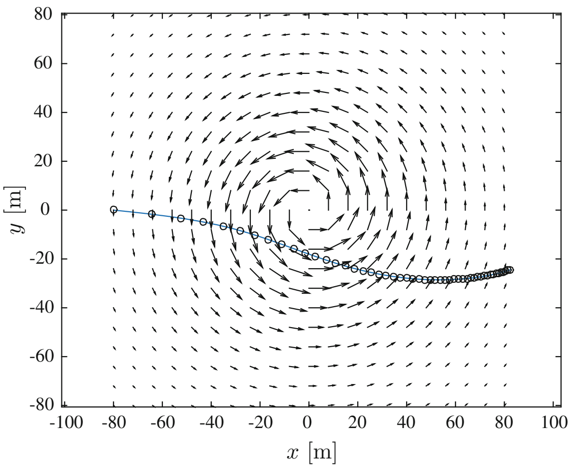


Fig. 6.10 Motion diagram for the probe. The *circles* illustrates the positions of the probe at 0.1 s intervals. The velocity field of the tornado is included for illustration

higher precision and is more stable than Euler method. We will therefore usually use this method here.

In **Euler-Cromer's method** we find the position vector, $\mathbf{r}(t_i)$, and the velocity vector, $\mathbf{v}(t_i)$, as a function of time by a stepwise summation of the acceleration vectors $\mathbf{a}(t_i)$, starting from $\mathbf{v}(t_0) = \mathbf{v}_0$ and $\mathbf{r}(t_0) = \mathbf{r}_0$:

$$\begin{aligned} \mathbf{v}(t_i + \Delta t) &\simeq \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(t_i) \\ \mathbf{r}(t_i + \Delta t) &\simeq \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t) \end{aligned} \tag{6.78}$$

We have implemented this method in a short program that finds the velocities and positions of the probe. You can find the components of the acceleration vector in the file tornado.d[6] recorded at an interval $\Delta t = 0.01$ s, where each line contains a time (in seconds) and the x- and y-components of the accelerations (a_x and a_y) given in m/s^2:

```
0.0000000e+00  -6.8310810e+02   4.1438567e+01  -2.8305553e+02
1.0000000e-02  -6.3476299e+02   3.6551439e+01  -2.6340926e+02
2.0000000e-02  -5.9144164e+02   3.2315870e+01  -2.4574236e+02
```

We read this data into Python calculate the timestep from the first few timesteps, $\Delta t = t_2 - t_1$, and apply Euler-Cromer's method to find the trajectory:

```
from pylab import *
t,x,y,z=loadtxt('tornado.d',usecols=[0,1,2,3],unpack=True)
n = length(t)
dt = t[1] - t[0]
a = zeros((n,3),float)
a[:,0] = x
a[:,1] = y
```

[6] http://folk.uio.no/malthe/mechbook/tornado.d.

```
a[:,2] = z
v = zeros((n,3),float)
r = zeros((n,3),float)
r[0] = array([-80.0,0.0,0.0])
v[0] = array([184.9,-18.49,73.96])
for i in range(0,n-1):
    v[i+1] = v[i] + a[i]*dt
    r[i+1] = r[i] + 0.5*(v[i+1]+v[i])*dt
    t[i+1] = t[i] + dt
hold('on')
ddt = 1.0
it = round(ddt/dt)
i = r_[1:it:n]
plot(r[i,0],r[i,1],'o')
hold('off')
```

The resulting motion is shown as motion diagram and trajectory in Fig. 6.10.

Formal Integration

If we know a mathematical expression for the acceleration vector, we can find the velocity and position vector by integration. We start from the acceleration and integrate from t_0 to t to find the velocity:

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}(t) \;\Rightarrow\; \int_{t_0}^{t} \frac{d\mathbf{v}}{dt}\,dt = \mathbf{v}(t) - \mathbf{v}(t_0) = \int_{t_0}^{t} \mathbf{a}(t)\,dt \;. \tag{6.79}$$

This allows you to find the velocity from the acceleration. If the acceleration is on component form, you integrate each component to find the velocity:

$$\begin{aligned} \int_{t_0}^{t} \mathbf{a}(t)\,dt &= \int_{t_0}^{t} \left(a_x(t)\,\mathbf{i} + a_y(t)\,\mathbf{j} + a_z(t)\,\mathbf{k}\right)\,dt \\ &= \left(\int_{t_0}^{t} a_x(t)\,dt\right)\mathbf{i} + \left(\int_{t_0}^{t} a_y(t)\,dt\right)\mathbf{j} + \left(\int_{t_0}^{t} a_z(t)\,dt\right)\mathbf{k} \;. \end{aligned} \tag{6.80}$$

When we have the velocity, we find the position by integration:

$$\int_{t_0}^{t} \frac{d\mathbf{r}}{dt} dt = \mathbf{r}(t) - \mathbf{r}(t_0) = \int_{t_0}^{t} \mathbf{v}(t)dt \;\Rightarrow\; \mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)dt \;. \tag{6.81}$$

We insert the result for $\mathbf{v}(t)$ from (6.79) to find the position:

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\,dt \\ &= \mathbf{r}(t_0) + \int_{t_0}^{t} \left(\mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t)dt\right) dt \\ &= \mathbf{r}(t_0) + \mathbf{v}(t_0) \int_{t_0}^{t} dt + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t)dt\right) dt \\ &= \mathbf{r}(t_0) + \mathbf{v}(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t)dt\right) dt \end{aligned} \tag{6.82}$$

These two equations provide the **integration method** to solve the equations of motion and find the position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ vectors from the acceleration vector, $\mathbf{a}(t)$:

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t) dt \\ \mathbf{r}(t) &= \mathbf{r}(t_0) + \mathbf{v}(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \mathbf{a}(t) dt \right) dt \end{aligned} \tag{6.83}$$

Again there is no need to memorize these equations. They follow from your knowledge of calculus. Instead, you should learn how to use your experience from calculus to find these equations when you need them. In this way, it is simpler for you to remember this result simply as a special case: The solution of the initial value problem when the acceleration is only a function of time.

Motion with Constant Acceleration

The integration method can use used to find the motion of an object moving with a constant acceleration, $\mathbf{a}(t) = \mathbf{a}_0$, starting from $\mathbf{r}(t_0) = \mathbf{r}_0$ and with initial velocity $\mathbf{v}(t_0) = \mathbf{v}_0$:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}_0\, dt = \mathbf{v}_0 + \mathbf{a}_0\,(t - t_0) \ . \tag{6.84}$$

and

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\, dt \\ &= \mathbf{r}_0 + \int_{t_0}^{t} (\mathbf{v_0} + \mathbf{a}_0\,(t - t_0))\ dt \\ &= \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}_0\, dt + \int_{t_0}^{t} \mathbf{a}_0\,(t - t_0)\ dt \\ &= \mathbf{r}_0 + \mathbf{v}_0\,(t - t_0) + \frac{1}{2}\mathbf{a}_0\,(t - t_0)^2 \ . \end{aligned} \tag{6.85}$$

Differential Equations

In mechanics we want to calculate the motion of an object based on the forces acting on the object. Therefore, you learn to gradually build more advanced models for the forces acting on an object. Using these models, you can apply Newton's second law to find the acceleration of the object. Finally, you find the velocity and position vector of the object as a function of time based on the expression you have for the acceleration and the initial values of the position and velocity. We called this general procedure "the structured problem-solving approach", and it is illustrated in Fig. 6.11.

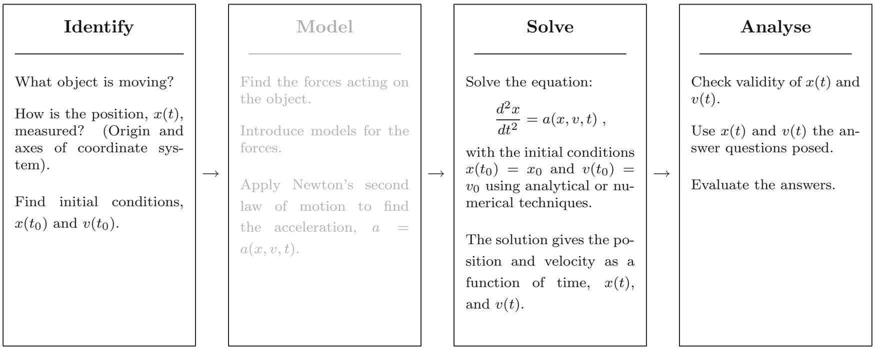


Fig. 6.11 Illustration of the structured problem-solving approach

So far, we have addressed the case when you have measured the acceleration vector, $\mathbf{a}(t)$, either for only a discrete number of time steps, or you can have a theoretical prediction for the acceleration for all times. However, usually, we do not know how the acceleration of an object varies in time, but we rather have models for how the acceleration depends on the position of the object, or the velocity of the object. For a probe scrambling through a tornado, it is the position and velocity of the probe relative to the tornado that determine the forces acting on it and therefore its acceleration. In this case, we cannot integrate the acceleration, because the acceleration also depends on velocity and position.

Generally, modeling the forces and applying Newton's second law leads to an expression:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}(\mathbf{r}, \mathbf{v}, t) = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right) \tag{6.86}$$

For example, a viscous force model leads to an acceleration:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} = -c\mathbf{v} = -c\frac{d\mathbf{r}}{dt} \ , \tag{6.87}$$

and a spring force model leads to an acceleration on the form:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a} = -Cr\frac{\mathbf{r}}{r} \ . \tag{6.88}$$

We see that the unknown function, $\mathbf{r}(t)$, occurs on both sides of the equality—therefore we cannot simply integrate over time to find the solution. These are examples of differential equations. In some cases they can be solved using analytical techniques, but in most cases we will need to turn to numerical methods. Fortunately, in many cases the numerical solution of these differential equations can be done using methods identical to the methods we have developed for direct integration.

One of the major steps in the structured problem-solving approach is the "Solver", where you find the time evolution of the velocity and position vectors from an equation for the acceleration and the initial conditions:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right) \ , \ \mathbf{v}(t_0) = \mathbf{v}_0 \ , \ \mathbf{r}(t_0) = \mathbf{r}_0 \ . \tag{6.89}$$

The result of the "Solver" step is the velocity and position as a function of time, either as continuous, mathematical functions, $\mathbf{v}(t)$, and $\mathbf{r}(t)$, or as the numerical solutions of the equations calculated at discrete time steps, $\mathbf{v}(t_i)$ and $\mathbf{r}(t_i)$. However, unless the time resolution, t_i, is determined by the time intervals of an experimental data-set, your numerical solution can be found at any time resolution you like.

Numerical Solution

We have seen several examples of how to find the numerical solution when the acceleration or velocity vectors are given functions of time. Let us now illustrate a general approach that also works for a differential equation of the form in (6.89). We know the initial position and velocity at $t = t_0$. The acceleration at $t = t_0$ is therefore given by (6.89). Let us to find the velocity and position at $t = t_1$ after a small time interval Δt at $t_1 = t_0 + \Delta t$. We use Euler-Cromer's method to find the new velocity vector based on the acceleration vector at $t = t_0$, and then to find the new position based on the position at $t = t_0$ and the velocity at $t = t_1$:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t \ \mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{6.90}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t \ \mathbf{v}(t_0 + \Delta t) \ . \tag{6.91}$$

We can continue this method iteratively, finding in sequence first t_1, then t_2 and so on, until we have reached the time t.

In **Euler-Cromer's** method we solve the (second order) initial value problem:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}\left(\mathbf{r}, \frac{d\mathbf{r}}{dt}, t\right) \ , \ \mathbf{v}(t_0) = \mathbf{v}_0 \ , \ \mathbf{r}(t_0) = \mathbf{r}_0 \ . \tag{6.92}$$

using the following iterative procedure:

$$\begin{aligned} \mathbf{v}(t_i + \Delta t) &\simeq \mathbf{v}(t_i) + \Delta t \, \mathbf{a}(t_i, \mathbf{r}(t_i), \mathbf{v}(t_i)) \\ \mathbf{r}(t_i + \Delta t) &\simeq \mathbf{r}(t_i) + \Delta t \, \mathbf{v}(t_i + \Delta t) \end{aligned} \tag{6.93}$$

You should in general think of the "Solver" as a call to a numerical function that returns the position and velocity vectors given the functional form of the acceleration and the initial conditions. When you grow up to become a professional physicist, you will have built your own set of methods and tools, numerical and analytical, to "Solve" problems. In particular, we advice you to use a fourth order Runge-Kutta method as your preferred method of numerical integration, although in this book we will focus on clarity and simplicity instead, and typically use Euler-Cromer's method, unless this produces significant errors.

6.3.1 Example: Feather in the Wind

In this example you learn to find the position and velocity from the acceleration, when the acceleration is given by a simplified mathematical model, and when it is given by a differential equation.

You are planning to reproduce the introductory film in Forrest Gump—by capturing the motion of a feather caught in the wind. You plan to start with a feather dropped from a lift, at a height h above the ground. A slight wind is blowing. Our task is to find the motion of the feather, given its acceleration.

Sketch: As always, a good sketch solves half the problem. For simplicity we assume that the motion is two-dimensional. Figure 6.12 shows a sketch of the path of the feather, including the velocity of the wind.

Simplified model—Free fall: The simplest, and least realistic, model for the falling feather is to assume that it falls without air resistance. We release the feather with a horizontal velocity equal to that of the wind, $\mathbf{v}_0 = \mathbf{w}$, and then assume that it has constant acceleration, $\mathbf{a} = -g\mathbf{j}$. We can then find the motion by integration.

$$\mathbf{a} = -g\,\mathbf{j} \tag{6.94}$$

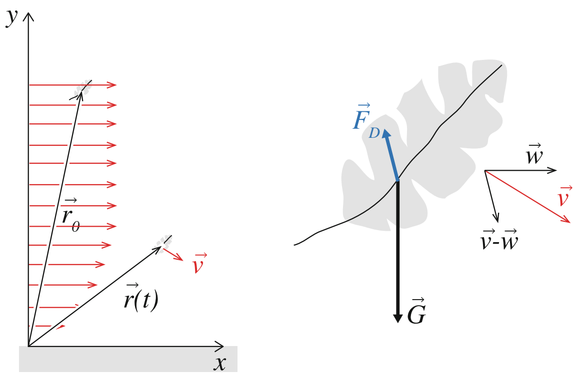


Fig. 6.12 Sketch of a feather falling to the ground. The velocity field, $\mathbf{w}$, of the air is illustrated by the *arrows*

and we release the feather at time $t_0 = 0\,\mathrm{s}$ wt $\mathbf{r}(0) = h\,\mathbf{j}$ with $\mathbf{v}(0) = \mathbf{v}_0 = \mathbf{w}$. We find the velocity by direct integration:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}\,dt = \mathbf{w} + \int_0^t -g\,\mathbf{j}\,dt = \mathbf{w} - gt\,\mathbf{j}\,. \tag{6.95}$$

We can simplify this a bit further if the wind is horizontal, $\mathbf{w} = w\,\mathbf{i}$:

$$\mathbf{v}(t) = w\,\mathbf{i} - gt\,\mathbf{j}\,. \tag{6.96}$$

We find the position by integrating once more over time:

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(t_0) + \textstyle\int_{t_0}^{t} \mathbf{v}(t)\,dt = \mathbf{r}_0 + \int_0^t (w\,\mathbf{i} - gt\,\mathbf{j})\ dt \\ &= h\,\mathbf{j} + wt\,\mathbf{i} - \tfrac{1}{2}gt^2\,\mathbf{j}\,. \end{aligned} \tag{6.97}$$

This gives us a complete solution for this simplified model, which is useful as a comparison when we address the full model.

Realistic model: A more realistic model includes two additional effects: Air resistance means that the acceleration depends on the velocity of the feather and the velocity of the wind, and the wind typically varies near the ground. A better model for the acceleration of the feather is:

$$\mathbf{a} = -g\,\mathbf{j} - C|\mathbf{v} - \mathbf{w}|\,(\mathbf{v} - \mathbf{w})\ , \tag{6.98}$$

where $\mathbf{w}$ is the velocity of the wind. A realistic model for the wind is

$$\mathbf{w}(\mathbf{r}) = \mathbf{w}_0 \left(1 - e^{-y/b}\right)\ , \tag{6.99}$$

In this case, $b = 5\,\mathrm{m}$ and $\mathbf{w}_0 = 3\,\mathrm{m/s}\,\mathbf{i}$ is the velocity of the wind. We drop the feather from rest from a position $\mathbf{r}_0 = h\,\mathbf{j}$, with $h = 10\,\mathrm{m}$.

Unfortunately, we cannot solve this equation exactly, but it is not difficult to solve numerically. We apply Euler-Cromer's method. We find the velocity and position at time $t_i + \Delta t$ from the position and velocity at t_i using:

$$\mathbf{v}(t_i + \Delta t) \simeq \mathbf{v}(t_i) + \Delta t\,\mathbf{a}(t_i, \mathbf{v}_i) \tag{6.100}$$

$$\mathbf{r}(t_i + \Delta t) \simeq \mathbf{r}(t_i) + \Delta t\,\mathbf{v}(t_i + \Delta t)\ . \tag{6.101}$$

In addition, you are told that $C = 30.0\,\mathrm{m}^{-1}$ for the feather.

Notice that the acceleration depends not only on the velocity $\mathbf{v}(t)$ of the feather, but also on its position, because the velocity of the air, $\mathbf{w}$, depends on the position of the feather. We therefore first calculate $\mathbf{w}(\mathbf{r})$ and then calculate $\mathbf{a}(t, \mathbf{r}, \mathbf{v})$. This is implemented in the following program. Notice the use of vector notation to find the acceleration.

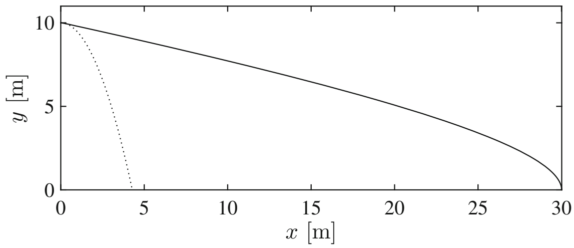


Fig. 6.13 The trajectory of the falling feather

```
# Physical constants
h = 10.0 # m
C = 30.0 # m^-1
w0 = 3.0 # m/s
b = 5.0  # m g = 9.8  # m/s
# Numerical constants
time = 20.0
dt = 0.001
n = int(ceil(time/dt))
t = zeros((n,1),float)
r = zeros((n,2),float)
v = zeros((n,2),float)
a = zeros((n,2),float)
# Initial conditions
t[0] = 0.0
r[0,:] = array([0.0,h])
v[1,:] = array([0.0,0.0])
# Find the motion
for i in range(n-1):
    w = w0*(1.0-exp(-r[i,1]/b))*array([1,0])
    a[i,:] = -g*array([0,1])-C*norm(v[i,:]-w)*(v[i,:]-w)
    v[i+1,:] = v[i,:] + a[i,:]*dt
    r[i+1,:] = r[i,:] + v[i+1,:]*dt
    t[i+1] = t[i] + dt
# Plot motion
i = find(r[:,1]>0.0)
plot(r[i,0],r[i,1],'-k');
axis('equal'), xlabel('x [m]'); ylabel('y [m]');
```

The expression we have used for $\mathbf{w}(\mathbf{r})$ is only value when $y > 0$, and the feather hits the ground when $y = 0\,\mathrm{m}$. We therefore only plot the trajectory when $y > 0$. This is ensured using the `find` command.

The resulting path of the feather is illustrated in Fig. 6.13. Notice that the feather almost immediately follows the wind—the drag forces rapidly reduces differences between the velocity of the feather and the velocity of the surrounding air.

We can compare these results with the simplified, analytical solution we found above. The simplified solution is illustrated by a dotted line, which clearly was not a good solution, because the vertical acceleration is greatly reduced when the feather moves fast in the horizontal direction.

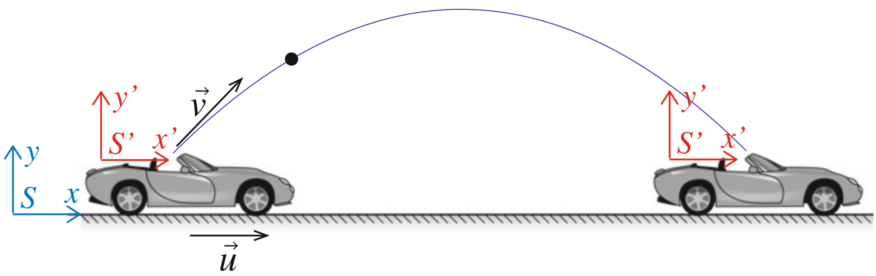


Fig. 6.14 An illustration of a person throwing a ball straight up from a convertible driving at a constant velocity as seen from a person on the sidewalk. The coordinate system S is fixed to the ground, and the coordinate system S' follows the car

6.4 Frames of Reference

Consider a person driving in an open convertible at constant velocity. She throws a ball straight up into the air. According to her, the ball moves straight up along the y-axis and falls straight down along the y-axis into her arms. A simple feat. However, a spectator standing by the road tells a different story. He observed the person in the car throwing the ball with an initial horizontal as well as vertical velocity component in a curved, parabolic path, and then she caught the ball exactly as it fell back to its original height. A more impressive feat as seen from the sidewalk. The two stories are illustrated in Fig. 6.14. In this section we address how we can relate these two observations.

The motion of the ball can be described in a reference system S, on the sidewalk, and in a reference system S' which moves with the car. The position of the ball as measured in system S at a time t (measured in S) is $\mathbf{r}(t)$. Similar, the position of the ball as measured in the system S' at a time t' (measured in S') is $\mathbf{r}'(t')$. The position of the system S' measured in system S is $\mathbf{R}$. We can therefore relate the positions in the two systems by:

$$\mathbf{r} = \mathbf{R} + \mathbf{r}' \ . \tag{6.102}$$

Let us assume that the time is the same in all reference systems. That is, we assume there is a universal time so that the time measured in system S is the same as the time measured in system S'. As we will learn later, this is only approximately true as long as the reference systems are not moving very fast relative to each other.

What is the velocity of the object measured in system S?

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(\mathbf{R} + \mathbf{r}'\right) = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'}{dt} \ . \tag{6.103}$$

System S' is moving with a velocity $\mathbf{u} = d\mathbf{R}/dt$ relative to system S—this is the velocity of the car (system S') in Fig. 6.14 relative to the sidewalk (system S). Since we have assumed that the time is universal, we have that:

$$\frac{d\mathbf{r}'(t')}{dt'} = \mathbf{v}' = \frac{\mathbf{r}'(t)}{dt} , \tag{6.104}$$

which is the vector velocity of the object measured in system S'. This corresponds to the vector velocity of the ball measured from the car in Fig. 6.14.

The relation between the velocities is therefore

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' . \tag{6.105}$$

If system S' moves at a constant velocity relative to system S, $d\mathbf{u}/dt = 0$, and the accelerations of the object are the same inboth systems:

$$\mathbf{a} = \mathbf{a}' . \tag{6.106}$$

This transformation is called the Gallileo-transformation. The acceleration of an object is the same in all systems moving at constant velocities relative to each other. This means that the acceleration is an invariant in the Gallileo-transformations. We call all systems moving at constant velocities **inertial systems**. Because Newton's laws determine the accelerations, Newton's laws are only valid in inertial systems.

If we now return to the case of the car, we can explain the behavior. The acceleration of the ball is the same in a system on the ground and in all systems moving at constant velocity relative to the ground. The car is such a system, moving with a constant velocity $\mathbf{u}$ relative to the ground. The acceleration of the ball is therefore the same in the sidewalk-system S and in the car's system S'. However, the initial velocities are different in the two systems. In system S' the ball is thrown straight up, which means that the initial velocity only has a component along the y'-axis of the S' system: $\mathbf{v}'_0 = v_0\,\mathbf{j}$. In system S, the initial velocity has both a vertical and horizontal component, because the relation between the two velocities are given by (6.105):

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' = u\,\mathbf{i} + v_0\,\mathbf{j} , \tag{6.107}$$

where u is the speed of the car in the horizontal direction. Seen from the car, it is obvious that the ball falls down into the car, since it was thrown with no horizontal velocity component relative to the car.

6.4.1 Example: Motion of a Boat on a Flowing River

In this example you will learn how to use transformations between frames of reference to address the motion of a boat moving relative to a moving background—a river.

You are driving a boat on a river that flows with a velocity of 1.0 m/s southwards. We will now discuss the motion of the boat in several scenarios: driving with a

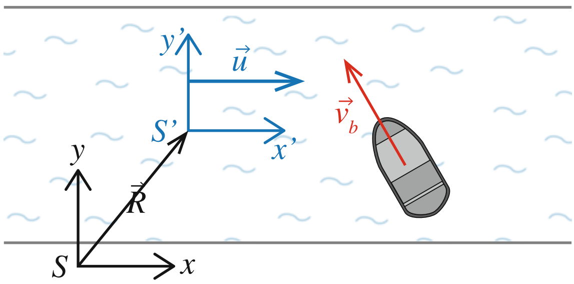


Fig. 6.15 Sketch of a boat driving in a flowing river. The velocity of the river is $\mathbf{u}$ and the velocity of the boat is $\mathbf{v}_b$

constant speed relative to the water and driving with a speed given by the boat speedometer, recorded at intervals of $\Delta t = 0.1$ s in the file boatvelocity.d.[7]

Sketch: We start by sketching the system: We sketch the boat and the surrounding river in Fig. 6.15. It is important to draw all the coordinate systems we will be using: We use a coordinate system S with axes x and y, which is on the riverbank, and a system S', with axes x' and y' which is following the motion of the river. The river is flowing in the x-direction, and the x and x' directions are parallel.

Position of the boat: The position of the boat in system S, which is stationary on the riverbank, is $\mathbf{r}(t)$. The position of the boat in the system S', which is following the water in the river, is $\mathbf{r}'(t)$ and the two positions are related by:

$$\mathbf{r}(t) = \mathbf{R}(t) + \mathbf{r}'(t) \ , \tag{6.108}$$

where $\mathbf{R}(t)$ is the position of system S' measured relative to system S.

Velocity of the boat: If we know the velocity of the boat relative to the water, $\mathbf{v}'_b$, then we find the velocity of the boat relative to the bank by taking the time derivative of (6.108):

$$\mathbf{v}_b(t) = \frac{d\mathbf{r}_b}{dt} = \frac{d\mathbf{R}}{dt} + \frac{d\mathbf{r}'_b}{dt} = \mathbf{u} + \mathbf{v}'_b \ . \tag{6.109}$$

Driving straight across the river: We can use this result to find what velocity the boat must have in order to drive straight across the river. To drive straight across the river means that the boat should have no velocity along the riverbank, that is, $v_{b,x} = 0$. Since we know that $\mathbf{u}$ is directed along the x-axis, $\mathbf{u} = u\,\mathbf{i}$, the velocity of the boat is:

$$\mathbf{v}_b = u\,\mathbf{i} + \mathbf{v}'_b \ , \tag{6.110}$$

and on component form:

[7] http://folk.uio.no/malthe/mechbook/boatvelocity.d.

$$v_{b,x} = u + v'_{b,x} \ , \ v_{b,y} = v'_{b,y} \ . \tag{6.111}$$

Straight across means that $v_{b,x} = 0$, that is that

$$u + v'_{b,x} = 0 \ \Rightarrow \ v'_{b,x} = -u \ , \tag{6.112}$$

and that $v'_{b,y}$ can have any value you like.

Driving with constant velocity: If the boat is driving with a constant velocity $\mathbf{v}'_b$, what is the position of the boat after a time t? We can find the position in either system S or in system S'—the result will be the same. If we find the position in system S' we need to transform the velocity first. The velocity of the boat in system S is:

$$\mathbf{v}_b = \mathbf{u} + \mathbf{v}'_b \ , \tag{6.113}$$

and the position is therefore:

$$\begin{aligned}\mathbf{r}_b(t) &= \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}_b(t)\,dt = \mathbf{r}(t_0) + \int_{t_0}^{t} \left(\mathbf{u} + \mathbf{v}'_b\right)\,dt \\ &= \mathbf{r}(t_0) + \left(\mathbf{u} + \mathbf{v}'_b\right)(t - t_0) \ .\end{aligned} \tag{6.114}$$

Driving with varying velocity: The velocity of the boat relative to the river, $\mathbf{v}'_b(t)$, is recorded by the speedometer. We can therefore use (6.108) to find the velocity of the boat relative to the ground. We notice that the velocity of the river is:

$$\mathbf{u} = -1.0\,\text{m/s}\,\mathbf{j} \ . \tag{6.115}$$

We integrate $\mathbf{v}(t_i)$ numerically to find the position of the boat relative to land using Euler's method:

$$\mathbf{r}_b(t_i + \Delta t) = \mathbf{r}_b(t_i) + \Delta t\,\mathbf{v}_b(t_i) \ , \tag{6.116}$$

where

$$\mathbf{v}_b(t_i) = \mathbf{u} + \mathbf{v}'_b(t_i) \ , \tag{6.117}$$

where we find $\mathbf{v}'_b(t_i)$ in the data set in boatvelocity.d.[8] This is implemented in the following program:

```
from pylab import *
u = [0.0,-5.0]
t,x,y=loadtxt('boatvelocity.d',usecols=[0,1,2],unpack=True)
n = length(t)
dt = t[1] - t[0]
v = zeros((n,2),float)
v[:,0] = x
```

[8] http://folk.uio.no/malthe/mechbook/boatvelocity.d.

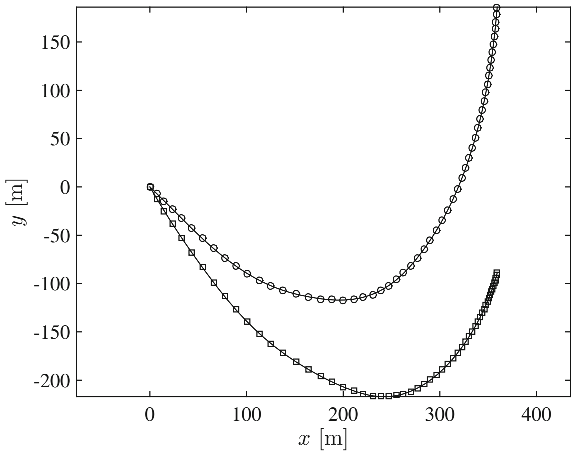


Fig. 6.16 The trajectory of the boat on the river (*blue*) compared with the trajector of the boat on a stationary lake (*red*). The *circles* illustrates the motion diagram of the boat, with a time interval of $\Delta t = 1$ s between each point

```
v[:,1] = y
r = zeros((n,2),float)
r[0] = array([0.0,0.0])
for i in range(1,n):
  r[i] = r[i-1] + (v[i-1]+u)*dt
plot(r[:,0],r[:,1])
xlabel('x [m]'), ylabel('y [m]')
axis('equal'), show()
```

The trajectory of the resulting motion is compared with the trajectory of the motion of a boat on a stationary lake in Fig. 6.16.

Summary

Motion:

- The motion of an object is described by the position vector, $\mathbf{r}(t)$, as a function of time, measured in a specified coordinate system.
- The velocity vector of the object is defined as

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \ .$$

- The acceleration vector of the object is defined as

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \ .$$

Problem-solving approach:

- We solve problems in kinematics using a structured approach.
- In the "Solver" we solve the equation:

$$\frac{d^2\mathbf{r}}{dt^2} = a\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right) \ .$$

  with the initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.
- *Numerically*: We solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$\mathbf{v}(t_i + \Delta t) = \mathbf{v}(t_i) + \Delta t\, \mathbf{a}(\mathbf{r}(t_i), \mathbf{v}(t_i), t_i)$$
$$\mathbf{r}(t_i + \Delta t) = \mathbf{r}(t_i) + \Delta t\, \mathbf{v}(t_i + \Delta t) \ .$$

- *Analytically*: When the acceleration, $\mathbf{a} = \mathbf{a}(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t)\, dt \ ,$$
$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t)\, dt \ .$$

  A typical example is motion with constant acceleration.
- The derivative of a vector function such as $\mathbf{r}(t)$ is found by componentwise derivation. If

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \ ,$$

  the derivative is:

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k} \ .$$

- The integral of a vector function is found by componentwise integration. For

$$\mathbf{v}(t) = v_x(t)\,\mathbf{i} + v_y(t)\,\mathbf{j} + v_z(t)\,\mathbf{k} \ ,$$

  the integral is

$$\int_{t_0}^{t} \mathbf{v}(t) dt = \left(\int_{t_0}^{t} v_x(t) dt\right)\mathbf{i} + \left(\int_{t_0}^{t} v_y(t) dt\right)\mathbf{j} + \left(\int_{t_0}^{t} v_z(t) dt\right)\mathbf{k} \ .$$

Exercises

Discussion Questions

6.1 Pendulum motion. A pendulum oscillates back and forth following the path of a vertical circle. What is the direction of the acceleration of the pendulum when the pendulum is at its upper position? At its lower? Explain.

6.2 Packet from a plane. A plan flies at constant velocity and altitude, and drops a packet. Describe trajectory of the packet as seen from the ground and from the plane.

6.3 U-turn. A train drives along a U-shaped turn at constant speed. When does it have maximum acceleration?

6.4 Magic hat. A magic hat is placed on a trolley that moves at a constant velocity along a straight track. How would you shoot a doll-rabbit from the trolley so that you can be sure that it falls into the hat?

6.5 Constant velocity. Can a motion occur at constant speed (constant magnitude of the velocity vector) and non-zero acceleration? Can the acceleration vector also be constant?

6.6 Remote robot. You are remote-controlling a Mars rover from a simplified control panel with four button that makes the rover move a given length North, South, East or West. You are given a complicated route consisting of a sequence of button to be pressed but make a mistake in the order. Does this matter?

6.7 No motion. If the average velocity for a motion is zero, is the average displacement also zero? If the average acceleration for a motion is zero, is the average displacement zero?

Problems

6.8 Curving swallow. A swallow is making a quick turn to catch a fly. Its motion in the horizontal plane is captured by a camera attached to a balloon. The horizontal positions were recorded at 0.1 s intervals:

t (s)	0.0	0.1	0.2	0.3	0.4	0.5
x (m)	10.00	11.00	11.75	12.25	13.00	14.00
y (m)	15.0	15.00	15.50	16.50	17.00	17.00

(**a**) Draw the motion diagram and the displacements for this motion.
(**b**) Use the motion diagram to find the average velocity vectors.

(**c**) Use the motion diagram to find the average acceleration vectors.
(**d**) When is the speed and the acceleration maximum?

6.9 Penalty shot. As a research assistant for the national soccer team, you have mounted a videocamera to record the path of the soccer ball during a penalty shot. You use a video-analysis software to extract the position of the football in the horizontal plane for each picture frame, taken at $\Delta t = 0.01$ s intervals. You find an example of a penalty shot in00000 penalty.d.[9] Each line in the file consists of a time, t_i, followed by x_i and y_i, the x- and y-positions of the ball respectively.
(**a**) Draw a motion diagram of the motion.
(**b**) Plot the x- and y-positions as function of time.
(**c**) Does the ball hit the goal, located at $x = 0$ m between $y = 25.0$ m and $y = 36.0$ m?
(**d**) Plot the components of the average velocities as functions of time. When is the x- and y-components of the velocity largest?
(**e**) Estimate the initial speed of the ball. And the speed of the ball when it reaches $x = 0$ m.
(**f**) Plot the components of the average accelerations as functions of time.
(**g**) Draw some of the accelerations in the same figure as the motion diagram. What do you think causes the acceleration?

6.10 Vertical loop. A small airplane is making a vertical loop. Sketch a motion diagram describing the motion of the airplane.

6.11 Unknown motion. A motion is described by the data-set in the file discrete-motion06.d.[10] Each line in the file consists of a time, t_i, followed by x_i, y_i and z_i, the x-, y-, and z-positions of the object respectively.
(**a**) Read in the data, and plot a motion diagram of the motion.
(**b**) What physical process do you think this motion describes?

6.12 Alpha particle. An alpha particle is ejected from an atom. The alpha particle moves with a constant velocity $\mathbf{v} = 1000\,\text{m/s}\,\mathbf{i} + 2000\,\text{m/s}\,\mathbf{j}$. After 1 s it hits another atom. We use the atom it is ejected from as the origin.
(**a**) What is the speed of the alpha particle?
(**b**) Find the position of the alpha particle as a function of time.
(**c**) How far have the alpha particle travelled in 1 s?

6.13 Airplane collision. An F-16 jet fighter is leaving from Rygge airfield, which we use as the origin of our coordinate system, at $t = 0.0$ s, and travels with a constant velocity $\mathbf{v}_1 = 1700.0\,\text{km/h}\,\mathbf{j}$ towards the North. At the same time, an Airbus A310 airplane is passing over Oslo, which is located at $\mathbf{r}_1 = -10\,\text{km}\,\mathbf{i} + 80\,\text{km}\,\mathbf{j}$. The Airbus travels with a constant velocity of $\mathbf{v}_2 = 105\,\text{km/h}\,\mathbf{i} + 905\,\text{km/h}\,\mathbf{j}$. They are both travelling at the same height.

[9] http://folk.uio.no/malthe/mechbook/penalty.d.
[10] http://folk.uio.no/malthe/mechbook/discretemotion06.d.

(**a**) Find the position of the jet fighter as a function of time.
(**b**) Find the position of the Airbus as a function of time.
(**c**) Sketch the trajectories of both planes in the same diagram. (You can do this on your computer if you like.)
(**d**) Will the airplanes collide?
(**e**) If the airplanes are within a distance of 1 km of each other, an alarm will sound in the plane, and an evasive maneuver will be attempted. Will the planes pass that close to each other?

6.14 Motion of spaceship. A spaceship is floating in free space with an initial velocity $\mathbf{v}_0 = 1000\,\mathrm{m/s}\,\mathbf{i}$. Suddenly, the spaceship turns its thrusters on, giving the spaceship a constant acceleration $\mathbf{a}_0 = 10\,\mathrm{m/s^2}\,\mathbf{j}$ for 10 s. (The acceleration does not change during this time).
(**a**) Sketch the path of the spaceship without doing a detailed calculation.
(**b**) Find the velocity of the spaceship as a function of time.
(**c**) Find the position of the spaceship as a function of time.
(**d**) Plot the path of the spaceship and compare with your initial sketch.

6.15 Controlling the electron beam. An electron is shot through a varying electrical field. Initially, the electron is moving in the x-direction with a velocity $v_x = 100\,\mathrm{m/s}$. The electron enters the field when it passes the origin. The field varies with time, causing an acceleration of the electron that varies in time:

$$\mathbf{a}(t) = \left(-20\,\mathrm{m/s^2} - 10\,\mathrm{m/s^3}\,t\right)\mathbf{j}\,. \tag{6.118}$$

(**a**) Find the velocity as a function of time for the electron.
(**b**) Find the position as a function of time for the electron.
The field is only acting inside a box of length $L = 2\,\mathrm{m}$.
(**c**) How long time is the electron inside the field?
(**d**) What is the displacement in the y-direction when the electron leaves the box. (We call this the deflection of the electron).
(**e**) Find the angle the velocity vector forms with the horizontal as the electron leaves the box.

6.16 Accelerometer reading. Your high precision pedometer contains a very precise accelerometer that measures the acceleration of your body as you are running. The reading from the accelerometer is recorded in the file pedometer.d,[11] where each line contains the time, t_i, measured in seconds, followed by the acceleration in the x and y direction respectively, measured in $\mathrm{m/s^2}$.
(**a**) Read the data from the file. Find the velocity vector as a function of time.
(**b**) Find the position vector as a function of time. Plot the results.
(**c**) Given an interpretation of the motion in its two distinct phases.

[11] http://folk.uio.no/malthe/mechbook/pedometer.d.

6.17 Running inside a bus. A bus is driving with constant velocity $v_x = 50$ km/h in the x-direction.
(**a**) If you are running towards the back of the bus at a speed of 10 km/h. How fast are you running relative to the ground?
(**b**) If you are running towards the front of the bus at a speed of 10 km/h. How fast are you running relative to the ground?

6.18 Jumping onto a running train. In your early career as a stuntwoman, you task was to jump from a bridge onto a running train. The train was running at 36 km/h.
(**a**) What was your velocity relative to the train when you landed on the train?

You were rolling and stumbling for 2 s before coming to rest at the train. You can assume that you were experiencing a constant acceleration in this time.
(**b**) Find the acceleration.
(**c**) Find your velocity as a function of time relative to the train from you landed and for the first 10 s.
(**d**) Find your velocity as a function of time relative to the ground from you landed and for the first 10 s.

6.19 A plane in crosswinds. You are trying to steer an airplane towards the north. The airspeed of your plane is 300 km/h. However, there is a strong wind from the west, with a wind speed of 60 km/h.
(**a**) In what direction should you direct the plane so that it travels towards the north? Illustrate your argument with a diagram.
(**b**) What is the speed of the plane relative to the ground?

Projects

6.20 Motion capture. In this project we will study the motion of an object that is fitted with an accelerometer, and it is your task to figure out what physical phenomenon we are observing. You will need to find the motion from the acceleration of the object, and then interpret the motion in physical terms.

First, we will guide your intuition by starting with an introductory, analytical exercise.

A car rolling down a hill with an inclination θ with the horizon experiences an acceleration $a = g\sin(\theta)$ along the surface of the hill. Here $g = 9.81\,\text{m/s}^2$ is the acceleration of gravity. The car is released from rest at the time $t = t_0 = 0$ s.
(**a**) Sketch a motion diagram of the motion of the car.
(**b**) Find the position s and the velocity v of the car along the hill after a time t.

We will now introduce a reference system S oriented with the x-axis in the horizontal direction and the y-axis in the vertical direction—that is in the direction gravity is acting. Let us assume that the car starts in the position $x = 0$ m, $y = h$, where h is the height of the car, and that the car moves in the positive x-direction.
(**c**) Sketch the system and the coordinate system.

(**d**) Find the position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ of the car after a time t.

We will now address the motion captured by the accelerometer. The data-set motion1.d[12] contains time (in s) and acceleration (in m/s^2) of an object, given as a sequence of points t_i, $a_{x,i}$ and $a_{y,i}$ taken at regular time intervals Δt.

```
t0 ax0 ay0
t1 ax1 ay1
```

(**e**) Find the velocity $\mathbf{v}$ and position $\mathbf{r}$ of the motion using numerical methods. (For example using a simple Euler scheme for integration). Plot the path of the object. (Your answer should include a listing of the program used.)
(**f**) Can you give a physical interpretation of the motion, that is, can you describe a physical system that you would expect to behave in this manner?
(**g**) Where is the magnitude of the acceleration the maximum?

Another, similar, experiment was performed, giving the motion data in motion2.d.[13]
(**h**) Find the position $\mathbf{r}_2$ of the motion, and plot it in the same plot as the motion in `motion1.d`.
(**i**) Can you give a physical interpretation of the motion?
(**j**) Where is the magnitude of the acceleration the maximum? How do you interpret this?

[12] http://folk.uio.no/malthe/mechbook/motion1.d.

[13] http://folk.uio.no/malthe/mechbook/motion2.d.

Chapter 7
Forces in Two and Three Dimensions

We have now introduced a vectorized description of motion that allows us to discuss motion not only in one dimension, but also for two- and three-dimensional systems. However, in order to predict and calculate the motion, we need to extend Newton's laws to two and three dimensions, and we need to introduce force models that are applicable in two and three dimensions. This is the focus of the current chapter.

The structured problem-solving approach used to address problems in mechanics has exactly the same form for one-, two-, and three-dimensional problems (See Fig. 7.1 for an illustration). The first step is to *identify* what objects we are studying, how we characterize their position, and what reference system we use to describe the motion. Second, we *model* the system by finding the forces acting on the object, we introduce models for the force, and use Newton's second law to find the acceleration of the object. Third, we *solve* the equations of motion, and determine the position and velocity of the objects as functions of time. Finally, we *analyze* the resulting motion, use the solution to answer the questions posed, and check the validity of the solutions.

In this chapter, we discuss how to identify forces, how to apply Newton's laws in two- and three-dimensions, and we generalize all force models to two- and three-dimensional motion.

7.1 Identifying Forces

In Chap. 5 we introduced a general method to identify and name the forces acting on an object, by drawing the free-body diagram. This method is the same in one-, two-, or three-dimensional systems. The only difference is that for the one-dimensional case we have so far not included all forces in order to ensure the problem was indeed one-dimensional. Now, we loosen that constraint and include all forces in the free-body diagram.

Let us illustrate the main principles of the free-body diagram for a fully three-dimensional problem by developing the free-body diagram of a car driving up a hill

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Identify	Model	Solve	Analyse
What object is moving? How is the position, $x(t)$, measured? (Origin and axes of coordinate system). Find initial conditions, $x(t_0)$ and $v(t_0)$.	Find the forces acting on the object. Introduce models for the forces. Apply Newton's second law of motion to find the acceleration, $a = a(x, v, t)$.	Solve the equation: $\frac{d^2x}{dt^2} = a(x, v, t)$, with the initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$ using analytical or numerical techniques. The solution gives the position and velocity as a function of time, $x(t)$, and $v(t)$.	Check validity of $x(t)$ and $v(t)$. Use $x(t)$ and $v(t)$ the answer questions posed. Evaluate the answers.

Fig. 7.1 The structured problem solving approach

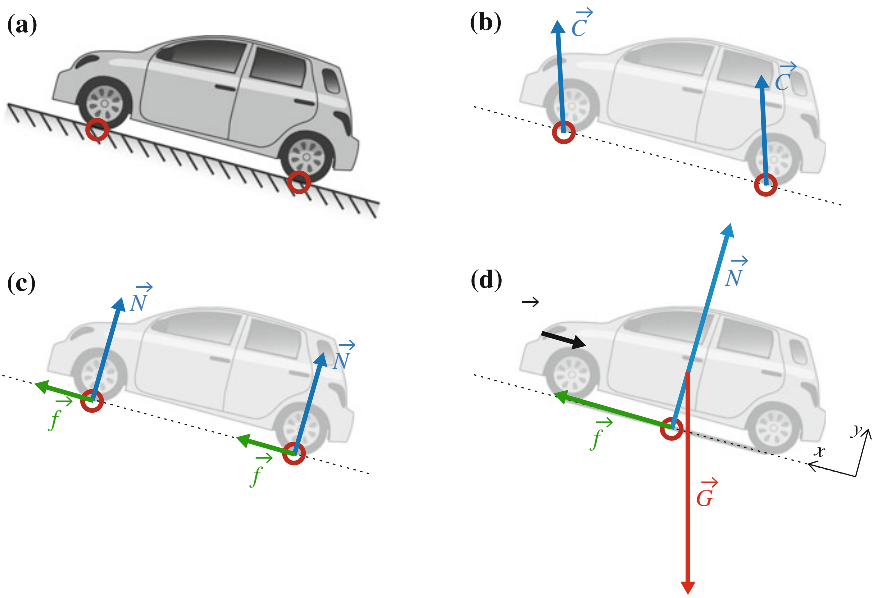

Fig. 7.2 Illustration of a car driving up an inclined slope

as illustrated in Fig. 7.2a. We follow the enumerated steps from the general method, while commenting on how it is applied.

1. Divide the problem into *system* and *environment*.

First, we define precisely what object we are studying the motion of—we discern between the *system* and the *environment*. We find the forces acting *on* the system. Everything that is not the *system* is the environment.

In this case the system is the car, and the environment is everything else, such as: the ground, the Earth, and the air around the car.

The external forces may occur either at the contact between the car and the environment. We call such forces *contact forces*. Or they may be *long-range forces*. Typically, we find the contact forces first and leave the long-range forces to last. We find the contact forces by addressing the physical processes occurring along the external surface of the object:

2. Draw a figure of the object and everything in contact with the object.
3. Draw a closed curve around the system.
4. Find contact points—these are the points where contact forces may act.
5. Give names and symbols to all the contact forces.

The car is in contact with the ground at the points where the four wheels are touching the ground. We have already discussed the possible ambiguity of a contact point. In many cases the contact is not in a point, but distributed over an area. However, on a large scale, we can usually still represent the force as acting in a point.

What forces are acting in these points? Your answer to this question will depend on your insight into and knowledge about force models—what kinds of macroscopic forces are acting between objects.

Contact forces: Previously, we argued that we should separate the identification of forces from the modeling of forces. Unfortunately, this is not really possible. In practice, we cannot separate the two steps. As we identify forces acting on a body, we are actually also identifying the interactions mechanisms, and we are already making assumptions on how to model the interactions.

To make this rather abstract point more concrete, let us look at the example of the car in detail. What forces are acting on the contact points between the car and the ground?

As a first approximation, we could introduce a single force vector, the contact force $\mathbf{C}$, acting in the point of contact and having a component both normal to the ground and along the ground as illustrated in Fig. 7.2b. While this is a correct description—there is a force acting on the car from the ground, and it may have components both normal and parallel to the ground—it is not a very useful way to identify the forces. Why? Because we will not have a force model for this general contact force. We could therefore introduce this general contact force here, but we would need to decompose it into different physical models when we addressed how to model this force.

Decomposing the contact force: Instead, what we mean when we say that we identify the forces acting on the car in the contact point, is that we identify the different force models, and that we introduce an individual force for each of the models. For the contact between the ground and the car we could identify two different force models: We recognize one force as due to the deformation of the wheel and the ground. This is the normal force, $\mathbf{N}$ from the ground on the car as illustrated in Fig. 7.2c. We recognize another force as due to the sticking or sliding of the wheel relative to the ground. This is the friction force from the ground on the car, $\mathbf{f}$. Notice that when we identify forces, we are really identifying mechanisms or processes that result in a force, which is the first step in identifying a model to describe the force.

One or many contact forces: The contact forces from the ground on the car are therefore the normal force, **N**, and the friction force, **f**. One force acts on each of the four contact points of the car. We may represent each of these forces individually, or we could instead describe the whole car as one block with only a single contact point with the ground, and then represent the four contact forces only by a single force. In this case we should redraw our figure so that there is only one contact point, as shown in Fig. 7.2d, and only one normal force, **N**, and one friction force, **f**. Later on, when we study the equilibrium of an object, we will see that we need to use more than one contact point to determine if the car is rotating or not.

Choosing the direction of the vectors: We draw the normal force as a vector **N** pointing upwards, since we expect this to be the direction of the force. If the y-axis is the direction normal to the hill, as illustrated in Fig. 7.2d, the normal force is $\mathbf{N} = N\mathbf{j}$. This simply means that if the normal force actually points in this direction, then N is positive. Does it matter if we draw the vector in the "wrong direction"? What would happen if we instead drew the normal force pointing downwards? In this case, we would have $\mathbf{N} = N(-\mathbf{j})$, and a positive value of N would mean that the normal force was acting downwards. We are therefore free to draw the vector in whatever direction we want, the resulting numerical values for the components of the vector will tell us in what direction the force is acting at a particular time. However, you have to be consistent: When you have drawn the vector in a particular direction, you need to stick to your choice.

Similarly, we draw a vector **f** pointing up along the hill to represent the friction force. With the x-axis pointing up along the hill, this means that: $\mathbf{f} = f\mathbf{i}$. The friction force may still act in the opposite direction for $f < 0$.

What other contact forces are acting on the car? The car is in contact with the surrounding air. We should therefore also add a force due to air resistance on the car, **D**, as illustrated in Fig. 7.2d.

6. Identify the long-range forces.

Finally, we find and draw the long-range forces acting on the car. The only long-range force is gravity from the Earth, which acts on the car and point down towards the center of the Earth. Gravity is drawn as the force **G** in Fig. 7.2d.

Now, the next two points in the general method are:

7. Make a drawing of the *object*. Draw the forces as arrows, vectors, starting where the force is acting. The direction of the vector indicates the (positive) direction of the force. Try to make the length of the arrow indicate the relative magnitude of the forces.
8. Draw in the axes of the coordinate system. It is often convenient to make one axis parallel to the direction of motion. When you choose direction of the axis you also choose the positive direction for the axis.

When you are more experienced, you will probably follow the expert's method to find the free-body diagram, but you should not progress to this stage before you have practiced the first method on several examples and found that it is too simple and elaborate for your taste.

Expert's method for drawing a free-body diagram: Follow these steps to find, identify, and draw all the forces acting on an object in a free-body diagram.

1. Identify the *system*, and make a drawing of the system. You may sketch the environment in different colors or with a dotted line to help you.
2. Contact forces are acting at the contact points between the system and the environment. Identify, name, and draw each contact force as a vector starting at the point of contact.
3. Choose a coordinate system and draw the axes of the coordinate system in the same figure as the system. It is often convenient to make one axis parallel to the direction of motion.

7.2 Newton's Second Law

We have already introduced Newton's laws of motion on a vector form, therefore, we do not need a new formulation for two- and three-dimensional problems.

Newton's second law relates the acceleration of an object to the net force acting on the object:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a} \; , \tag{7.1}$$

where the sum is over all the *external* forces acting on the system.

The external forces are exactly the forces you have included in the free-body diagram of the object. Often we refer to the sum of all the external force as the *net external force*:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j^{\text{ext}} \; . \tag{7.2}$$

Inertial system: Newton's second law is only valid in an *inertial system*. If the references system is accelerated—either linearly or rotating—we cannot use Newton's law.

Net external force: Newton's second law is related to the *net external force*. External forces are forces that have a cause outside the system, as we insisted on when you

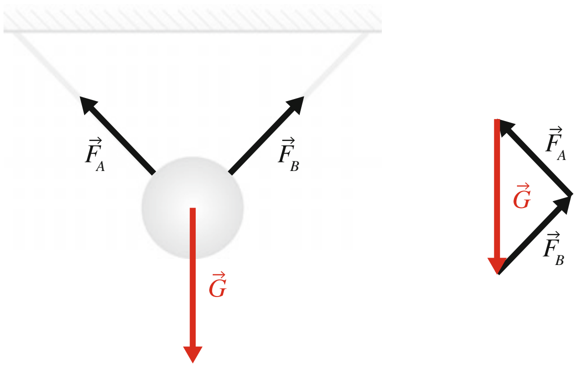

Fig. 7.3 A *sphere* is hanging from two ropes that are attached to the roof. In this case, the net force is zero even when none of the forces point in the same direction, as shown by the graphical vector summation to the right

drew the free-body diagram of the system. You can therefore not use the law for a single force alone—it is the net force that is causing the acceleration of the object, not one individual force acting on the system. For example, for the car driving up an inclined slope discussed above, the net force on the car is:

$$\mathbf{F}_{\text{net}} = \sum_j \mathbf{F}_j = \mathbf{N} + \mathbf{f} + \mathbf{D} + \mathbf{G} \,. \tag{7.3}$$

Net force is a vector sum: The net force is a *vector sum* of all the external forces. Notice that the net force can be zero even if none of the force vectors point in the same direction, as illustrated in Fig. 7.3.

Vector equation: Newton's second law is a *vector equation*. This means that it is valid for each of the vector components independently:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a} \,, \tag{7.4}$$

implies that:

$$\sum_j F_{j,x}^{\text{ext}} = ma_x \,, \quad \sum_j F_{j,y}^{\text{ext}} = ma_y \,, \quad \sum_j F_{j,z}^{\text{ext}} = ma_z \,, \tag{7.5}$$

where $F_{j,x} = \mathbf{F}_j \cdot \mathbf{i}$ and $a_x = \mathbf{a} \cdot \mathbf{i}$, and similarly for the other two components.

We can decompose Newton's law along any set of axes we like. We will see that it is often useful to choose the axes wisely, for example by ensuring that the net force along one of the axis direction is zero, so that there is no change in motion in this direction.

Notice that this means that the object can be accelerated in the x-direction, if the net force in this direction is non-zero, while it moves with constant velocity in the

y-direction, if the net force in this direction is zero. The behavior along orthogonal axes can therefore be completely decoupled.

Superposition principle: Forces are subject to the *superposition principle*. We can add together or decompose forces as we like. This allows us to subdivide a force, such as the surface interaction force, $\mathbf{f}$, for the car driving up the hill, into several forces, each representing a specific surface interaction term:

$$\mathbf{f} = \mathbf{f}_{\text{friction}} + \mathbf{f}_{\text{adhesion}} + \mathbf{f}_{\text{lubrication}} + \dots \ . \tag{7.6}$$

7.3 Force Model—Constant Gravity

According to Newton's law of gravity, there is a gravitational force between any two objects with gravitational masses. For an object close to the Earth's surface, the gravitational force on the object can be approximated as:

$$\mathbf{G} = -mg\,\mathbf{j} \ , \tag{7.7}$$

where m is the gravitational mass of the object, g is the acceleration of gravity, and the unit vector $\mathbf{j}$ points upwards. Upwards is indeed usually defined based on the direction of the force from gravitation. This **constant gravity** force model is valid as long as the object does not move far away from the surface of the Earth, and as long as the object does not move too far along the surface of the Earth, since this would lead to a change in the unit vector $\mathbf{j}$.

We notice that this force model is particularly simple: The force on an object due to gravity is a *constant*— both in magnitude and direction. Our discussion of the constant gravity force model may be extended to any constant force, such as the force on a charged particle in a homogeneous electric field.

If you throw a ball from the ground, the only forces acting on the ball after it has left your hand are the force from gravity, $\mathbf{G}$, and air resistance, $\mathbf{D}$. It we neglect the effect of air resistance, the only force acting on the ball is gravity. We can therefore apply Newton's second law to find the acceleration of the ball:

$$\sum_j \mathbf{F}_j = \mathbf{G} = mg\,\mathbf{j} = m\mathbf{a} \ \Rightarrow \ \mathbf{a} = -g\,\mathbf{j} \ . \tag{7.8}$$

Since we have wisely chosen the y-axis to correspond to the direction of gravity, the acceleration of the ball is non-zero only in y-direction. The acceleration in the x- and z-directions are zero, and the velocities in these directions do therefore not change. We call such a motion *decoupled* because the motion in the x- and y-axes are independent of each other. We will use this when we solve problems with constant forces.

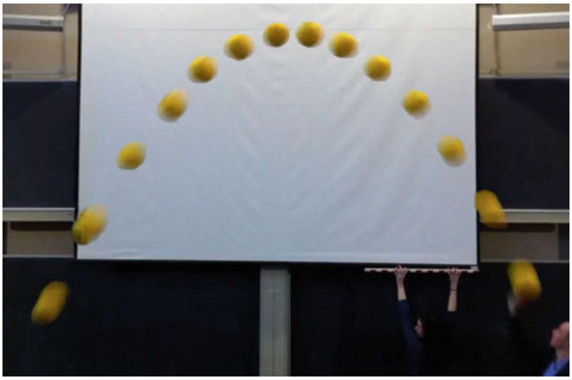

Fig. 7.4 The motion of a ball thrown across the lecture room

7.3.1 Example: Motion of a Ball with Gravity

Throughout this chapter we will follow a particular problem as we gradually increase the complexity of the physical model: The motion of a ball thrown across the classroom. The experiment is illustrated in Fig. 7.4, which illustrates the observed path of a ball in an experiment. How can we develop a realistic model for the motion of this ball? We start from the simplest description: The motion when affected by gravity alone.

Problem: A ball is thrown from a height h above the ground at an angle α with the horizontal with an initial speed v_0. Find the velocity and position of the ball at a time t.

Identify and Sketch: In this exercise we address the motion of the ball, described by the position $\mathbf{r}(t)$ as a function of time. At the time t_0, the ball was thrown. We place the coordinate system so that gravity is acting in the y-direction, and we place the x-axis so that the ball is thrown in the positive x-direction. The origin is placed at the ground, directly below the initial position of the ball at $t = t_0$. The initial position vector is therefore $\mathbf{r}(t_0) = \mathbf{r}_0 = h\,\mathbf{j}$. The initial velocity is directed at an angle α with the horizontal, this means that the initial velocity is $\mathbf{v}(t_0) = v_0 \cos(\alpha)\,\mathbf{i} + v_0 \sin(\alpha)\,\mathbf{j}$. The situation is illustrated in Fig. 7.5.

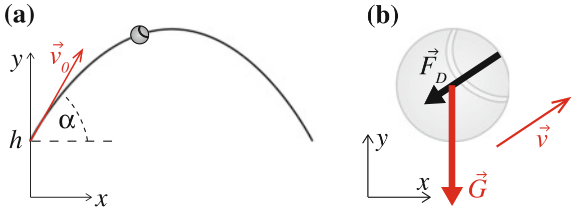

Fig. 7.5 **a** Illustration of the motion of the ball. **b** Free-body diagram of the ball

Model: The motion of the ball is determined by the forces acting on it. The only contact force acting on the ball is air resistance, $\mathbf{F}_D$, but we will here assume that this force is negligible. The only long-distance force acting on the ball is gravity, $\mathbf{G}$, as illustrated in the free-body diagram in Fig. 7.5.

Newton's second law is applied to both the x- and the y-component of the forces independently. In the x-direction Newton's second law gives:

$$\sum F_x = ma_x = 0 \ . \tag{7.9}$$

There are no horizontal forces. The sum of the forces in the horizontal, x-direction is therefore zero. Consequently, the acceleration in the x-direction, a_x is also zero.

Newton's law of motion in the y-direction gives:

$$\sum F_y = G = -mg = ma_y \ , \tag{7.10}$$

where we have used that the gravitational force from the Earth is mg, and that it acts in the negative y-direction.

The acceleration of the ball is therefore:

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j} \ , \tag{7.11}$$

and the initial conditions are $\mathbf{r}(t_0) = h\,\mathbf{j}$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.

Solve: We find the motion of the ball by solving the differential equation in (7.11). Since the acceleration is constant, we can solve it by direct integration for each of the components.

In the x-direction, the acceleration is zero, and the velocity in this direction is therefore constant.

$$v_x(t) = v_x(t_0) = v_0 \cos(\alpha) \ . \tag{7.12}$$

The x-position is given by direct integration:

$$v_x(t) = \frac{dx}{dt} \tag{7.13}$$

$$\int_{t_0}^{t} v_x(t)dt = \int_{t_0}^{t} \frac{dx}{dt}dt \tag{7.14}$$

$$\int_{t_0}^{t} v_0 \cos(\alpha)dt = \int_{x(t_0)}^{x} (t)dx \tag{7.15}$$

$$v_0 \cos(\alpha)(t - t_0) = x(t) - x(t_0) \tag{7.16}$$

that is, we have recovered motion with constant velocity:

$$x(t) = x(t_0) = v_0 \cos(\alpha)(t - t_0) \ , \tag{7.17}$$

In the y-direction, the acceleration is constant, $a_y = -g$. We can find the velocity by direct integration:

$$\int_{t_0}^{t} \frac{dv_y}{dt} dt = \int_{t_0}^{t} -g dt \tag{7.18}$$

$$v_y(t) - v_y(t_0) = -g(t - t_0) \tag{7.19}$$

which gives:

$$v_y(t) = (v_0 \sin(\alpha) - g(t - t_0)) \tag{7.20}$$

We find the position by integrating once more, using that $v_y(t) = dy/dt$, and that $v_{0,y} = v_0 \sin(\alpha)$:

$$\frac{dy}{dt} = \left(v_{0,y} - g(t - t_0)\right) \tag{7.21}$$

$$\int_{t_0}^{t} \frac{dy}{dt} dt = \int_{t_0}^{t} \left(v_{0,y} - g(t - t_0)\right) dt \tag{7.22}$$

$$y(t) - y(t_0) = \int_{t_0}^{t} v_{0,y} dt - g \int_{t_0}^{t} (t - t_0) dt \tag{7.23}$$

$$y(t) - y(t_0) = v_{0,y}(t - t_0) - \frac{1}{2} g (t - t_0)^2 \tag{7.24}$$

which gives

$$y(t) = h + v_0 \sin(\alpha)(t - t_0) - \frac{1}{2} g (t - t_0)^2 \ , \tag{7.25}$$

where $y(t_0) = h$ is the launch height of the projectile.

Analyze: We notice that the motion in the x- and y-directions are independent of each other. The motion in the x-direction is simply a motion with constant velocity. The motion in the y-direction is the same as for the one-dimensional problem. If the ground is flat, it is the motion in the y-direction that determines how long time it takes to reach the ground. We can therefore answer questions about flight time, and maximum height by just studying the one-dimensional motion along the y-direction.

We have now found the complete solution for the motion of a ball subject only to gravity. From this solution, we can answer any complicated question, such as how far the projectile travels or what choise of initial direction gives the maximum length.

7.4 Force Model—Viscous Force

For an object moving through a fluid, such as a projectile flying through the air, a meteor entering the Earth's atmosphere, or a tiny microrobot navigating through your bloodstream, there is a contact force on the object due to the motion of the object relative to the fluid. The fluid has to flow around the object when the object moves, as a result the fluid exerts a force on the object. This force is distributed: It acts everywhere on the surface of the object, and it may also vary in magnitude and direction along the surface of the object. Usually, we will simplify the effect of the this distribution of forces into a single force acting in a single point on the object, and we will call this force the drag force or the "fluid resistance" acting on the object. For most purposes this is a sufficiently precise description of the interaction with the fluid.

The form of the drag force depends on the velocity of the object relative to the fluid. We discern between a behavior at low velocities, where the drag force is proportional to the velocity, and high velocities, where the drag force depends on the square of the velocity:

The **drag force** on an object moving at a velocity $\mathbf{v}$ relative to a fluid moving with a velocity $\mathbf{w}$ is:

$$\mathbf{F}_D \simeq \begin{cases} -k_v\,(\mathbf{v}-\mathbf{w}) & \text{at small velocities} \\ -D\,|\mathbf{v}-\mathbf{w}|\,(\mathbf{v}-\mathbf{w}) & \text{at high velocities} \end{cases} \tag{7.26}$$

The constant k_v depends on the object's size, shape and surface, as well as on the (dynamic) viscosity of the fluid. For a sphere Stokes found that

$$k_v = 6\pi\, R\,\eta \tag{7.27}$$

where R is the radius of the sphere, and η is the viscosity of the fluid. The viscosity of air is $\eta = 1.82 \times 10^{-5}\,\mathrm{Nsm}^{-2}$ and for water it is $\eta = 1.00 \times 10^{-3}\,\mathrm{Nsm}^{-2}$, both at room temperature.

Experimental data indicates an approximative value for D for a spherical object:

$$D \simeq 12.0\,\rho\,R^2 \ . \tag{7.28}$$

where $v = |\mathbf{v}|$, ρ is the density of the fluid, and R is the radius of the sphere.

Versatility of the viscous force model: The viscous force model

$$\mathbf{F}_D = -k_v\mathbf{v}\ , \tag{7.29}$$

is much more versatile than suggested by its application to fluid drag forces. It is often used as a general damping term—a term reducing relative motion that also introduces dissipation and heat generation. For example, you will find that the viscous force model used to model damping of vibrations in solid object, to model the damping of vibrations in macroscopic objects and macroscopic springs, and to model surface forces in nano-scale surface contact. The viscous force model is a first order model to study any velocity-dependent force that tends to reduce velocity differences.

7.4.1 Example: Path Through a Tornado

You are part of a tornado-chaser team—a group of scientists trying to discover the inner workings of tornadoes. An important part of this work is to develop methods to measure the pressure and wind velocity inside the tornado. Your plan is to use many tiny projectiles with small accelerometers inside. You plan to shoot the projectiles through the tornado, pick them up afterwards, and read the recorded accelerations. Here, we will assume that the accelerometers record the acceleration in the x, y, and z-direction during flight. Here, we will develop a model for the flight of the projectile, and calculate realistic trajectories in order to learn how to launch the projectiles.

Sketch and Identify: Our task is to determine the motion of a projectile, characterized by its position $\mathbf{r}(t)$. For the calculations, we use a coordinate system with the origin in the center of the tornado at ground level. The z-axis points in the vertical direction, with the positive direction upwards. We will launch the projectile from the position $\mathbf{r}(t_0) = \mathbf{r}_0$ with an initial velocity, $\mathbf{v}(t_0) = \mathbf{v}_0$ at $t_0 = 0.0$ s.

Model: While the projectile is in the air, the only contact force affecting the object is the force from the surrounding air, $\mathbf{F}_D$, in addition to gravity, $\mathbf{G}$, as illustrated in Fig. 7.6.

Since the projectile will be moving fast, we use the square-law force model for the air resistance. However, in this case it is important to realize that the force depends on the velocity of the projectile, $\mathbf{v}$, relative to the velocity of the wind, $\mathbf{u}(\mathbf{r})$. The square-law force model is therefore:

$$\mathbf{F}_D = -D\,(\mathbf{v} - \mathbf{u})\,|\mathbf{v} - \mathbf{u}| \ , \tag{7.30}$$

where for a spherical object we have that the prefactor is $D \simeq 3.0\rho d^2$, where d is the diameter of the sphere and ρ is the density of the surrounding air. Here, we will assume that the density of the surrounding air does not change significantly, and we

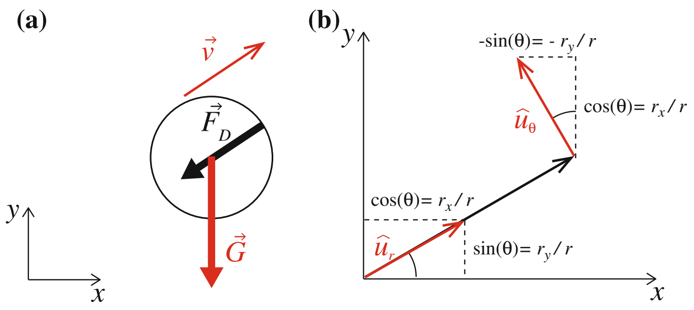

Fig. 7.6 **a** Free-body diagram of the projectile. **b** Illustration of the tangential direction in the tornado

will use $\rho = 1.293\,\mathrm{k/m^3}$. Let us also assume that the projectile has a diameter of $d = 0.02\,\mathrm{m}$, and that its mass is $m = 0.1\,\mathrm{kg}$.

The force from gravity is $\mathbf{G} = -mg\,\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

Newton's second law: Newton's second law gives the acceleration of the projectile:

$$m\mathbf{a} = \sum_j \mathbf{F}_j = \mathbf{F}_D + \mathbf{G}\ . \tag{7.31}$$

The acceleration is therefore:

$$\mathbf{a} = -\frac{D}{m}\left(\mathbf{v} - \mathbf{u}\right)\left|\mathbf{v} - \mathbf{u}\right| - g\,\mathbf{k}\ . \tag{7.32}$$

and the initial conditions are $\mathbf{r}(t_0) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}_0$.

Model of wind velocity: However, in order to test and analyze the path of the projectile, we need a model for the velocity $\mathbf{u}$ in the tornado. Since we do not know this velocity-field, we will here use a model for the velocity taken from an analoguous situation. We can make an experimental tornado by rotating a thin cylinder in a fluid. For this case, we know that the velocity in the fluid will have the form:

$$\mathbf{u}(\mathbf{r}) = u(r)\hat{u}_\theta\ , \tag{7.33}$$

where the center of the cylinder is at the origin, and the speed, $u(r)$, depends only on the distance to the center of the cylinder, and the unit vector $\hat{u}_\theta$ points in the tangential direction, as illustrated in Fig. 7.6. The speed $u(r)$ has the following form:

$$\mathbf{u}(r) = \begin{cases} u_0\,(r^*/r) \text{ for } r > r^* \\ u_0\,(r/r^*) \text{ for } r < r^* \end{cases}\ , \tag{7.34}$$

where r^* is the radius of the cylinder.

We use this as a model for the velocity field inside the tornado to estimate the path of the projectile. Let us assume that we study a tornado with a radius of $r^* = 10\,\mathrm{m}$, and with a maximum wind speed of $u_0 = 50.0\,\mathrm{m/s}$, which corresponds to a category F1 tornado.

Numerical solution: We can now use our theoretical model to find the motion of the projectile. We use a Euler-Cromer method to find the velocity and position vectors as function of time, starting from $t = t_0$, and continuing until the projectile hits the ground.

Euler-Cromer's method consists of the following steps:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t\,\mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{7.35}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t\,\mathbf{v}(t_0 + \Delta t)\ . \tag{7.36}$$

This method is implemented in the following program:

```
from pylab import *
m = 0.2
diam = 0.025
rho = 1.293
D = 3.0*rho*diam**2
Dm = D/m
g = array([0.0,0.0,9.8])
u0 = 50.0
rast = 5.0
r0 = array([-100.0,0.0,0.0])
alpha = 45.0*pi/180.0;
v0 = 100.0*array([cos(alpha),0,sin(alpha)])
time = 10.0
dt = 0.001
n = int(round(time/dt))
r = zeros((n,3),float)
v = zeros((n,3),float)
a = zeros((n,3),float)
t = zeros(n,float)
r[0] = r0
v[0] = v0
i = 1
while (r[i,2]>=0.0) and (i<n):
    rr = norm(r[i])
    if (rr>rast):
        U = u0*(rast/rr)
    else:
        U = u0*rr/rast
    u = U*array([-r[i,1]/rr,r[i,0]/rr,0.0])
    vrel = v[i] - u
    aa = -g - Dm*norm(vrel)*vrel
    a[i] = aa
    v[i+1] = v[i] + dt*aa
    r[i+1] = r[i] + dt*v[i+1]
    t[i+1] = t[i] + dt
    i = i + 1
imax = i
ii = r_[1:imax]
```

Notice how we have used that the tangential vector to a point at (x, y) points in the direction $(-y/r, x/r)$, where $r^2 = x^2 + y^2$.

Analysis: We now have a tool to start addressing the motion of the projectile inside the tornado. Let us use this to test the path of a projectile launched from a distance of 100 m towards the center of the tornado with an initial speed $v = 100$ m/s.

Since the tornado is symmetric, we launch the projectile from the position $\mathbf{r}_0 = -100\,\mathrm{m}\,\mathbf{i}$. We fire the projectile at an angle of 45° with the horizon, which corresponds to an angle of $\pi/4$ in radians, since this gives the maximum length when there is no air-resistance. The initial velocity is therefore $\mathbf{v}_0 = 100\,\mathrm{m/s}\,\cos(\pi/4)\,\mathbf{i} + 100\,\mathrm{m/s}\,\sin(\pi/4)\,\mathbf{j}$. The resulting path is shown in Fig. 7.7.

The trajectory is hardly affected by the tornado. How can we change the trajectory to make it more sensitive to the wind speed? We could shoot it at an angle with the center, and not directly towards the center. This is attempted by introducing the angle $\boldsymbol{\theta}$, which gives the deviation from the line straight into the center. The initial velocity is now.

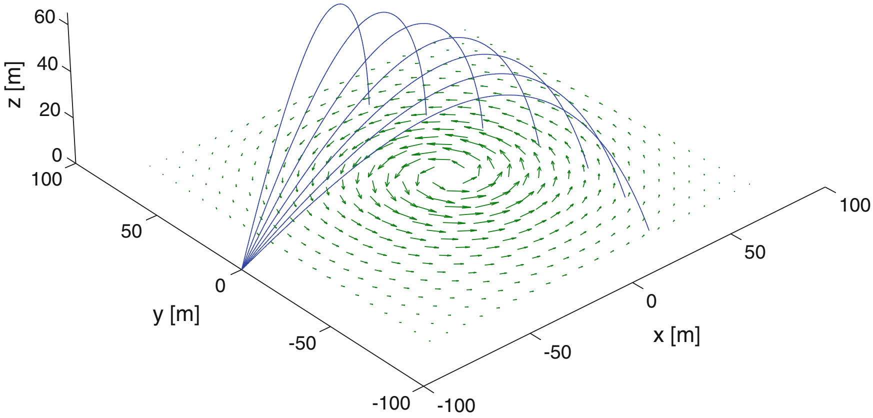

Fig. 7.7 Simulated trajectories of projectiles launched at different angles. The velocity field of the tornado is illustrated by the arrows

$$\mathbf{v}_0 = v_0\left(\cos\alpha\cos\boldsymbol{\theta}\,\mathbf{i} + \cos\alpha\sin\boldsymbol{\theta}\,\mathbf{j} + \sin\alpha\,\mathbf{k}\right) \ , \tag{7.37}$$

Simulations for various values of $\boldsymbol{\theta}$ are given in Fig. 7.7.

Notice the change in trajectories for the various angles. The next step would be to see how easy it would be to find the velocity field from these trajectories or from a set of trajectories, but we will not pursue this direction here. From our discussion you should be able to see how we can use this model to address how to launch the projectile.

Test your understanding: Based on this analysis, would you recommend launching the projectile by shooting it into the tornado, or by ejecting them vertically as the tornado is passing from a container ledged to the ground?

7.5 Force Model—Spring Force

When we introduced the spring model in one dimension, it was used to represent two different concepts:

- To model the force from a linearly elastic spring
- To represent the simplest position-dependent force model

Both of these interpretations are still important in two- and three-dimensional systems. We may use a spring model as a model for deformation, and we may use the spring model as a simplified model for a position-dependent force. However, we have more degrees of freedom, and we can therefore define a spring model in several different ways.

Full Spring Model

Let us first see how the force due the deformation of a spring can be generalized in two- and three-dimensions. From one-dimensional experiments, we expect the force from a spring on the object attached to the spring to depend on the elongation of the spring and act in the direction of the spring. The situation is illustrated in Fig. 7.8. A spring is characterized by its equilibrium length, L_0, and its spring constant, k. The force from the spring on the object is:

$$\mathbf{F} = -k\,(L - L_0)\,\hat{u}_r \ , \tag{7.38}$$

where L is the length of the spring, and the unit vector $\hat{u}_r$ points from the spring towards the object. (Check for yourself that the sign is indeed correct). If the object is located at the position $\mathbf{r}$, and the other end of the spring is attached to the point $\mathbf{R}$, as in Fig. 7.8, the length of the spring is:

$$L = |\mathbf{r} - \mathbf{R}| \ , \tag{7.39}$$

and the unit vector pointing from $\mathbf{R}$ toward $\mathbf{r}$ is:

$$\hat{u}_r = \frac{\mathbf{r} - \mathbf{R}}{|\mathbf{r} - \mathbf{R}|} \ . \tag{7.40}$$

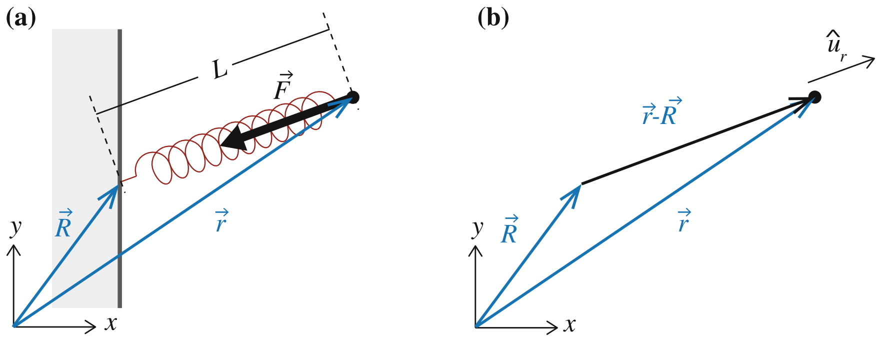

Fig. 7.8 **a** Illustration of a spring attached to a wall at **R** and to a small particle at **r**. **b** Illustration of the force **F** from the spring on the particle

In the **full spring model**, the force from a spring attached to one end at $\mathbf{R}$ and at the other end to an object at $\mathbf{r}$ is:

$$\mathbf{F} = -k\,(L - L_0)\,\hat{u}_r \ , \tag{7.41}$$

where $L = |\mathbf{r} - \mathbf{R}|$ is the length of the spring, and L_0 is the equilibrium length of the spring.

Often, we will place the origin at the attachment point of the spring, so that $\mathbf{R} = 0$, and the full model simplifies to:

$$\mathbf{F} = -k\,(r - L_0)\,\frac{\mathbf{r}}{r} \ , \tag{7.42}$$

where the length of the spring, $L = r = |\mathbf{r}|$, is the distance from the origin to the particle.

We have named this model the "full model" because it most closely represents the behavior of a real, physical spring. This force model is versatile and general and can be widely applied. For example, it can be used to model the deformation of an elastic body, or the force between two atoms in a molecule. This model will be our preferred model for contact forces such as forces due to deformation in two- and three-dimensional systems.

Notice that the force model has a spherical symmetry: When we choose the origin at the attachment point ($\mathbf{R} = 0$), the force from the spring on the attached object always acts along a line through the origin, and the magnitude of the force depends on the distance r to the origin. This means that force on the particle from the spring in the x-direction is:

$$F_x = -k\,(r - L_0)\,\frac{x}{r} = -k\left(\sqrt{x^2 + y^2 + z^2} - L_0\right)\frac{x}{\sqrt{x^2 + y^2 + z^2}} \ . \tag{7.43}$$

That is, the force in the x-direction, depends not only on the x-position, but also on the y and z coordinates. If we apply Newton's second law of motion to such a system, the acceleration of the object in the x-direction, will depend on the x, y, and z coordinates of the object: We call such a system *coupled*, and notice that it may not be that simple to find the solution to the equations of motion in this case.

Lattice Spring Model

While the full spring model has a simple physical interpretation—it models the behavior of a physical spring—the resulting equations of motion are not that simple

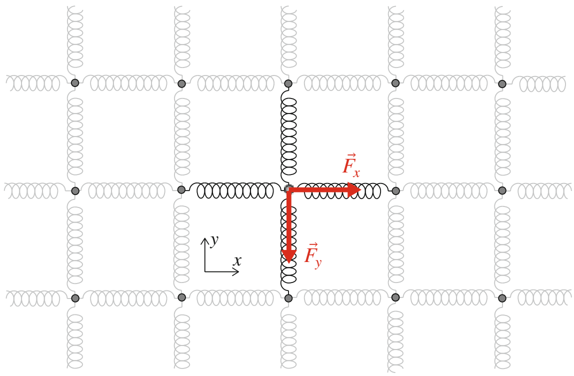

Fig. 7.9 An atom in a lattice is attached to four neighbors by linear springs, so that the springs in the x-direction only act in the x-direction, and the springs in the y-direction only act in the y-direction

from a mathematical point of view, because the motion in the x, y, and z-directions are coupled. It is therefore common to introduce a *decoupled* spring model. The physical analogue of a decoupled system is found in a crystalline lattice: A central atom is attached with linear springs to four neighboring atoms, as illustrated in Fig. 7.9. Let us assume that the neighboring atoms do not move.

In the **lattice spring model**, the force on an object attached with springs in all directions is:

$$\mathbf{F} = -k_x\,(x - x_e)\,\mathbf{i} - k_y\,(y - y_e)\,\mathbf{j}\,, \tag{7.44}$$

where x_e and y_e is the position of the atom when it is in equilibrium, and k_x and k_y are the spring constants in the x- and y-directions respectively. This model is often referred to as the *harmonic oscillator* model .

In this model, the forces in the x- and y-directions are independent of each other, which also means that the motion in the x- and y-directions are decoupled. For example, if the atom is only affected by the lattice spring force, Newton's second law gives:

$$\sum \mathbf{F} = k_x\,(x - x_e)\,\mathbf{i} + k_y\,(y - y_e)\,\mathbf{j} = m\left(a_x\,\mathbf{i} + a_y\,\mathbf{j}\right) \tag{7.45}$$

equating the components on each side therefore gives two, independent equations of motion for motion along the x- and y-axes:

$$k_x\,(x - x_e) = ma_x\;,\;\; k_y\,(y - y_e) = ma_y \tag{7.46}$$

This problem is mathematically much simpler than the full model: We can simply use the results we already found in the one-dimensional case.

7.5.1 Example: Motion of a Bouncing Ball with Air Resistance

In this example we study the motion of an object subject to a constant force, a velocity-dependent force, and a position-dependent force. We solve the problem numerically and discuss the results following a workflow similar to what you will find in many practical problems.

Let us continue to refine the description of a ball thrown in the classroom. So far we have introduced gravity and air resistance. But what happens when the ball hits the floor? We need to also include a force model for the normal force from the floor on the ball. The simplest approach to such a contact force model is a spring model: We model the interaction between the floor and the ball as a single spring. But the normal force is zero when there is no contact. In this problem we demonstrate how to include such effects in our models.

Problem: A ball is thrown from a height h above the ground with an initial velocity $\mathbf{v}_0$. Find the velocity and position of the ball as a function of time t. Include the normal force from the floor while the ball is in contact with the floor.

Identify and Sketch: We describe the position of the ball by $\mathbf{r}(t)$, measured in a coordinate system with origin at the floor. The initial position and velocity of the projectile is $\mathbf{r}(t_0) = h\,\mathbf{j}$ and $\mathbf{v}(t_0) = \mathbf{v}_0 = v_{x,0}\,\mathbf{i} + v_{y,0}\,\mathbf{j}$.

Model: The motion of the ball is determined by the forces acting: air resistance, $\mathbf{F}_D$, the normal force $\mathbf{N}$ from the floor, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$, as illustrated in the free-body diagram in Fig. 7.10. We use a square law for air resistance:

$$\mathbf{F}_D = -Dv\mathbf{v} \; . \tag{7.47}$$

The normal force from the floor on the ball is represented by a spring force. This is a strong simplification of the actual deformation process occurring at the contact between the ball and the floor due to the deformation of both the ball and the floor.

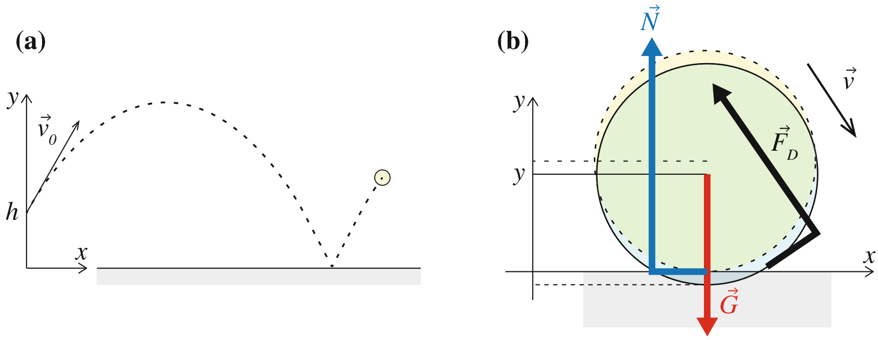

Fig. 7.10 **a** Sketch of the path of the ball. **b** Free-body diagram for the ball when in contact with the floor

The deformed region corresponds roughly to the region of "overlap" between the ball and the floor in Fig. 7.10. The depth of this region is $\Delta y = R - y(t)$, where R is the radius of the ball, which corresponds to the compression ΔL of the spring:

$$\mathbf{N} = -k\,(R - y(t))\,\mathbf{j}\ . \tag{7.48}$$

We check that the sign is correct: The normal force must act upward when $y < R$, hence the sign must be negative.

However, we must also ensure that the normal force only acts when the ball is in contact with the floor, otherwise the normal force is zero. The full formation of the normal force is therefore:

$$\mathbf{N} = \begin{cases} -k\,(R - y(t))\,\mathbf{j} & \text{when } y(t) < R \\ \mathbf{0} & \text{when } y(t) \geq R \end{cases}\ . \tag{7.49}$$

Newton's second law: Newton's second law is now

$$\sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{F}_D + \mathbf{N} = m\mathbf{a}\ , \tag{7.50}$$

which gives

$$\mathbf{a} = -\,(D/m)\,v\,\mathbf{v} - g\,\mathbf{j} + \mathbf{N}/m\ , \tag{7.51}$$

with the initial conditions: $\mathbf{r}(t_0) = \mathbf{r}(0\,\mathrm{s}) = \mathbf{r}_0$ and $\mathbf{v}(t_0) = \mathbf{v}(0\,\mathrm{s}) = \mathbf{v}_0$. While it is difficult to determine the motion analytically, we may be able to find analytical solutions for parts of the motion. However, we can determine the motion numerically by integrating (7.51) using Euler-Cromer's method:

$$\mathbf{v}(t_i + \Delta t) \simeq \mathbf{v}(t_i) + \Delta t\,\mathbf{a}(t_i, \mathbf{r}(t_i), \mathbf{v}(t_i)) \tag{7.52}$$

$$\mathbf{r}(t_i + \Delta t) \simeq \mathbf{r}(t_i) + \Delta t\,\mathbf{v}(t_i + \Delta t)\ . \tag{7.53}$$

The implementation is straight-forward:

```
from pylab import *
m = 0.2      # kg
g = 9.81     # m/s^2
vT = 20.0    # m/s
h = 2.0      # m
R = 0.1      # m
k = 1000.0   # N/m
r0 = array([0.0,h])
v0 = array([10.0,10.0])
time = 20.0 # s
dt = 0.001   #
s n = int(round(time/dt))
r = zeros((n,2),float)
v = zeros((n,2),float)
t = zeros(n,float)
r[0] = r0
v[0] = v0
```

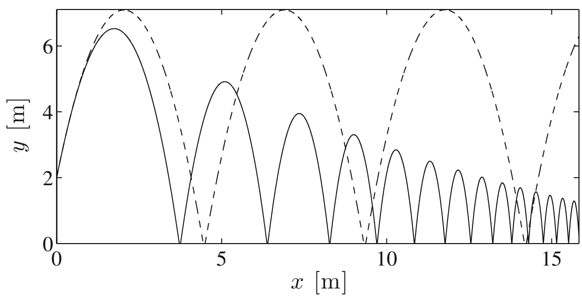

Fig. 7.11 Plot of the trajectory of the ball calculated using Euler-Cromers method for quadratic air resistance (*solid line*), compared with the trajectory without air resistance (*dashed line*)

```
t[0] = 0.0
# Simulation loop
for i in range(n):
   if (r[i,1]<R):
      N = k*(R-r[i,1])*array([0,1])
   else:
      N = array([0,0])
   FD = - D*norm(v[i])*v[i]
   G = -m*g*array([0,1])
   Fnet = N + FD + G
   a = Fnet/m
   v[i+1] = v[i] + a*dt
   r[i+1] = r[i] + v[i+1]*dt
   t[i+1] = t[i] + dt
plot(r[:,0],r[:,1])
xlabel('x [m]'), ylabel('y [m]')
```

where we have used the $v_T = 10.0$ m/s to calculate D, and $m = 0.2$ kg. We have chosen to use the spring constant $k = 1000$ N/m, a number we have largely guessed for now. (The effective spring constant may be measured by experiment or calculated if we know the material properties of the ball and the floor). The resulting path is illustrated in Fig. 7.11, where it is compared with the path of the ball without air resistance.

Detailed analysis of wall contact: While the behavior in Fig. 7.11 looks reasonable at first, a closer examination shows that something is wrong, at least according to our intuition. When there is no air resistance, the ball bounces back to the same height! We know that a real ball would not behave like this. What is wrong?

Figure 7.12 shows a magnification of the behavior of the ball during a bounce when there is no air resistance. The red lines in the figures mark the positions and the times when the ball comes in contact with the floor. We see that the horizontal velocity, v_x, does not change at all during the bounce. This is not surprising, since the normal force only has a vertical component. No horizontal forces means no horizontal acceleration. However, we also see that the vertical velocity component, v_y, simply reverses during the bounce. This is the part that strikes us as unrealistic. And indeed it is. How can we modify the force model for the normal force to ensure that the vertical speed is smaller after the collision? This question we will return to later when we discuss energy and collisions. The short answer is that we need to introduce a force that is velocity dependent: Any force model that only depends on

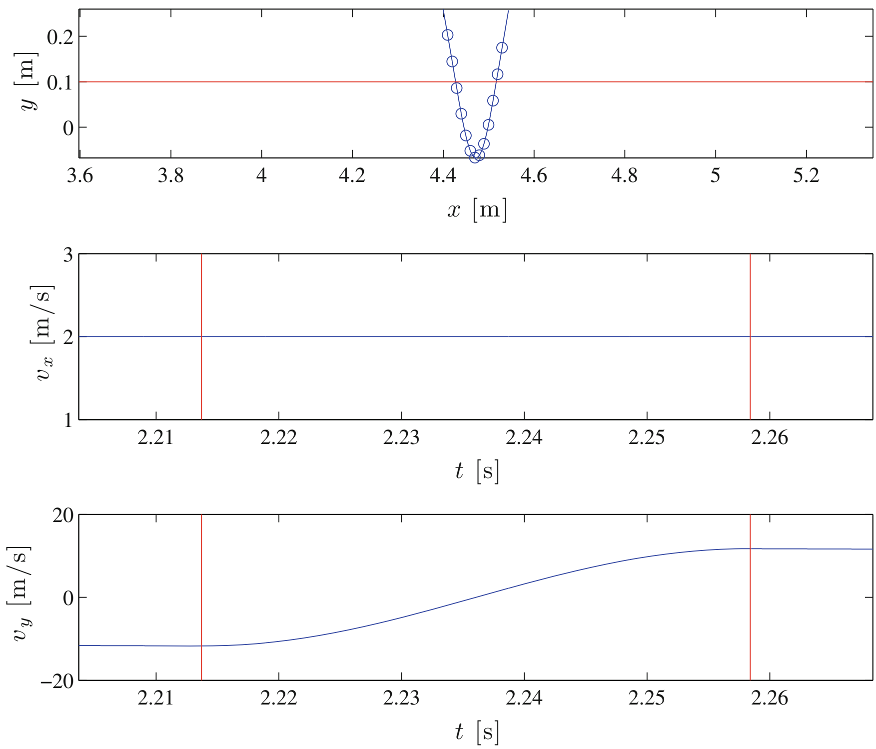

Fig. 7.12 Plot of the trajectory of the ball without air resistance during its collision with the floor. The red lines illustrates the points where the ball comes in contact with the floor

the vertical position $y(t)$ of the ball will always result in a reversal of the velocity! By introducing a viscous force, a linear velocity dependent force model, during the collision with the floor, the outgoing vertical velocity will be reduced. But more on this later! (See Chap. 12.)

Comments: We have now built a model for a bouncing ball, including both air resistance and a normal force from the floor. You should notice the simple structure followed systematically: as long as we can develop force models for the interactions, we can model the motion. This method is robust and no different in one-, two-, or three dimensions. You should also notice that the normal force changes: It does not always just balance the gravitational force—it is indeed larger than the gravitational force in parts of the collision. Otherwise the normal force would not be able to change the direction of the ball. This simple point is a clear result of the force model approach, and also of a careful physical analysis, but represents a classical misunderstanding.

7.6 Force Model—Central Force

The long-distance forces of gravitation and the electrostatic interaction (Coulomb's law) are examples of central forces: A force between two objects that:

- acts in the center of the objects
- acts along a line connecting the two objects
- depends on the distance between the two objects

A **central force** has the form:

$$\mathbf{F} = F(r)\frac{\mathbf{r}}{r} = F(r)\,\hat{u}_r \ , \tag{7.54}$$

where the magnitude $F(r)$ of the force is a function of the distance r only.

Both gravitation and the electrostatic force has the same form for the central force, the *inverse square law*:

$$\mathbf{F} = \frac{C}{r^2}\,\hat{u}_r = C\frac{\mathbf{r}}{r^3} \ . \tag{7.55}$$

where for $C = -GmM$ this corresponds to Newton's law of gravitation.

The central force model is a force between two objects, where we have placed one of the objects in the origin, and we are only interested in the motion of the other object. This corresponds to the case where the object in the origin is very massive, so that it does not move significantly, or where it is in some way attached, so that it does not move.

Notice that the full spring model also is a central force model, but it does not display the inverse square law. We can use the central force model to describe not only gravitation and electrostatic interactions, but also for many interatomic forces, and as we have seen, for spring forces.

7.6.1 Example: Comet Trajectory

A comet of mass m is affected by the gravitational force from the Sun

$$\mathbf{F}(\mathbf{r}) = -G\,m\,M\,\frac{\mathbf{r}}{r^3} \ , \tag{7.56}$$

where $\mathbf{r}$ is the position of the comet in a coordinate system centered on the Sun. We assume that the Sun does not move. How can we find the motion of the comet? (Fig. 7.13)

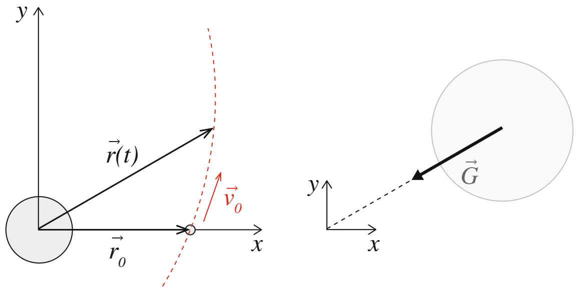

Fig. 7.13 Sketch of a comet moving around the Sun

Sketch: We start from a simple sketch of the system. The comet starts at $\mathbf{r} = R\mathbf{i}$, with an initial velocity $\mathbf{v}_0$. But we do not still know how it will move, so we have only made a guess for its trajectory. We have also added the coordinate system in the center of the Sun.

Newton's second law: Newton's second law gives us the acceleration of the comet:

$$m\mathbf{a} = -G\, m\, M\, \frac{\mathbf{r}}{r^3} \;\Rightarrow\; \mathbf{a} = -G\, M\, \frac{\mathbf{r}}{r^3}\;, \tag{7.57}$$

which is independent of the mass of the comet. The trajectory of a small comet and a large planet will therefore be the same with the same initial conditions.

Integration of motion: We find the motion of the comet by solving (7.57) numerically. Generally, this problem requires a more advanced numerical solution method in order to avoid inaccuracies—you should use a fourth order Runge-Kutta with adaptive time step—but we will here employ the Euler-Cromer scheme with a very small time step because of its transparent implementation.

The Euler-Cromer scheme allows us to find the velocity and time after a small time-step Δt, starting from the initial condition at time $t = t_0$:

$$\mathbf{v}(t_0 + \Delta t) \simeq \mathbf{v}(t_0) + \Delta t\, \mathbf{a}(t_0, \mathbf{r}(t_0), \mathbf{v}(t_0)) \tag{7.58}$$

$$\mathbf{r}(t_0 + \Delta t) \simeq \mathbf{r}(t_0) + \Delta t\, \mathbf{v}(t_0 + \Delta t)\;. \tag{7.59}$$

We use realistic numbers for the masses of the Sun, and choose the initial radius R and the initial velocity to correspond to that of the Earth: v_0: $M = 1.99 \times 10^{30}$ kg, $\mathbf{R} = 1.5 \times 10^{11}$ m, $v_0 = 3 \times 10^4$ m/s, and $G = 6.673 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}. We choose the direction of the initial velocity to be along the y-axis. The implementation is then straight-forward:

```
from pylab import *
# Physical values
M = 1.99e30   # kg
R = 2e11      # m
v0mag = 3e4   # m/s
G = 6.673e-11 # m^3\,kg^-1 s^-2
# Initial conditions
r0 = R*array([1,0])
```

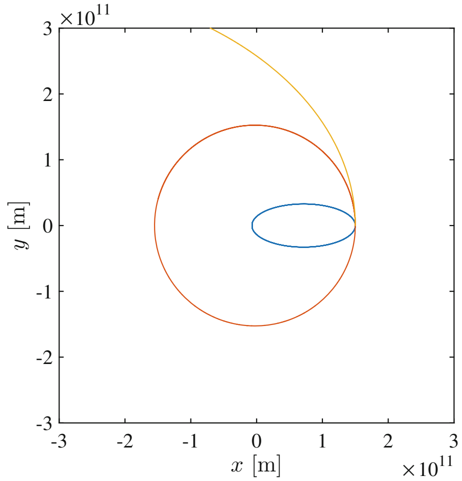

Fig. 7.14 Trajectories for a comet following Earth's trajector (*solid line*), with 1/3 of the initial velocity (*dashed line*), and with 1.5 times the initial velocity (*dotted line*)

```
v0 = v0mag*array([0,1])
# Numerical values
time = 60*60*24*365*5 # s
dt = 100 # s
# Setup Simulation
n = ceil(time/dt)
r = zeros((n,2),float)
v = zeros((n,2),float)
t = zeros((n,1),float)
r[0] = r0 # vectors
v[0] = v0 # vectors
GM = G*M
# Calculation loop
for i in range(n-1):
    rr = norm(r[i,:])
    a = -GM*r[i]/rr**3
    v[i+1] = v[i] + dt*a
    r[i+1] = r[i] + dt*v[i+1]
    t[i+1] = t[i] + dt
plot(r[:,0],r[:,1])
xlabel('x [m]'); ylabel('y [m]'); axis equal
```

Analysis: The resulting trajectory in Fig. 7.14 shows that the comet is moving in a circular orbit! What happens if we change the initial conditions a bit? If we reduce the initial velocity by a factor 3, the resulting trajectory is no longer a circle, but looks more like an ellipse. What if we increase the initial velocity by a factor 1.5? Then the resulting trajectory is no longer a closed loop—the comet leaves the solar system!

You will learn more about planetary motion later on. For now we realize that we can calculate the motion of planet using Newton's second law and the gravitational force model.

Summary

Newton's second law:

- Newton's second law relates the acceleration of an object to the net forces acting on it: $\sum_j \mathbf{F}_j = m\mathbf{a}$, where the sum is over all forces acting on the object, and m is the inertial mass.
- All forces acting on *a system* has a source in the *environment*.
- Forces can be *contact forces* acting on the boundary between the system and the environment.
- Forces can be *long range forces* from an object in the environment.
- Forces are drawn as vectors starting at the point where the force is acting, pointing in the direction of the force, and with a length indicating the length of the force.
- The force may be a given quantity, $\mathbf{F}$.
- The gravitational force acts between all objects. On the surface of the Earth the gravitational force on an object is $\mathbf{W} = -mg\,\mathbf{j}$, where $\mathbf{j}$ is a unit vector pointing upwards, g is the acceleration of gravity, and m is the gravitational mass. The gravitational mass is equal to the inertial mass.
- The contact force from a fluid on a moving object depends on the velocity of the object relative to the fluid. The simplest force model is the viscous force, $\mathbf{D} = -k_v\mathbf{v}$, where the constant k_v depends on the viscosity of the fluid and the size of the object.
- The contact force from a solid depends on the distance between the object and the solid. The simplest force model that depends on the position of an object is the independent spring force model: $\mathbf{F} = -k_x(x - x_e)\,\mathbf{i} - k_y(y - y_e)\,\mathbf{j} - k_z(z - z_e)\,\mathbf{k}$. Here, x_e, y_e, z_e are equilibrium positions in the x, y, and z-directions, and k_x, k_y, and k_z are spring constants. The spring model is one of the most fundamental force models because it is the first order Taylor expansion of any position-dependent force.

Problem-solving approach:

- We **identify** the object and its initial conditions.
- We **model** the behavior by find the forces acting on the object, introducing force models for all the forces, and applying Newton's second law to find an equation for the acceleration of the object.
- We **solve** the problem by finding the position and velocity from the acceleration and the initial conditions using numerical or analytical techniques.
- We **analyze** the solution to validate it, and use the solution to answer the original question posed.

Exercises

Discussion Questions

7.1 Free kick. A soccer player is making a free kick and the opposing team is making a wall to protect their goal. Is it always theoretically possible for the kicker to hit the goal in an ideal situation with only a vertical acceleration due to gravity?

7.2 Flying ball. A projectile is shot through the air. Can you think of any situation where the projectile may experience an upward acceleration?

7.3 Bouncing ball. A basket ball is thrown in a long arc and bounces off the floor. We assume that the contact with the floor can be modelled as a spring force acting normal to the floor. Describe how the horizontal and vertical components of the velocity change during the collision.

7.4 Earth and Sun. The force from the Sun on the Earth acts directly towards the Sun, yet the Earth does not fall into the Sun. Explain.

7.5 Rope magic. You tie a long, strong rope between your car and a tree in order to exert a large force on the car. How can you pull the rope to ensure that you pull at the car with a much larger force than you can exert on the rope?

7.6 Curving the ball. You may know from soccer that you can curve a ball by spinning it. Can you explain this by the physics you have learned so far?

7.7 Suspension. The suspension of a car consists of both a spring and a dashpot. The dashpot provides a viscous force response. Why is it not sufficient with a spring alone?

Problems

7.8 Paraglider. Samantha is jumping from an airplane.
(**a**) Identify the forces acting on Samantha and draw a free-body diagram of her before she has pulled the chord.
(**b**) Identify the forces acting on Samantha and draw a free-body diagram of her after she has pulled the chord.
(**c**) Samantha hits the water instead of the boat she was aiming for. Identify the forces acting on Samantha and draw a free-body diagram of her as she slows down in the water.

7.9 Boat on a lake. A boat is sailing at constant velocity over a small lake. Identify the forces acting on the boat and draw a free-body diagram of the boat.

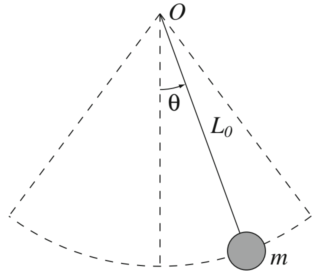

Fig. 7.15 Illustration of a pendulum consisting of a ball of mass m attached to rope of length L_0. The other end of the rope is attached at the point O

7.10 Force on the Moon. Identify the forces acting on the Moon, and draw a free-body diagram of it.

7.11 Chandelier. A chandelier of mass $m = 200$ kg is hanging in 4 wires of equal length. The wires are attached on the corners of a square in the ceiling. The distance between each attachment point in $L = 4$ m. The chandelier is suspended a length h below the ceiling.
(**a**) Draw a free-body diagram of the chandelier.
(**b**) Find an expression for the wire tension if the chandelier is not moving.
(**c**) The wires can sustain a maximum tension of 10, 000 N. How far down can the chandelier be suspended?

7.12 Three-pointer. You are throwing for a three-pointer. The ball leaves your hand with a velocity of 9.4 m/s at and angle of 60° with the horizon. You score from a horizontal distance of 7 m. The height of the basket is 3.5 m. You can ignore air resistance (Fig. 7.15).
(**a**) Draw a free-body diagram of the ball.
(**b**) Find the position and velocity of the ball as a function of time.
(**c**) At what height was the ball released?
(**d**) What is the velocity in the vertical direction as the ball hits the goal?

7.13 Hitting an apple. You are aiming your bow directly at an apple placed on top a high pole 50 m away. The arrow leaves the bow with a horizontal velocity of 50 m/s. You can ignore air resistance.
(**a**) Draw a free-body diagram of the arrow while in the air.
(**b**) Find the position and velocity of the arrow as a function of time.
(**c**) How far does the arrow fall below a horizontal path in the first half of the motion?
(**d**) How far does the arrow fall down in the second half of the motion? Why is it not the same as you found above?
(**e**) The apple is 4 cm in radius. How far from the apple can you stand and still hit it?

7.14 Hitting the target. You are trying to make a winning play in urban terrain golf—you are standing on top of the physics building, 10 m above ground, and try to hit a hole located 5 m out from the building. What initial speed should you give the ball in order for it to hit the hole? You can ignore air resistance.

7.15 Long jump world record. In 1991 Mike Powell beat the long-standing world record of Bob Beamon by reaching a length of 8.95 m. His maximum speed is 9.5 m/s. What is his maximum range?

7.16 Adjusting the aim of a rifle. You are adjusting the aim of your rifle by shooting at a target 100 m away. When you have adjusted your rifle at this length, your start shooting at a target 200 m away. How far above the target do you need to aim in order to hit the target? You know that the bullet leaves your rifle with a speed of 1000 m/s. You can ignore air resistance.

Projects

7.17 Ball in a spring. In this project you will study an advanced model of a pendulum. The pendulum consists of a ball in a massless rope moving in a vertical plane. The ball has mass m. You can neglect air resistance. We describe the position of the ball by the position vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. In this project we will introduce a *model* for the pendulum by assuming that the rope can be modelled as a spring with a spring constant k and an equilibrium length L_0.
(**a**) Identify the forces and draw a free-body diagram of the ball.
(**b**) Show that the net external force acting on the ball can be written as:

$$\sum_j \mathbf{F} = -mg\,\mathbf{j} - k\,(r - L_0)\,\frac{\mathbf{r}}{r}\,, \tag{7.60}$$

where $r = |\mathbf{r}|$ is the length of the (stretched) rope, and the origin of the coordinate system is chosen to be the attachment point, O, of the rope.
(**c**) Rewrite the expression of the external force on component form by writing the force components, F_x, and F_y, as functions of the components x and y of the position vector, $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$.

In this project, we will not assume that the ball is following a particular path, such as a circle, but we will instead use Newton's second law to determine the motion of the ball from the forces acting on it. Using our model, we can measure the tension in the rope, as well as the motion of the ball, and analyze these to learn about the motion.
(**d**) For a pendulum, it is customary to describe the position of the pendulum by its angle $\boldsymbol{\theta}$ with the vertical. Does the angle $\boldsymbol{\theta}$ give a sufficient description of the position of the ball in this case? Explain your answer.
(**e**) If the ball is at rest at $\boldsymbol{\theta} = 0$ with no velocity ($\mathbf{v} = \mathbf{0}$) and no acceleration, what is the position of the ball? What happens if you increase the value of k for the rope?

We will now study a specific pendulum, consisting of a ball with a mass of 0.1 kg, and a rope of equilibrium length $L_0 = 1$ m with a spring constant $k = 200$ N/m, which corresponds to a rather elastic rope. Initially, you can assume that the ball starts

with zero velocity at an angle $\boldsymbol{\theta} = 30°$ at a distance L_0 from the origin. We want to study the motion of the ball by integrating the equations of motion numerically.
(**f**) Find an expression for the acceleration, **a**, of the ball. You should write it both on vector form, where there acceleration vector is a function of the position vector **r** and its length, r, and on component form, where the components a_x and a_y are functions of the x and y components of the position vector.
(**g**) What is the mathematical initial value problem you need to solve in order to find the motion of the ball? Include both the differential equation you need to solve and the initial conditions in your answer.
(**h**) How can you solve this problem numerically? Write down a set of equations that find the position and velocity at a time $t + \Delta t$ given the position and velocity at t using Euler-Cromer's method. Insert your expression for the acceleration from above. Mark the terms in your equations that vary in time.
(**i**) Write a program that "solves" the problem by finding the motion of the ball. The program should plot the position of the ball in the xy-plane for the first 10 s of the motion. *Hint 1*: You may write the mathematical expression almost directly into your program if you use a vector notation and vector operations in your code. *Hint 2*: Remember that $r = r(t) = |\mathbf{r}(t)|$ varies in time! *Hint 3*: Do *not* use $\boldsymbol{\theta}(t)$ to describe the position of the ball. Describe the motion using $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ and use your results from above for the acceleration.
(**j**) Use the program to find the behavior for the given initial conditions using a time-step of $\Delta t = 0.001$. Plot the resulting motion. Describe what you see.
(**k**) What happens if you increase Δt to $\Delta t = 0.01$ and $\Delta t = 0.1$? Can you explain this? Test what happens if you use Euler's method with $\Delta t = 0.001$ instead of Euler-Cromer's method.
(**l**) Rerun the program with $k = 20$ and $k = 2000$. Describe the motion in these cases and compare with $k = 200$ case. Are your results reasonable? Based on this, can you suggest how to use this method to model a pendulum in a stiff rope? What do you think would be the limitation of this approach? (Test what happens if you use $k = 2\,10^6$ in your program).
(**m**) Rewrite your program to ensure that the rope tension is zero if the spring is compressed, because the rope cannot sustain compression. Use this program to determine the motion with the initial conditions $\mathbf{v}_0 = 6.0\,\text{m/s}\,\mathbf{i}$ and $\mathbf{r}_0 = -L_0\,\mathbf{j}$. What happens?

7.18 Weather balloon. In this project we will develop a model to determine the motion of a weather balloon released from the ground. We start from a simplified model and gradually add features to make to model more realistic.

After the balloon is released, it is driven by buoyancy. Initially, we will assume that the buoyancy force is a constant, B.
(**a**) Draw a free-body diagram of the balloon. Identify the forces and introduce symbols. Indicate the relative magnitudes of the forces by the length of the vectors.

First, let us neglect air resistance.
(**b**) What is the acceleration of the balloon?
(**c**) Find the position and velocity of the balloon as a function of time.

Let us now introduce air resistance, using a quadratic law: $\mathbf{F}_D = -D\,v\,\mathbf{v}$.

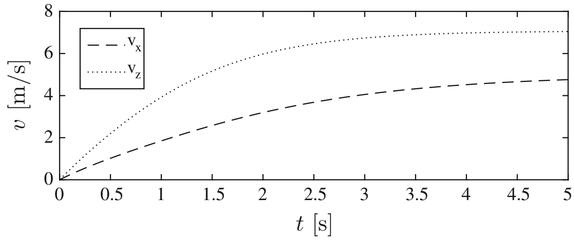

Fig. 7.16 Result of a simulation

(**d**) Show that the acceleration of the balloon in the upward (z) direction is $a_z = (B/m) - g - (D/m)\ |v_z|\, v_z$, where v_z is the velocity in the z-direction.
(**e**) Sketch the acceleration and velocity as a function of time for the model including air resistance.
(**f**) Find the asymptotic (terminal) velocity of the balloon.

The balloon is released on a windy day, with a wind blowing with a velocity $\mathbf{w} = w\,\mathbf{i}$ along the horizontal x-axis.
(**g**) How does the wind modify the air resistance force $\mathbf{F}_D$ on the balloon?
(**h**) Draw a free-body diagram for the balloon in this case. Indicate the magnitude of the forces by the relative lengths of the vectors.
(**i**) Find an expression for the acceleration **a** of the balloon. What are the initial conditions for the motion of the balloon?
(**j**) Why do we call the motion in the z and the x direction "coupled" in this case? Can you determine the motion of the balloon analytically?
(**k**) We can determine the motion of the balloon using numerical methods. Write a program to find the velocity $\mathbf{v}(t)$ and position $\mathbf{r}(t)$ as functions of time. (It is sufficient to only include the main integration step in your answer - that is the part that determines $\mathbf{v}(t + \Delta t)$ and $\mathbf{r}(t + \Delta t)$ given $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

Figure 7.16 shows the result of a simulation.
(**l**) Describe the motion of the balloon. Illustrate by relevant sketches.
(**m**) Find the asymptotic (terminal) velocity of the balloon.

In a real situation, the wind velocity is smaller near the ground and increases gradually to the full velocity w_0 as the balloon moves upward. Typically, the velocity of the wind can be described by $\mathbf{w} = w_0\,(1 - \exp(-z/d))\ \mathbf{i}$, where $d = 10\,\text{m}$ is a length determining the transition.
(**n**) Rewrite your program to include this effect.
(**o**) What is the terminal velocity of the balloon now?

Chapter 8
Constrained Motion

In many cases, the object we are studying is not free to move in every direction. For example, a bead on a wire can only move along the wire. The shape and position of the wire determine the path of the bead, but the bead is still free to move in many different ways along that wire. In this case we call the motion "constrained". The constraints can be strong, by restricting the motion to be along a given path, or the constraints can be weak, such as for a bead caught between two parallel glass plates, or for a car driving on the terrain. Constraints may also arise because objects are connected to each other: The motion of an individual atom in a large rigid molecule is constrained by the motion of the whole molecule, and a small part of a spinning wheel is constrained to follow the motion of the wheel.

We have learned that the motion of an object can be determined from the forces acting on the object. This is, of course, also the case for constrained motion. But in many cases it is not practical to include the forces that restrict the motion, either because we do not have good models for them, or because we would rather like to determine these constraining forces from what we know about the motion—from the fact that the object follows a particular path. For example, a bead moving along a wire will be affected by normal forces from the wire on the bead, and it is these forces that cause the bead to follow the path given by the wire. However, we may not want to model these forces in detail, instead we are interested in the consequences of constraints—we want to find the normal force when we know that the bead follows the wire.

In this chapter, we will discuss constrained motion. Starting from the simplest case of motion constrained to a line, through circular motion, to motion constrained to be along a general curve.

8.1 Linear Motion

The simplest case of a constrained motion is that of a bead moving along a straight, rigid wire (see Fig. 8.1) or a car moving along a straight road. We call such a motion linearly constrained—because the motion is constrained to follow a given line.

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_8

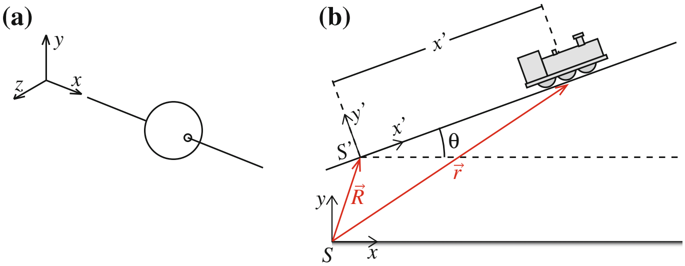


Fig. 8.1 **a** The motion of a bead moving along a straight wire is constrained to be along the line described by the wire. **b** The motion of a small train moving up along an inclined track is also constrained by the track. The motion can be described both in a coordinate system with axes along the track, and in a coordinate system with axis in the *horizontal* and *vertical* direction

How can we formulate a mathematical constraint corresponding to the motion of a bead on a wire as in Fig. 8.1? The simplest way is to choose the x-axis to be along the wire. Then the bead can only move along the x-axis, and the y-, and z-components remain zero throughout the motion:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = x(t)\,\mathbf{i} \; . \tag{8.1}$$

Because we are free to choose the coordinate system, we can always make such a choice for linear motion.

We can also formulate the constraint without fixing the x-axis to be along the direction of motion, such as for the train in Fig. 8.1. The train moves along a slope that forms the angle θ with the horizon. We can describe the slope by the point $\mathbf{R}$ and the unit vector $\hat{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ which points along the slope. The position of the train is then given by how far along the slope it has moved, $s(t)$:

$$\mathbf{r}(t) = \mathbf{R} + \hat{u}\,s(t) \; . \tag{8.2}$$

We call $s(t)$ the distance, and notice that it would correspond to the coordinate x' of a coordinate system oriented along the slope. Since $\mathbf{R}$ is a constant, the velocity of the train is

$$\frac{d\mathbf{r}}{dt} = \underbrace{\frac{d\mathbf{R}}{dt}}_{=0} + \hat{u}\frac{ds}{dt} = \hat{u}\frac{ds}{dt} \; , \tag{8.3}$$

where ds/dt is the velocity measured *along the track*, which is what you would measure from a speedometer. The distance $s(t)$ traveled reflects the motion of the bead. It does not have to be only increasing, but may reflect a complicated motion along the wire.

8.2 Curved Motion

We can use the insight from motion along a straight wire to understand curved motion, such as the circular motion of an atom in a rotating, rigid molecule or the motion of a roller-coaster car following its track.

Position

For a train running along a track as illustrated in Fig. 8.2 we can use the same description as we used above: We describe the position, $\mathbf{r}(s(t))$, of the train by the distance, $s(t)$, travelled *along the track*. In Fig. 8.2 the train moves with constant speed along the track. A sequence of times at constant intervals, $t_i = i\Delta T$, as shown with circles.

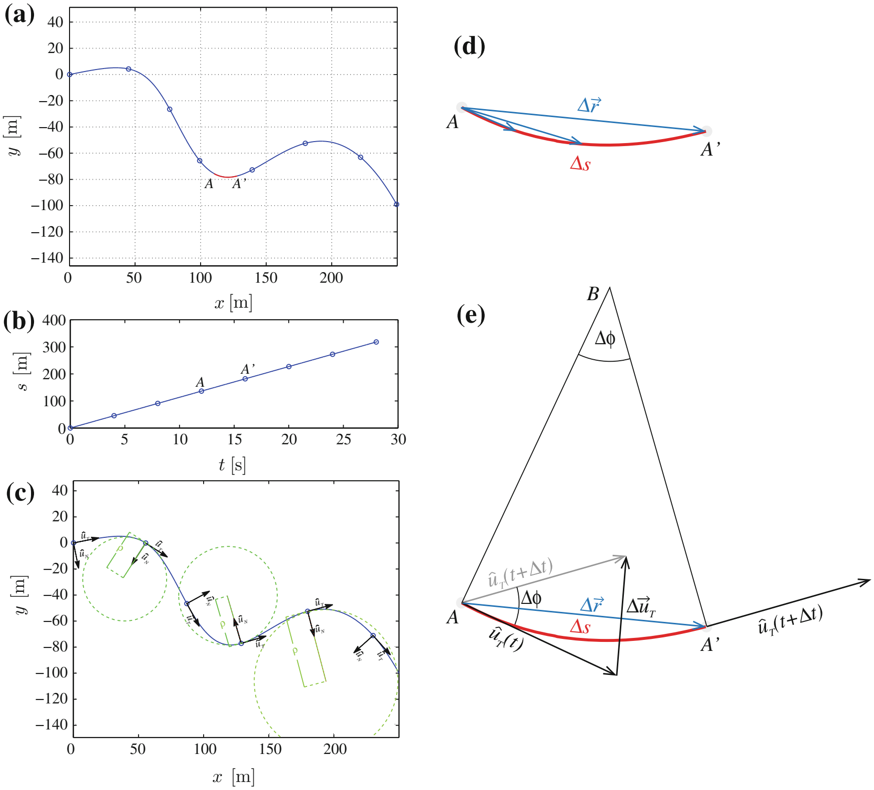


Fig. 8.2 Illustration of a motion along a curved path. **a** A train track. The train starts at $\mathbf{r} = 0$ m at $t = 0$ s. Circles mark the positions at times t_i. **b** Plots of the length $s(t)$ along the track. Circles indicate the times t_i. **c** The velocity in A approaches a tangent to the line as the time interval Δt decreases. **d** Illustration of the change in tangential vector $\hat{u}_T$. **e** Illustration of the velocity, acceleration, and normal vectors along the track, as well as the local curvature of the track illustrated by the radius of the circles

Velocity Vector

What is the velocity of the train? The velocity vector is the time derivative of the position vector:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \ . \tag{8.4}$$

We see from Fig. 8.2c that as the time interval Δt decreases, the displacement $\Delta \mathbf{r}$ becomes tangential to the curve, just as we found in Chap. 6:

$$\mathbf{v}(t) = v(t)\hat{u}_T \ , \tag{8.5}$$

where $\hat{u}_T(t)$ is the *unit tangent vector*, $\hat{u}_T = \mathbf{v}/|\mathbf{v}|$. We also see that as Δt becomes small, the arc length along the curve, Δs becomes approximately the same as the displacement $\Delta r = |\Delta \mathbf{r}|$:

$$\Delta r \to \Delta s \text{ when } \Delta t \to 0 \ . \tag{8.6}$$

The magnitude of the velocity therefore approaches

$$v = \frac{|\Delta \mathbf{r}|}{\Delta t} = \frac{\Delta r}{\Delta t} \to \frac{\Delta s}{\Delta t} \to \frac{ds}{dt} \ . \tag{8.7}$$

when $\Delta t \to 0$. Now, we know both the direction and the magnitude of the velocity:

Velocity of motion along a curve:

$$\mathbf{v}(t) = v(t)\,\hat{u}_T(t) = \frac{ds}{dt}\,\hat{u}_T(t) \ . \tag{8.8}$$

Notice that the unit vector $\hat{u}_T$ is not a constant, but changes with time as the object moves: $\hat{u}_T = \hat{u}_T(t)$.

Acceleration Vector

The acceleration of the train can be found by taking the time derivative of the velocity vector:

$$\mathbf{a} = \frac{d}{dt} v(t)\hat{u}_T(t) = \frac{dv}{dt}\hat{u}_T + v(t)\frac{d\hat{u}_T}{dt} \ . \tag{8.9}$$

We recognize the first component as the rate of change of the magnitude of the velocity—this is how the speedometer changes as the train accelerates.

What about the second term? We have illustrated a small part of a motion in Fig. 8.2d. We are interested in the change in the tangent vector from the point A at $\mathbf{r}(t)$ to the point A' at $\mathbf{r}(t + \Delta t)$. We have illustrated the tangent vectors, and in the small inset, we have illustrated the change in the tangent vectors, $\Delta \hat{u}_T$. For a small increment Δt, we argue that the length of the $\Delta \hat{u}_T$ vector is approximately given as the length of the arc between the two tangent vectors, which is the angle $\Delta \phi$ between the two vectors multiplied by the length of the vectors, which is 1. Now, we can relate the angle $\Delta \phi$ to the local geometry of the curve by drawing a line AB normal to the tangent vector in A, and a line $A'B$ normal to the tangent vector in A'. This forms a triangle that is congruent with the small triangle formed by the tangent unit vectors, the angle spanned by the two lines AB and AB' is therefore $\Delta \phi$. The length of AB and AB' is called the radius of curvature, ρ. The angle $\Delta \phi$ is also given as the arc length from A to A', which is the distance Δs along the trajectory, divided by the local radius ρ:

$$\Delta \phi = \frac{\Delta s}{\rho} \ . \tag{8.10}$$

The direction of the change in the unit tangent vector is toward the center of the circle, that is, toward the point B. We call this direction the normal direction, and we call the unit vector pointing in this direction for the unit normal vector, $\hat{u}_N$. The change in the tangent vector is therefore:

$$\Delta \hat{u}_T = |\Delta \hat{u}_T| \hat{u}_N = \Delta \phi \hat{u}_N = \frac{\Delta s}{\rho} \hat{u}_N \ . \tag{8.11}$$

The time derivative of the tangent vector is therefore:

$$\frac{d\hat{u}_T(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta u_T}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\rho \Delta t} \hat{u}_N = v(t) \frac{1}{\rho(t)} \hat{u}_N(t) \ . \tag{8.12}$$

Now, we insert this expression back into the expression for the acceleration in (8.9), getting

$$\mathbf{a}(t) = \frac{dv}{dt} \hat{u}_T(t) + \frac{v^2(t)}{\rho(t)} \hat{u}_N(t) \ . \tag{8.13}$$

Here, we write that the radius of curvature, $\rho(t)$, is a function of time, because the radius of curvature is a function of the position along the curve, $\rho(s)$, and the position along the curve for the object is a function of time, $s = s(t)$: $\rho(t) = \rho(s(t))$.

We have therefore shown that the acceleration vector can be written as

$$\mathbf{a}(t) = a_T(t) \hat{u}_T(t) + a_N(t) \hat{u}_N(t) \ , \tag{8.14}$$

where the tangential acceleration is

$$a_T(t) = \frac{dv}{dt} , \tag{8.15}$$

and the normal acceleration is

$$a_N(t) = \frac{v^2(t)}{\rho} . \tag{8.16}$$

The magnitude of the normal acceleration, $a_N = v^2/\rho$ is often called the *sentripetal acceleration* of the object.

Uniform Circular Motion

These results can be simplified for uniform circular motion, where an object moves in a circle with a constant speed, v. In this case, the speed $v(t)$ does not change, and the radius of curvature is constant and equal to the radius, R, of the circle. The acceleration is therefore $a_N = v^2/R$, directed in towards the center of the circle. This is illustrated by the motion diagram in Fig. 8.2.

Non-uniform Circular Motion

Figure 8.3 illustrates a motion diagram for a non-uniform circular motion: One case where the speed is increasing and one case where the speed is decreasing. In both cases, the radius of curvature is constant and equal to the radius, R, of the circle. The acceleration is therefore

$$\mathbf{a} = \frac{dv}{dt}\,\hat{u}_T + \frac{v^2}{R}\,\hat{u}_N . \tag{8.17}$$

Angular Coordinates

We can also specify a position on the circle by the angle $\phi(t)$ as illustrated in Fig. 8.3. How are the distance $s(t)$ and the angle $\phi(t)$ related? The distance $s(t)$ corresponds to the arc length along the circle:

$$s(t) = R\phi(t) . \tag{8.18}$$

The two representations $s(t)$ or $\phi(t)$ can both be used to represent the position, and you may use whatever representation you find most practical.

The speed can be related to the rate of change of the angle, ϕ, since

$$\frac{ds}{dt} = \frac{d}{dt}\left(R\phi(t)\right) = R\frac{d\phi}{dt} = R\omega(t) , \tag{8.19}$$

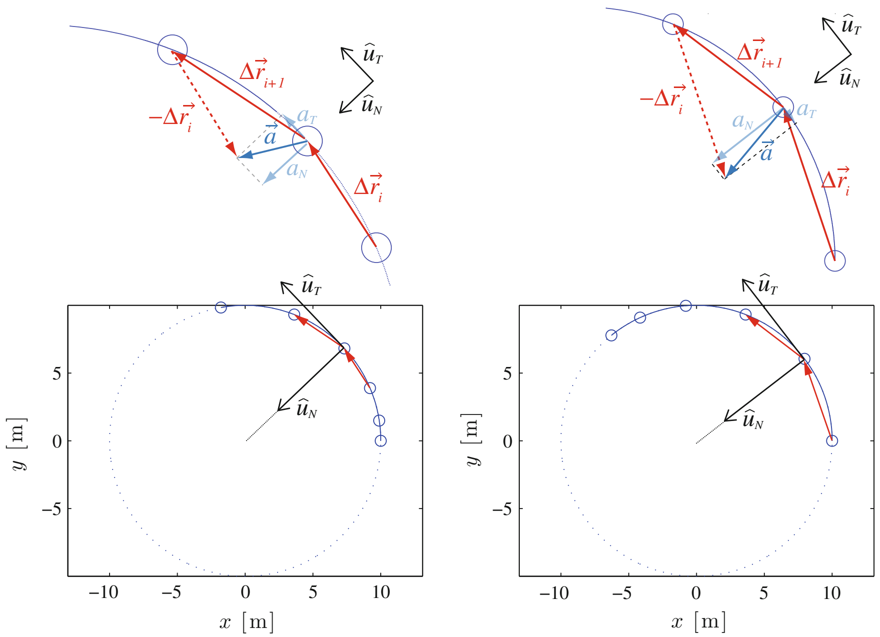


Fig. 8.3 Illustration of how the acceleration vector is decomposed into a tangential and normal component for a motion with increasing speed (*left*) and decreasing speed (*right*)

where the time derivative of the angle is called the angular velocity, ω. We will return to this description of rotational motion when we address rotations in Chap. 14.

8.2.1 Example: Acceleration of a Matchbox Car

Problem: A matchbox car is spun around a horizontal, semi-circular track with a radius of 20 cm. The car has an approximately constant speed $v = 0.5$ m/s throughout the turn. (a) Find the acceleration of the car during the motion. (b) If you make the circle half the diameter, will the acceleration of the car increase or decrease?

Solution: The car is moving with constant speed along a circular track. The motion of the car is therefore constrained to follow a circular path. Since the car is moving with constant speed, we argued above that the car only has a radial acceleration in the direction toward the center of the circle, and with a magnitude:

$$a_N = \frac{v^2}{R} = \frac{(0.5\,\mathrm{m/s})^2}{0.2\,\mathrm{m}} = 1.25\,\mathrm{m/s^2} \ . \tag{8.20}$$

The acceleration is inversely proportional to the radius of the circle. If we reduce the radius to one half, the acceleration will increase by a factor 2.

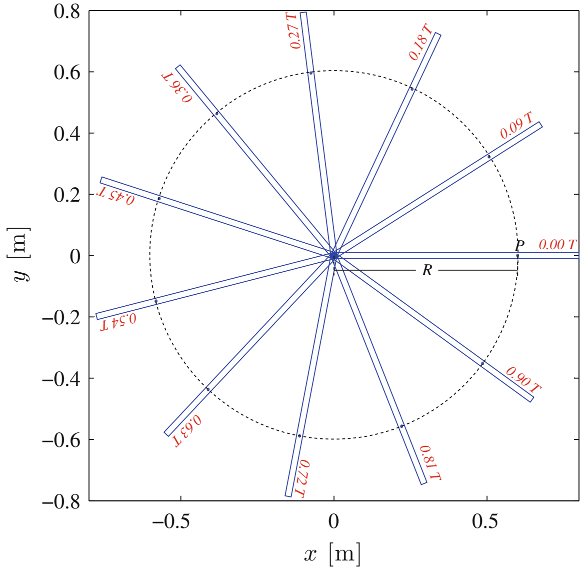


Fig. 8.4 Illustration of a rod of length $L = 0.8\,\text{m}$ rotating at a constant rate around an axis through one of the ends of the rod. The position of the rod is illustrated for various times given as fractions of T, the time of a complete revolution, called the period of the rotation. The point P is at a distance R from the rotation axis, and the circular path traveled by the point is illustrated by the *dashed circle*

8.2.2 Example: Acceleration of a Rotating Rod

Problem: A rod of length L is rotating around an axis through one of its ends. The rod is rotating at a constant speed so that it takes a time T for the rod to complete a revolution. (a) Find the speed of a point P on the rod located a distance R from the rotation axis. (b) Find the acceleration of the point P.

Solution: The rotating rod is illustrated in Fig. 8.4. The point P is located a distance R from the rotation axis. Since the rod is rotating at a constant speed, the speed of the point P on the rod is also constant, and the average speed is equal to the instantaneous speed. We find the average speed over a complete revolution from the length the point has traveled during a complete revolution and the time a revolution takes. The rod completes a revolution in a time T, and the point P has moved a distance $s = 2\pi R$ along the whole, closed circular loop. We find the average speed of P as:

$$v = \frac{s}{T} = \frac{2\pi R}{T} = \frac{2\pi}{T} R \ , \tag{8.21}$$

The quantity $2\pi/T$ is often called the angular frequency, or, as we will see later, the angular velocity, and we usually use the symbol ω for this:

$$\omega = \frac{2\pi}{T} \ , \tag{8.22}$$

we can therefore write:

$$v = R\omega \ . \tag{8.23}$$

We find the acceleration of the point P by realizing that the point moves along a circular path of radius R and with a speed v. The acceleration is therefore directed in toward the center of the circle, which is also the rotation axis, and the magnitude of the acceleration is given by the centripetal acceleration term:

$$a = \frac{v^2}{R} = \frac{(R\omega)^2}{R} = R\omega^2 = R\left(\frac{2\pi}{T}\right)^2 . \tag{8.24}$$

8.2.3 Example: Normal Acceleration in Circular Motion

Problem: A ball is spun in a string in a circular path with radius $R = 1\,\mathrm{m}$. (a) If the ball is spun at a constant speed of $v = 2\,\mathrm{m/s}$, find the tangential and normal acceleration of the ball. (b) If the ball starts at rest and increases its speed at a constant rate, $dv/dt = 0.1\,\mathrm{m/s^2}$, find the tangential and normal accelerations of the ball as a function of time.

Solution (a): The acceleration of the ball has a tangential and a normal component:

$$a_T = \frac{dv}{dt} \, , \; a_N = \frac{v^2}{\rho} \, . \tag{8.25}$$

Because the speed is constant, we have that $dv/dt = 0$, and the tangential acceleration is zero. The radius of curvature, ρ for a circular path with radius R is the radius of the circle, $\rho = R$. The tangential acceleration is therefore:

$$a_N = \frac{v^2}{\rho} = \frac{(2\,\mathrm{m/s})^2}{1\,\mathrm{m}} = 4\,\mathrm{m/s^2} \, . \tag{8.26}$$

Solution (b): The speed of the ball is $v(t)$. We know that the speed of the ball is $v(0) = 0\,\mathrm{m/s}$ when $t = 0\,\mathrm{s}$, and that $dv/dt = 0.1\,\mathrm{m/s^2}$. We find the speed by integration:

$$v(t) - v(0) = \int_0^t \frac{dv}{dt} dt = \int_0^t 0.1\,\mathrm{m/s}\,dt = 0.1\,\mathrm{m/s} \int_0^t dt = 0.1\,t\,\mathrm{m/s} \, . \tag{8.27}$$

We use this to find the tangential and normal acceleration:

$$a_T = \frac{dv}{dt} = 0.1\,\mathrm{m/s} \, , \; a_N = \frac{v^2}{\rho} = \frac{\left(0.1\,t\,\mathrm{m/s^2}\right)^2}{1\,\mathrm{m}} = 0.01\,t^2\,\mathrm{m/s^4} \, . \tag{8.28}$$

We see that $a_T = a_N$ when $0.1\,\mathrm{s}^2 = 0.01\,t^2$, that is for $t = t^* = \sqrt{10}\,\mathrm{s}$. When $t < t^*$ the tangential acceleration is larger than the normal acceleration, and when $t > t^*$ the normal acceleration is larger than the tangential acceleration.

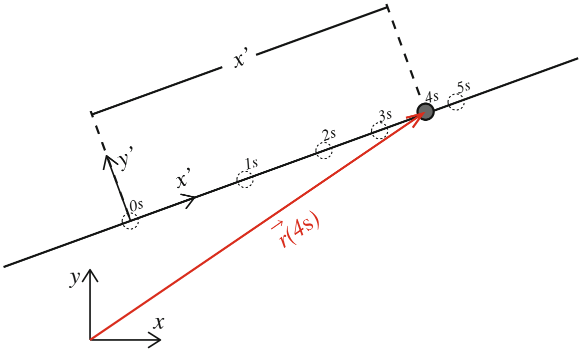
Summary

Constrained motion:

- *Constrained* motion is motion along a given path, for example, a bead on a wire is constrained to follow the wire.
- We describe motion along a curve by the distance, $s(t)$, along the path, as a function of time, t.
- Mathematically, any motion $\mathbf{r}(t)$ can be decomposed into a path, $\mathbf{r}(s)$, and motion along the path, $s(t)$.

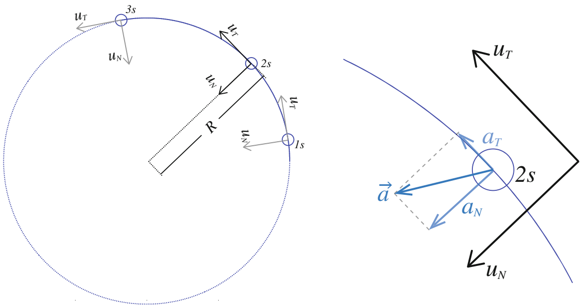
Linear motion:

- In *Linear motion*, the motion of an object is restricted to a straight line.
- We orient the x-axis along the line, and describe the position of the object with $x(t) = s(t)$.
- The velocity and acceleration of the object are also directed along the line: $v(t) = (ds/dt) = (dx/dt)$, $a(t) = (dx/dt)$.



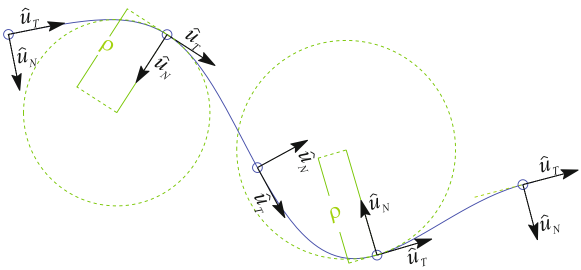
Circular motion:

- In circular motion the object is constrained to move along a circle of radius R.
- The velocity of the object is tangential to the circle (and normal to the radius): $\mathbf{v}(t) = v\ \hat{u}_T(t)$
- For motion with constant speed, the acceleration is directed in toward the center of the circle.
- For motion with varying speed, the acceleration has both tangential and normal components: $\mathbf{a}(t) = (dv/dt)\ \hat{u}_T(t) + (v^2(t)/R)\ \hat{u}_N(t)$



General motion:

- We describe a constraining curve by the $\mathbf{r}(s)$, parameterized by the path length, s, along the curve. The position of an object is $\mathbf{r}(s(t))$, where $s(t)$ is the position along the curve.
- The velocity of the object is tangential to the curve: $\mathbf{v}(t) = v(t)\ \hat{u}_T(t)$
- The acceleration of the object has both tangential and normal components: $\mathbf{a}(t) = (dv/dt)\ \hat{u}_T(t) + (v^2(t)/\rho(t))\ \hat{u}_N(t)$ where $\rho(s(t))$ is the radius of curvature of the curve at $\mathbf{r}(s(t))$.



Exercises

Discussion Questions

8.1 Vertical loop. You spin a ball in a string through a vertical loop, keeping the speed constant throughout. Where is the acceleration the greatest, at the bottom or at the top of the loop. Explain your answer.

8.2 On the surface of the Earth. You drive around the world with an accurate accelerometer. Sketch the direction of the acceleration you measure: at the North Pole, in London, on the equator, in New Zealand, on the south pole.

8.3 In the plane. If your motion is restricted to be along a flat plane, may your acceleration be out of the plane? Explain. If your motion is restricted to be on a surface, is your acceleration restricted to be along the surface?

8.4 Loose branch. A small monkey is climbing far out on a branch when it suddenly breaks. The branch does not snap off, but start to rotate about the point where it is broken. The monkey clings to the branch. What is the direction of the acceleration of the monkey?

Problems

8.5 Skier pulled up a slope. A skier is pulled up a hill with an inclination α with the horizontal. He is pulled with a constant acceleration of $a = 2\,\text{m/s}^2$ along the hill and starts from rest at the bottom of the hill.
(**a**) Find the speed, $v(t)$, of the skier measured along the slope as a function of time, t.
(**b**) Find the position, $s(t)$, of the skier measured as a distance from the starting point after a time t.
(**c**) Find the position, $\mathbf{r}(t)$, of the skier in the xy-coordinate system, where x is the horizontal axis and y is the vertical axis.
(**d**) Use the vector position, $\mathbf{r}(t)$, to find the speed of the skier, and compare with the results you found above.

8.6 Skiing down a slope. You are skiing down a planar skislope with an inclination α with the horizontal. Your acceleration down along the slope is $a = g\,\sin\alpha$. You start from a height h.
(**a**) Find your speed, $v(t)$, measured along the slope as a function of time, t.
(**b**) Find your position, $s(t)$, along the slope as a function of time, t.
(**c**) Find your position, $\mathbf{r}(t)$, relative to the point you started at.
(**d**) How long time does it take until you reach the ground at $y = 0$?

8.7 Bead on a line. A bead is inserted onto a thin line with an inclination α with the vertical. When the bead is released, its acceleration along the line is $a = g\cos\alpha$.
(**a**) Find the speed, $v(t)$, of the bead as a function of time.
(**b**) Find the position $s(t)$ of the bead along the line as a function of time, t.
(**c**) Find the height, $h(t)$ of the bead as a function of time, t.

8.8 Acceleration of 200 m sprinter. During a 200 m race, a sprinter is running with a speed of 10 m/s through the first curve. The length of the curve is 100 m.
(**a**) Find the radius, R, of the curve (it is a perfect half-circle).
(**b**) Find the magnitude and direction of the acceleration a of the sprinter.

8.9 Velocity of point on helicopter rotor blade. The rotor blade of an helicopter has a radius of 5 m and is rotating 200 times a minute.
(**a**) What is the velocity of the outer tips of the rotor blade?
(**b**) What is the acceleration of the outer parts of the rotor blade?

8.10 Turning a high-speed train. A high speed train holds a constant speed of 200 km/h. Your job is to design a 90° turn. Let us assume that you design the turn as a part of a circle.

(**a**) Find an expression for the acceleration of the train while turning.
(**b**) How large must the radius of the circle be for the acceleration to be smaller than $0.1\,g$, where $g = 9.8\,\text{m/s}^2$?
(**c**) How long time does the turn take?

8.11 Acceleration on the equator. You are standing on the equator of the Earth. The radius of the Earth is $R = 6378\,\text{km}$.
(**a**) What is your velocity in space due to the rotation of the Earth?
(**b**) What is your acceleration due to the rotation of the Earth? How large is this compared to g, the acceleration of gravity.

8.12 Artificial gravity in space travel. Your spaceship has been designed with a large rotating wheel to give an impression of gravity. The radius of the wheel is $R = 50\,\text{m}$.
(**a**) How many rotation per minutes must the wheel execute for the acceleration at the outer end of the wheel to correspond to the acceleration of gravity at the Earth, $g = 9.8\,\text{m/s}^2$?
(**b**) What is the difference in acceleration of your feet and your head if you are standing with your feet at the outer end of the rotating wheel? You can assume that you are approximately 2 m high.

8.13 Probe in tornado. A probe caught in a tornado is moving in a circular path in the horizontal plane with approximately constant speed. You have three observations of the position of the probe:

t	$\mathbf{r}$
0.0 s	$35.7\,\text{m}\,\mathbf{i} + 35.6\,\text{m}\,\mathbf{j}$
1.0 s	$12.2\,\text{m}\,\mathbf{i} + 49.3\,\text{m}\,\mathbf{j}$
2.0 s	$-14.6\,\text{m}\,\mathbf{i} + 44.9\,\text{m}\,\mathbf{j}$

(**a**) Find the average acceleration of the probe.
(**b**) Find the center of the circle and the radius of the circle. You can use approximations as you see fit.
(**c**) Find an expression for the position of the probe as a function of time.

8.14 Bead on ring. A bead is attached to a ring of radius R. The ring is rotating n times per minute around an axis that intersects the ring at two positions and that also passes through the center of the ring, as illustrated in Fig. 8.5. The bead is attached at a position given by the angle θ.
(**a**) Find the velocity of the bead as a function of θ.
(**b**) Find the acceleration of the bead as a function of θ. Indicate the direction of the acceleration in the figure.

8.15 Acceleration of Jupiter (Open). Find the magnitude of the acceleration of Jupiter in its motion around the Sun.

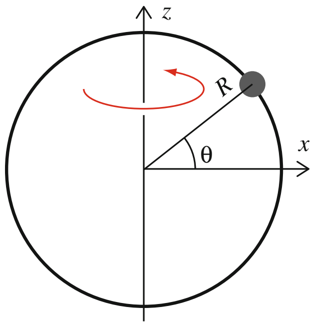


Fig. 8.5 A bead attached to a rotating ring

8.16 Car in a wire. High speed model cars are often run in circular paths by attaching them to a wire. Here we address a car attached to a steel wire of length 8 m. The car starts from rest and accelerates with a tangential acceleration $a_t = 0.5\,\mathrm{m/s}^2$.
(**a**) Find the speed of the car as a function of time.
(**b**) Find the radial acceleration of the car as function of time.
(**c**) At what speed is the radial acceleration 100 times larger than the tangential acceleration?

8.17 Driven pendulum. A rigid pendulum consists of a 1 m long rod with a weight attached at the end. The motion of the pendulum is fixed by a motor driving the rod. The position of the weight along its circular path is $s(t)$:

$$s(t) = A \sin \frac{2\pi t}{T} \ , \tag{8.29}$$

(**a**) Show that the position of the weight can be written as:

$$\mathbf{r}(t) = R \left(\cos \frac{s}{R} \text{ß} + \sin \frac{s}{R} \text{æ} \right) \ . \tag{8.30}$$

What position on the rod does the origin correspond to?
(**b**) What is the velocity of the weight?
(**c**) What is the speed of the weight? Where is the speed maximum, and where it is minimum?
(**d**) What is the acceleration of the weight?
(**e**) Decompose the acceleration into a tangential and a normal acceleration. Compare the two accelerations throughout the motion and comment on the results.

Chapter 9
Forces and Constrained Motion

You have now learned to determine the motion of an object based on an analysis of the forces acting on the object and the application of Newton's second law. We use the structured problem-solving approach to first *identify* the moving object, *model* the forces acting to produce an equation of motion, *solve* the equations of motion to find the motion, and *analyze* the results to answer questions about the motion. This method is very robust—it works as long as we have good models for all the forces.

For example, if you drop a steel box onto a wooden table, you have learned to use a spring model to approximate the normal force from the table on the box while the two are in contact, as illustrated in Fig. 9.1. The normal force, N, depends on the position, x, of the bottom of the box:

$$N = \begin{cases} -kx & x < 0 \\ 0 & x \geq 0 \end{cases} \tag{9.1}$$

However, if the box is lying at rest on the table, we do not need a force model to find the normal force on the box. We can instead use our knowledge about the motion of the box—we know that it is not moving—and apply Newton's second law to determine the normal force, N, to be equal to the gravitational force, G, on the box:

$$\sum F = N - G = ma = 0 \ \Rightarrow \ N = G \ . \tag{9.2}$$

In this case the motion of the box is constrained: We know that it is not moving in the direction of gravity. For constrained motion we can therefore calculate some of the forces without a force model!

First, why is this interesting? If we already know that the box is not moving in the direction of gravity, why would we need to know the forces acting on the box in this direction? Was not the whole point of introducing quantitative models for the forces that we could use this to determine the motion? It turns out that some force models, such as the model for friction, depends on the normal force on the object. In this case, we need to know the normal force to find the force acting in the direction

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_9

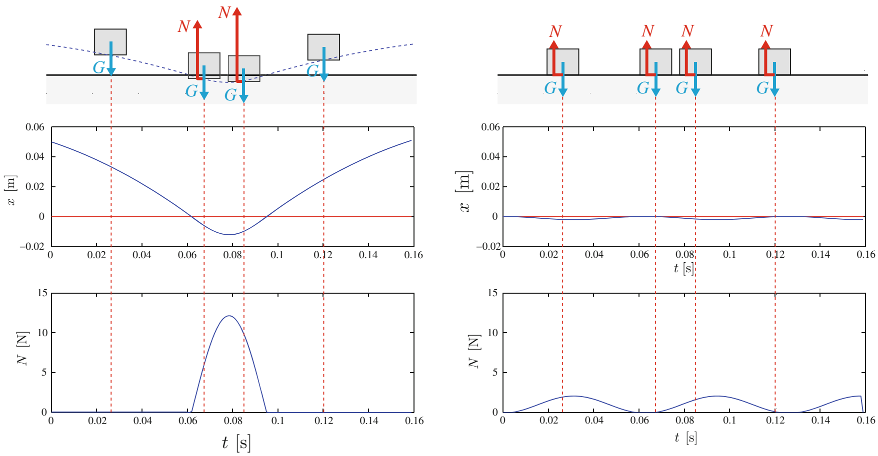

Fig. 9.1 Illustration of a box in contact with a surface. *Left figure* shows the gravitational, G, and normal force, N, on the box when the box is released from a small height above the floor. *Right figure* shows the gravitational and normal force when the box is lying (almost) still on the floor. The forces and vertical motion of the box has been exaggerated

of the motion. Also, there are many cases where the motion is constrained only for a certain range of normal forces: For example, the attachment of a roller coaster cart may only sustain a certain maximum force before breaking, or the motion of a ball in a rope is only constrained by the rope as long as the rope is tight. In both these cases it is an advantage if we can find forces such as the normal force without introducing a force model.

Second, you may object that it is not really true that the box is not moving in the direction of gravity. The box is never completely at rest on the table, it still oscillates slightly up and down, and the normal force varies slightly. That is correct, but we always make many similar approximations. Generally, if the oscillations are very small, we ignore them, and the corresponding variation in forces, and assume that the motion is constrained.

In this chapter we will discuss the use of constrained motion to simplify problems. First, we demonstrate how we try to choose coordinate systems wisely so that motion only occurs along some of the axes. This technique is often called “decomposition of forces”, and it simplifies the analysis of a problem, because only force components in the directions that the object can move can contribute to the acceleration of the object along its path.

We will also introduce a new force model, the friction force model, which allows us to model forces during the relative motion of two solids. This model has a long history and allows us to study many classical examples, even though the physical origin of friction forces is poorly understood.

We will also study more complicated constrained motions, such as the circular motion of an object attached to a rope, or the motion of a car driving along a curved road or a hilltop. In these cases the constraints are only valid for a limited range of the normal forces, and care must be taken to find when the limits are exceeded.

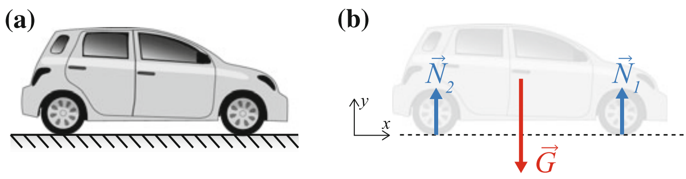

Fig. 9.2 Illustration of a car at rest on a flat surface. **a** Illustration of the car and the surface. **b** Free-body diagram for the car

9.1 Linear Constraints

No Motion—Statics

Let us start with the simplest example of a constrained motion—an object that is not moving at all. In this case, we are not interested in determining the motion of the object, but instead we want to find the forces acting on the object from the constraint that the object is not moving, and that the net force on the object therefore is zero. Such problems are typically called equilibrium problems, or *static* problems.

For example, let us analyze a car standing on the ground, as illustrated in Fig. 9.2a. We follow the first steps in the structured problem-solving approach. First, we *identify* the object, which in this case is the car.

Second, we *model* the interactions between the object and the environment by identifying the forces acting on the object and by introducing quantitative models for the forces. The free-body diagram for the car is illustrated in Fig. 9.2b. The only contact forces are acting on the wheels of the car from the ground, and we call these contact forces $\mathbf{N}_1$ and $\mathbf{N}_2$.[1] In addition, there is a gravitational force, $\mathbf{G}$. Both the normal forces and the gravitational force act in the y-direction only.

We have a quantitative model for the gravitational force, $\mathbf{G} = -mg\,\mathbf{j}$, where m is the mass of the car. However, we do not know the normal forces. Instead, we will use Newton's second law to determine the forces given the constraint we have on the motion—the car is not moving.

Because the car is not moving, the acceleration of the car is zero, $\mathbf{a} = \mathbf{0}$. Newton's second law therefore gives:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} = m\,\mathbf{a} = \mathbf{0}\,. \tag{9.3}$$

$$\mathbf{N}_1 + \mathbf{N}_2 = -\mathbf{G} = mg\,\mathbf{j}\,. \tag{9.4}$$

[1] On a real car there are of course four contact forces acting on the four wheels of the car.

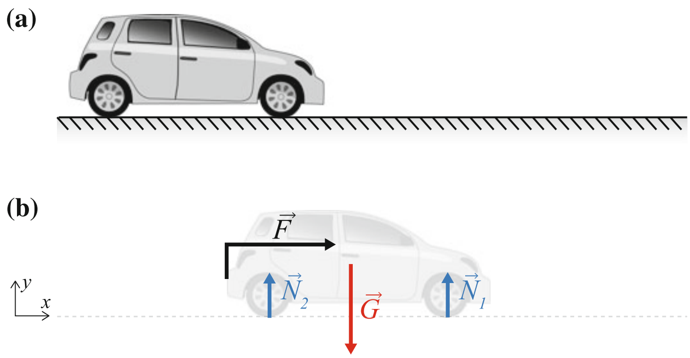

Fig. 9.3 **a** Illustration of a car pushed along a horizontal surface. **b** Free-body diagram for the car

We have found that the *sum* of the normal forces equals the opposite of the gravitational force.

Generally, we do not know how the total normal force is divided between the normal forces.[2] For example, if the car has a very large load in the luggage room in the back, we expect the back suspensions to be more compressed than the front suspensions, which suggest that the normal force on the back of the car is larger than on the front of the car. On the other hand, if the engine in the front of the car is very heavy, we would expect the front wheels to experience a larger load than the back wheels.

It is therefore common to only draw a single contact point for the car, where the sum, $\mathbf{N}$, of the normal forces are acting:

$$\mathbf{N} = \mathbf{N}_1 + \mathbf{N}_2 \ . \tag{9.5}$$

Equilibrium Along One Axis, Motion Along the Other

Let us now look at a slightly more complicated problem, a car moving in a straight line along a flat surface, as illustrated in Fig. 9.3. In this case the motion of the car is constrained to a line, and the forces acting on the car are either directed along the line or are normal to the line. Notice that since the car is constrained to move along a line, the acceleration normal to the line is zero.

Let us analyze the motion of the car when a horizontal force $\mathbf{F}$ is applied, as show in Fig. 9.3. The forces acting on the car are illustrated in Fig. 9.3. The contact forces are the two normal force, $\mathbf{N}_1$ and $\mathbf{N}_2$, as we introduced above, the externally applied force, $\mathbf{F}$, acting horizontally, and the gravitational force $\mathbf{G}$. We assume that the effects of friction and air resistance are negligible.

We have a force model for the gravitational force, $\mathbf{G} = -mg\,\mathbf{j}$, and we assume that we know the value of the externally applied force, $\mathbf{F}$.

[2]As we will see later, we can determine each of the normal forces by making a similar assumption about the rotation of the car: the car is not rotating.

We can determine both the motion of the object and the normal forces by applying Newton's second law. We recall that Newton's law is a vector equation, and that it can be applied along each unit vector independently:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} + \mathbf{F} = m\mathbf{a} \ . \tag{9.6}$$

$$N_1\mathbf{j} + N_2\mathbf{j} - G\mathbf{j} + F\mathbf{i} = m\mathbf{a} = ma_x\,\mathbf{i} + ma_y\,\mathbf{j} \ . \tag{9.7}$$

This can be written as two sets of equations, one for motion along the x-axis, and one for motion along the y-axis:

$$\begin{aligned} F &= ma_x \\ N_1 + N_2 + G &= ma_y \end{aligned} \tag{9.8}$$

Since the car is moving along the x-axis, the acceleration along the y-axis is zero:

$$N_1 + N_2 + G = ma_y = 0 \ , \tag{9.9}$$

and

$$N_1 + N_2 = -G = mg \ . \tag{9.10}$$

The forces in the y-direction are therefore the same as we found above.

The motion in the x-direction is given by the forces acting in the x-direction, as we have found previously.

This analysis demonstrates how we *decompose* the motion into

- motion *along* the constraint, and
- motion *normal* to the constraint.

This is done through the analysis of force, where we similarly *decompose* the forces into

- forces *along* the constraint, and
- forces *normal* to the constraint.

In this particular case all forces are acting either along or normal to the line of constraint. Consequently, the analysis is particularly simple.

Decomposition of Forces

We are now ready to add one more complication, and address motion where the external forces are not simply parallel to or normal to the motion, but where there are

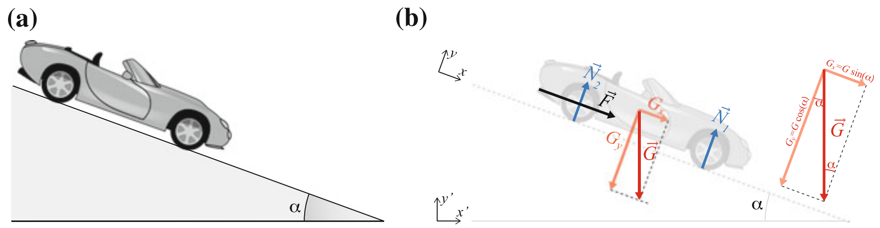

Fig. 9.4 **a** Illustration of a car rolling down an inclined slope. **b** Free-body diagram for the car

force components both parallel to and normal to the motion. In this case, the use of *decomposition* of forces becomes more important, and we will demonstrate how we use this technique combined with the application of Newton's second law through an example.

Let us study the motion of a car rolling down an inclined slope, as illustrated in Fig. 9.4. We realize that also in this case the car is constrained to move along the surface of the slope. We determine the motion of the car by identifying and modeling the forces acting on the car. The contact forces on the car are the normal force, $\mathbf{N}_1$ and $\mathbf{N}_2$. These act normal to the surface. We have chosen the coordinate system to be oriented along the inclined slope so that the x-axis points in the direction of motion—what we called along the constraint—and the y-axis points in a direction normal to the constraint. We assume that we can neglect the effects of friction and air resistance.

In addition, the car is subject to a gravitational force, $\mathbf{G}$. The gravitational force is directed along the vertical axis, which we have called the y'-axis in Fig. 9.4.

We decompose the forces in the x- and y-directions, parallel and normal to the direction of motion for the car. We notice that the normal forces are directed along the y-axis:

$$\mathbf{N}_1 = N_1\,\mathbf{j} \ , \ \mathbf{N}_2 = N_2\,\mathbf{j} \ . \tag{9.11}$$

The gravitational force, $\mathbf{G}$, is not parallel to the x- or the y-axis, but we can decompose it into two components along these directions. From Fig. 9.4 we see that the x- and y-components of gravity can be written as:

$$G_x = G\sin\alpha \ , \ G_y = -G\cos\alpha \ . \tag{9.12}$$

The gravitational force is therefore:

$$\mathbf{G} = G_x\,\mathbf{i} + G_y\,\mathbf{j} \ . \tag{9.13}$$

where the gravitational force model gives the value, $G = mg$, for the gravitational force on the car.

Notice that we have decomposed the forces in the direction of motion, which is given by the constraint, and in the direction normal to motion. This allows us to apply Newton's laws of motion in each direction, separating the motion along the surface from motion normal to the surface.

We apply Newton's second law of motion to the car:

$$\sum \mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{G} = m\,\mathbf{a}\ . \tag{9.14}$$

We write the vectors on component form:

$$N_1\,\mathbf{j} + N_2\,\mathbf{j} + G_x\,\mathbf{i} + G_y\,\mathbf{j} = ma_x\,\mathbf{i} + ma_y\,\mathbf{j}\ . \tag{9.15}$$

We can separate this into two equations, one equation for the x-axis and one equation for the y-axis:

$$G_x = ma_x \tag{9.16}$$

$$N_1 + N_2 + G_y = ma_y \tag{9.17}$$

Because we know that the car is only moving along the surface, the acceleration in the y-direction is zero, giving:

$$N_1 + N_2 = -G_y = G\cos\alpha = mg\cos\alpha\ . \tag{9.18}$$

We have therefore found that sum of normal forces on the car balances the component of the gravitational force that is normal to the surface.

We find that motion along the x-axis is determined by:

$$a_x = \frac{G_x}{m} = \frac{mg\sin\alpha}{m} = g\sin\alpha\ . \tag{9.19}$$

We can therefore use our tools to solve motion with constant acceleration to find the motion along the x-axis.

Notice the use of decomposition: We *choose a coordinate system so that one or more axes are directed in a direction where we know that there is no motion*. Above we chose the y-axis to be directed normal to the surface and the x-axis to be along the direction of motion. We apply Newton's second law in the directions where there are no motion to determine the other forces in this direction. Then, we find force components in the direction of motion, which determine the acceleration and hence the motion of the object along the surface. This approach is general, and can be applied to a wide range of problems where motion is constrained to a line.

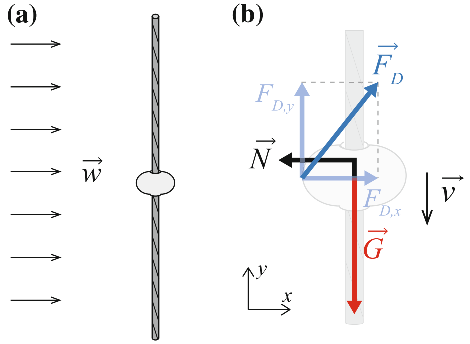

Fig. 9.5 **a** Illustration of bead on a vertical wire. **b** Free-body diagram of the bead

9.1.1 Example: A Bead in the Wind

Problem: A bead is sliding without friction along a vertical wire. A wind with a constant wind speed of w is blowing horizontally. Find the force from the wire on the bead and the acceleration of the bead. You can use a linear air resistance law for the bead.

Approach: We find the net force along the bead, including the effect of air resistance, and use this to find the acceleration along the wire and the forces normal to the wire from Newton's second law.

Identify: The center of the bead, given by $\mathbf{r}(t)$, must move along the wire. We therefore choose the y-axis to be along the wire in the vertical direction, and the x-axis to be directed in the direction the wind is blowing. (See Fig. 9.5a).

Model: The contact forces on the bead are the force from the wire on the bead, $\mathbf{N}$; the air drag force, $\mathbf{F}_D$; and gravity $\mathbf{G} = -mg\,\mathbf{j}$. The air drag force depends on the velocity $\mathbf{v}$ of the bead *relative to* the velocity $\mathbf{w}$ of the wind:

$$\mathbf{F}_D = -k_v\,(\mathbf{v} - \mathbf{w}) \ , \tag{9.20}$$

These forces are illustrated in the free-body diagram for the bead in Fig. 9.5b.

We decompose in the x- and y-direction. Gravity acts vertically, and therefore along the y-axis. The air drag depends on the velocity, $\mathbf{v}$, of the bead. Since the bead moves only in the vertical direction, along the wire, the velocity only has a vertical component:

$$\mathbf{v} = v_y\,\mathbf{j} \ . \tag{9.21}$$

The velocity of the air is in the positive x-direction:

$$\mathbf{w} = w\,\mathbf{i} \ . \tag{9.22}$$

The air drag force is therefore:

$$\mathbf{F}_D = F_{D,x}\,\mathbf{i} + F_{D,y}\,\mathbf{j}\ , \tag{9.23}$$

where we find F_x from:

$$F_{D,x} = \mathbf{F}_D\ \cdot\ \mathbf{i} = -k_v\,(\mathbf{v} - \mathbf{w})\ \cdot\ \mathbf{i} = -k_v\,(\mathbf{v}\cdot\mathbf{i} - \mathbf{w}\cdot\mathbf{i}) = -k_v\,(0 - w) = k_v w \tag{9.24}$$

And, similarly, we find the y-component:

$$F_{D,y} = \mathbf{F}_D \cdot \mathbf{j} = -k_v v_y\ , \tag{9.25}$$

We also notice that the normal force, $\mathbf{N}$, only has a component along the x-axis. We have now decomposed the forces, and can apply Newton's second law along the x- and y-directions.

Since the bead is constrained not to move in the x-direction, we know that the acceleration in the x-direction is zero. Newton's law along the x-axis therefore gives:

$$\sum F_x = F_{D,x} - N = ma_x = 0\ \Rightarrow\ N = F_{D,x} = k_v w\ . \tag{9.26}$$

The normal force is therefore constant for this particular model of the motion!

We find the acceleration of the bead by applying Newton's law in the y-direction:

$$\sum F_y = F_{D,y} - G = -k_v v_y - mg = ma_y\ \Rightarrow\ a_y = -\frac{k_v}{m} v_y - g\ . \tag{9.27}$$

From these equations you know how to find the motion of the bead. For this particular formulation of the problem, it is decoupled—the motion of the bead in the y-direction does not depend on the wind velocity.

Test your understanding: How would this problem change if you used a square-law for the air drag?

9.2 Force Model—Friction

We have now addressed constrained motion such as the motion of a block sliding along a surface as illustrated in Fig. 9.6. We have shown that we can use Newton's second law in combination with the constraint to determine the normal force on the block without including a force model for the contact. For the case illustrated in Fig. 9.6, we can apply Newton's second law in the vertical direction to find:

$$\sum F_y = N - G = N - mg = ma_y = 0\ , \tag{9.28}$$

$$N = mg\ , \tag{9.29}$$

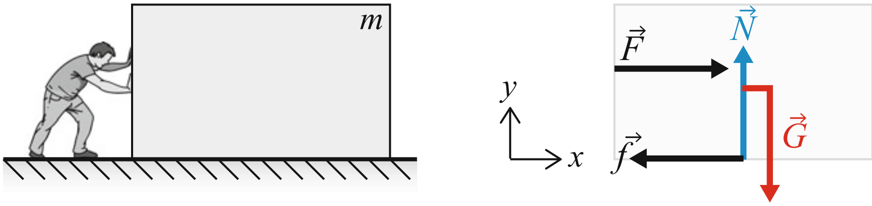

Fig. 9.6 *Left* Illustration of a crate on the floor. *Right* A free-body diagram of a crate on the floor. You are trying to push the crate with the force **F** directed along the floor

But so far there has been no coupling between the forces normal to the surface and the forces along the surface—the tangential forces. The magnitude of the normal force has no impact on the motion of the block.

In real systems there are several mechanisms that introduce a coupling between the normal and tangential forces. When we discuss macroscopic systems it is common to group these mechanisms into a common term: friction. Friction is a force acting on the contact between two objects, it acts tangential to the contact, and it depends on the magnitude of the normal force at this contact. The introduction of a force model for friction forces is therefore essential in order to get a realistic description of contact forces. Unfortunately, the concept of friction is not that well defined. The mechanisms of friction are currently not well understood, and are subjects of current research. There is also not only one, single mechanism responsible for the tangential forces acting on a contact—there are many different mechanisms acting all at the same time, including effects such as surface geometry and roughness, the effect of thin fluid films, surfactants, and dirt on the surfaces, the effect of adhesion forces and plastic deformation of small asperities on the contact surface, the effect of the external boundaries, and many more. In addition, it is also clear that the scale you are studying is important: For example on very small scales inter-atomic forces and the presence of individual atoms or molecules along the surface can affect the tangential forces acting.

But wait, you interrupt, I already know something about friction, and it is a simple law! Yes. Already in 1699 Amontons reported the experimental observations that we today also call the “laws of friction”:

Amonton’s laws of dynamic friction:

- The friction force is proportional to the normal force: $F = \mu N$.
- The friction force does not depend on the apparent contact area.

This is an experimental observation that turns out to be surprisingly robust, in particular considering that it includes so many different effects! In this section we will motivate and discuss the force model for friction as a contact force acting in the tangential direction along any contact between two solids. We will introduce a model for the static friction force for the case when the two surfaces are not moving relative to each other, and a model for the dynamic friction force when the surfaces are moving relative to each other.

Static Friction

We all have a good intuition for solid friction because it is an important force in our macroscopic world. If you want to push a heavy crate along the floor, you know that you need to apply a certain force to "get it going". If you do not push hard enough, it will not start moving. You may also know that if you want to keep it moving, you must constantly apply a force to it, otherwise it will stop moving. If you give the crate a hard push, it will start moving, but you know that it will decelerate and stop after some distance. These simple insights provides us with some of the basic properties of a the friction force

Sliding box model system: Let us analyze the motion of a crate on a flat floor in detail (See Fig. 9.6 for an illustration). You push on the crate with a horizontal force, $\mathbf{F}$. The motion of the crate is determined by the forces acting on it: The applied force, $\mathbf{F}$, the normal force $\mathbf{N}$ from the floor, and the gravitational force, $\mathbf{G}$. The crate is cannot move down through the floor—it is constrained not to move in the y-direction. We can therefore use Newton's second law in the y-direction to determine the normal force:

$$\sum F_y = N - G = N - mg = ma_y = 0 \;\Rightarrow\; N = mg \; . \tag{9.30}$$

We also know, from our experience or from staging a simple experiment, that when the applied force, $\mathbf{F}$, is small, the crate is not moving in the horizontal direction. However, according to Newton's second law of motion the horizontal acceleration is given by the net horizontal force:

$$\sum F_x = ma_x \; . \tag{9.31}$$

Because the crate is not moving, the sum of the forces must be zero. Consequently, the crate cannot only be affected by the applied force $\mathbf{F}$ in the horizontal direction. There must be an additional force, counteracting the applied force. This force must come from the contact with the floor. And the force has the apparently "magical" property that it is exactly equal to the applied, horizontal force! If you push with a very small force, $\mathbf{F}_0$, then the force from the floor will be exactly the same but in the opposite direction. If you push with a slightly larger force, $\mathbf{F}_1$, then the force from the floor will again be exactly the same, but in the opposite direction. In this respect the friction force is similar to the normal force from the floor on the crate, which in the static situation matches the force from gravity. We call the force from the floor on the crate the *static friction* force

Microscopic model for static friction: The word static indicates that it is the friction along a surface when the object does not move relative to the surface. We can develop a simple model for the origin of the static friction force from a microscopic picture of the contact. On the microscopic scale, the contact between the crate and the floor consists of many, small irregularities: small bumps on the surface of the floor and the surface of the crate. When the crate is placed on the surface, these irregularities are pressed together. One possible explanation is that in this process, the irregularities are "glued" together due to adhesive forces between the two materials. This glue acts like small springs, so that when we try to push at the crate, these small springs are extended, exerting forces that counteract the applied force. However, if the applied force on the crate becomes too large, the adhesive forces become so large that the contacts break, and the crate starts to slip.

It is reasonable to assume that the number of irregularities that are in contact, and possibly the contact area of each irregularity, increase with the normal force. This means that we should discern between the apparent area of contact, which is simply the size of the side of crate facing the floor, and the actual area of contact, which is given by the contact areas of all the small irregularities. If the actual contact area is proportional to the normal force, we expect the static friction force also to be proportional to the normal force.

The simple model we have developed now, based only on our intuition, is surprisingly close to the most usual description of static friction. By the word "law" here, we mean a model for the force — it is not a fundamental law of nature, but rather a model that can often, but not always, be applied to contact problems.

Coefficient of static friction: The upper limit of the static friction force is $F_m = \mu_s N$, where N is the normal force for the contact, and μ_s is called the coefficient of static friction. If the friction force exceeds this upper limit, the object will start moving relative to the surface, and our model for static friction is no longer valid.

We see that the concept of static friction is closely related to a constraint on the motion of an object:

> The **static friction force** is a tangential force acting on the interface between two solids in contact that are not moving relative to each other. The magnitude and direction of the force is so that the two object do not accelerate relative to each other. However, there is an upper limit on the magnitude of the static friction force
>
> $$F_s < F_{s,max} = \mu_s N \ , \tag{9.32}$$
>
> where N is the normal force at the contact.

In order to determine the static friction force we need to assume that the objects are not accelerating, that is, we need to assume a constraint on the motion, and use Newton's second law of motion to determine the static friction force. Then, we need

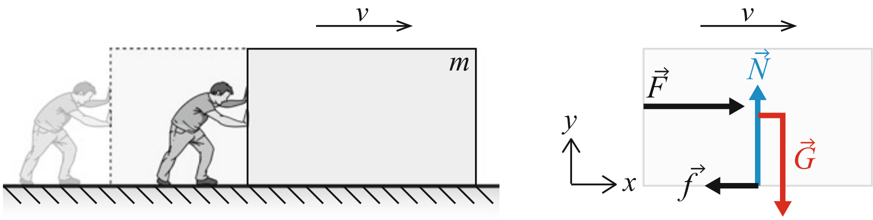

Fig. 9.7 *Left* Illustration of a crate moving along the floor. *Right* A free-body diagram of a crate on the floor. You are pushing the crate with the force **F** directed along the floor

to check that the force does not exceed the threshold. If it exceeds the threshold, the objects will start moving relative to each other, and we need another force model—the dynamic friction model.

This law is experimentally justified and it is only an approximation of the real behavior of materials, where many different effects contribute to bring about the effect of a static friction force.

Dynamic Friction

You are gradually increasing how hard you are pushing a crate on the floor. What happens when the crate slips? Our model so far has been for *static* friction. However, you know from your own experience that if you give the crate an initial velocity and release it along the floor, it slows downs and stops (see Fig. 9.7). That means that the crate decelerates. We can therefore use Newton's second law and conclude that because the crate decelerates, there must be a force acting along the floor in the direction opposite the movement, which is the cause of the deceleration. We call this force the dynamic friction force.

There is a surprisingly simple law for the dynamic friction force, $\mathbf{F}_d$, acting on an object in contact with another object:

> The **dynamic friction force**, F_d, is tangential to the surface of contact. The magnitude of the force is $F_d = \mu_d N$ where N is the normal force across the contact. The direction of the force is opposite the relative motion of the two objects.

This law is empirically based. This means that it is the result of measurements. The law states that the dynamic friction force depends only on the normal force and on the coefficient of friction, which is a property of the two materials in contact. The dynamic friction force does not depend on the area of contact. Neither does it depend on the velocity of the object relative to the floor. This is surprising—we

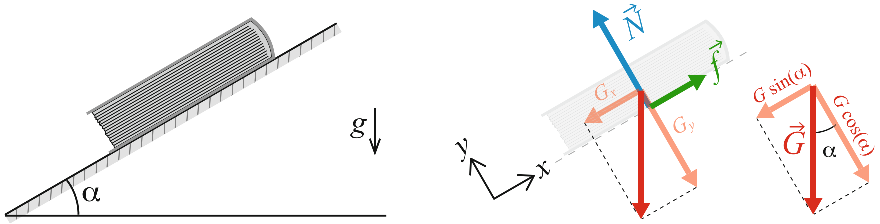

Fig. 9.8 *Left* Illustration of a book lying on an inclined table. *Right* Free-body diagram for the book

would expect the force to be higher for higher velocities. For high velocities, there will be a velocity dependence, but for most problems related to objects you typically find around you—macroscopic objects—the law holds reasonably well.

9.2.1 Example: Static Friction Forces

Problem: A book is lying on a table tilted an angle $\alpha = 30°$ with the horizontal. The coefficient of static friction between the book and the table is $\mu_s = 2/3$. What is the friction force on the book? At what tilting angle will the book start to slide?

Approach: Does the book move? We start from the assumption that the book is not moving—a special case of constrained motion—and check if the conditions for static friction are fulfilled. If they are not fulfilled, the book starts to slide.

Identify: As shown in Fig. 9.8 we choose the x-axis to be directed along the table so that forces tangential to the surface are along the x-axis and forces normal to the surface are along the y-axis.

Model: The contact forces from the table on the book are the normal force N and the friction force F. In addition, the book is affected by gravity, $G = mg$, as illustrated in the free-body diagram in Fig. 9.8.

We apply Newton's second law in the x- and y-direction, decomposing W along each direction as illustrated in Fig. 9.8. We have assumed no motion in the x-direction:

$$\sum F_x = f - G_x = ma_x = 0 \;\Rightarrow\; f = G_x = G \sin\alpha = mg \sin\alpha \;. \tag{9.33}$$

And, similarly no motion in the y-direction:

$$\sum F_y = N - G_y = ma_y = 0 \;\Rightarrow\; N = G_y = mg \cos\alpha \;. \tag{9.34}$$

As long as the book is not moving, F is a result of static friction. But there is a limit on the static friction force: $f < f_{\max} = \mu_s N$. If the static friction force exceeds this limit, the book starts to slide.

First, let us check if the static friction force is exceeded when $\alpha = 30°$ and $\mu_s = 2/3$. In this case $\cos\alpha = \sqrt{3}/2$,

$$N = mg\cos\alpha = \frac{\sqrt{3}}{2}mg \ , \tag{9.35}$$

and the maximum static friction force is:

$$f_{\text{max}} = \mu_s N = \frac{2}{3}\frac{\sqrt{3}}{2}mg = \frac{\sqrt{3}}{3}mg \ . \tag{9.36}$$

We found above that when the book is not moving the friction force is:

$$f = mg\sin\alpha = \frac{1}{2}mg \ , \tag{9.37}$$

which is smaller than the maximum friction force. The book does not slide.

Let us find at what angle the book starts to slide. Sliding occurs when the friction force becomes equal to the maximum friction force. For smaller angles than this, the book does not slide, but for an angle large than this, the book starts sliding. This angle is called the angle of marginal stability, α_m . It is found from

$$f = f_{\text{m}} \tag{9.38}$$

$$\sin(\alpha_m)\,mg = \mu_s\cos(\alpha_m)\,mg \tag{9.39}$$

$$\frac{\sin(\alpha_m)}{\cos(\alpha_m)} = \mu_s \tag{9.40}$$

$$\alpha_m = \arctan(\mu_s) \ . \tag{9.41}$$

We should check if this result is reasonable by testing the behavior for large and small values of μ_s. When μ_s is small, α_m is also small, which is the correct behavior, When μ_s becomes very large, α_m approaches $\pi/2$, which is also the correct result.

9.2.2 Example: Dynamic Friction of a Block Sliding up a Hill

Problem: In order to build a pyramid, in case you are planning for greatness, you need to push large blocks of rock up an inclined hill. (Unless, of course, you have access to a lift). It has been suggested that the Egyptians made ramps of sand in order to push and pull the rocks up to their place. Let us assume they were able to make a ramp with an inclination of $\alpha = 5°$. What force is needed to push a rock of mass 6000 kg up this hill? A typical coefficient of dynamic friction between rock (sandstone) and sand is $\mu_d = 0.6$.

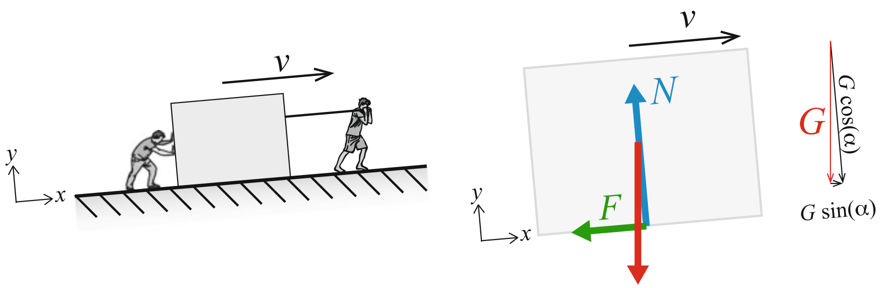

Fig. 9.9 *Left* Illustration of a rock block being pushed up along a hill. *Right* Free-body diagram for the block

Sketch and Identify: The problem is illustrated in Fig. 9.9. We have chosen the coordinate system so that x is along the inclined hill.

Model: The rock is in contact with a pushing and/or pulling mechanism: The rock may be pulled by ropes and pushed at the bottom. We call the sum of both of these forces, F, which acts along the slope, in the x-direction. The rock is also in contact with the sand slope. There is a normal force, N, from the slope on the rock, and there is friction force acting along the slope from the slope on the rock. Because the rock is moving relative to the slope, we use the dynamic friction model for the friction force, $f = \mu_d N$, acting in the negative x-direction.

We apply Newton's second law in each direction. The block does not move in the y-direction:

$$\sum F_y = N - G_y = ma_y = 0 \,, \tag{9.42}$$

where we find G_y from the inset in Fig. 9.9, $G_y = G\cos(\alpha)$, which gives

$$N = G_y = G\cos(\theta) = mg\cos(\theta) \,. \tag{9.43}$$

Newton's law of motion in the x-direction gives:

$$\sum F_x = F - f - G_x = ma_x \,. \tag{9.44}$$

Now, the smallest force exerted in order to move the block up the hill, is the force required to keep the block at constant velocity, that is, the force that makes the acceleration $a_x = 0$:

$$F = f + G_x \,. \tag{9.45}$$

From the inset in Fig. 9.9 we find that $G_x = G\sin(\alpha) = mg\sin(\alpha)$. We solve by inserting $f = \mu_d N$ into (9.44).

$$
\begin{aligned}
F &= f + G_x \\
&= \mu_d N + G_x = \mu_d mg\cos(\theta) + mg\sin(\theta) \\
&= mg\left(\mu_d\cos(\theta) + \sin(\theta)\right) \\
&= 6000\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\,\big(0.6\cdot\cos(5^\circ) + \sin(5^\circ)\big) \\
&\simeq 40.3\,\mathrm{kN}\,. \qquad (9.46)
\end{aligned}
$$

Test your understanding: The Egyptians knew that sliding a big block of rock on sand required too large forces, because the friction was too large. Therefore, they invented an improved method: They laid down wooden plates in front of the block, and poured water onto the plates. This reduced the effective dynamic coefficient of friction to $\mu_d = 0.2$. What is the force needed in this case?

9.2.3 Example: Oscillations During an Earthquake

The numerical implementation of the models for friction forces may seem straightforward, but sometimes poses unexpected problems. In this exercise you will learn to implement both dynamic and static friction in a numerical model of motion under friction.

Problem: A real, rough surface typically consists of many individual asperities—small protrusions that make up the real contacts along the surface. Here, we will address the motion of one such asperity along a contact surface during an earthquake. The system is sketched in Fig. 9.10. How can we make a simplified model of this system? The asperity will deform and slide as the two surfaces are moving relatively to each other during the earthquake. We propose a very simple model for this interaction: We model the asperity as a block of mass m, attached to the top surface with a spring of stiffness k. The block is pressed down onto the underlying surface with a force F_N. The attachment point x_0 of the spring follows the motion of the top surface. We will assume that the top surface, and therefore x_0, oscillates as $x_0(t) = A\sin\omega t$ during the earthquake. Find the motion of the asperity.

Identify and Sketch: We describe the position of the block by $x(t)$, its position relative to the bottom surface. We assume that the block starts at rest from $x(t) = 0$ at $t = 0$.

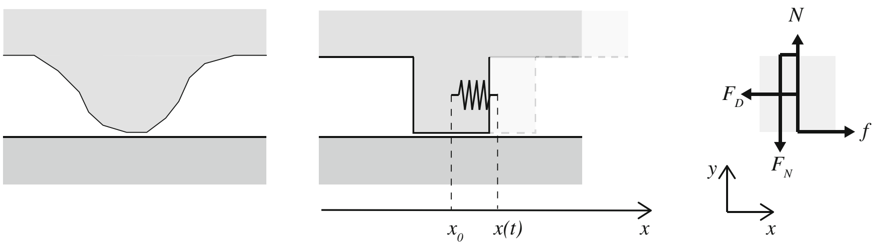

Fig. 9.10 Illustration of a single surface protrusion and its simplified model

Model: The block is affected by contact forces from the top surface: a force F_N and the spring force, F_D, due to the spring attached at point $x_0(t)$. The block is also affected by contact forces from the bottom surface: A normal contact force N and a friction force f acting along the surface. The free-body diagram is shown in Fig. 9.10.

We apply Newton's second law in the direction normal to the surface

$$\sum F_y = N - F_N = 0 \;\Rightarrow\; N = F_N \; . \tag{9.47}$$

Newton's law of motion along the surface gives:

$$\sum F_x = F_D - f = -k\,(x - x_0(t)) - f = ma \; . \tag{9.48}$$

However, we now also need a model for the friction force f, but that depends on whether the block is moving. If it is moving, the force model is

$$f = -\text{sign}(v)\,\mu_d F_N \; , \tag{9.49}$$

where μ_d is the coefficient of dynamic friction. However, if the block is not moving, the force f will correspond to the force F_D, unless it then exceeds the maximum static friction force $\mu_s\, F_N$.

Initiation of sliding: The block starts from rest and will start moving if the force f exceeds the maximum static friction force, that is, if:

$$f = F_D = -k\,(x - x_0(t)) > \mu_s F_N \; , \tag{9.50}$$

where $x = 0$ and $x_0(t) = A \sin \omega t$. The maximum value of f occurs when $x_0 = A$, which gives

$$kA > \mu_s F_N \; . \tag{9.51}$$

The block will therefore only start moving if $A > \mu_s F_N / k$.

Block dynamics: We find the motion of the block by integrating Newton's second law for the horizontal motion:

$$ma = F_D - f = -k\,(x - x_0(t)) - f \; . \tag{9.52}$$

We solve this equation numerically, using an Euler-Cromer scheme for integration:

$$v(t + \Delta t) = v(t) + a(t)\,\Delta t \tag{9.53}$$

$$x(t + \Delta t) = x(t) + v(t + \Delta t)\,\Delta t \; , \tag{9.54}$$

However, to implement the effect of the friction force f, we need to consider the *state* of the block. The block may be sliding; still; or transitioning from sliding to still or from still to sliding. If the block is sliding, we use the dynamic friction model. If the block is still, we use the static friction model. The force calculation is implemented as follows:

```
x0 = A*sin(omega*t[i])
FD = -k*(x[i]-x0)
if v[i]==0.0: # Static friction
    f = -FD
    if abs(f)>mus*N: # Slips
        f = -sign(FD)*mud*N
else:         # Dynamic friction
    f = -sign(v[i])*mud*N
Fnet = FD + f
#
```

First, we calculate the other forces, F_D, acting on the block. Then we check if the block is sticking, which occurs when the velocity is zero. In this case, the force will be the static friction force, F_s, which is equal to the other forces, $F_s = -F_D$, so that the net force is zero. However, this is only true if the static friction force is less than the friction threshold. If the force exceeds the friction threshold, the block will start moving in a direction given by the force F_D, and the block will experience a dynamic force acting in a direction opposite F_D and with a magnitude given by the dynamic friction force, $f = -\text{sign}(F_D)\mu_d N$.

Changing the frictional state: Notice that this method checks if the velocity is exactly equal to zero, but this will never occur during a simulation. During an integration step the velocity will move from being positive to being negative without ever being precisely zero. However, when the velocity of the block changes sign, the block will actually stop and start sticking to the surface with a static friction force. We therefore set the velocity to be exactly zero when the velocity changes sign. (Alternatively, you could have introduced a state variable, telling if the block is in a sliding or a sticking state). This check needs to be done during integration:

```
v[i+1] = v[i] + a*dt
if (v[i]!=0.0) and (sign(v[i+1])!=sign(v[i])):
    v[i+1] = 0.0 # The block has stopped
```

We now have all the components needed for a program to solve the motion of the block:

```
# Initialize
m = 2e-12; # kg
A = 1e-5;  # m
k = 10.0;  # N/m
N = 1e-4;  # N
T = 0.01;  # s
omega = 2*pi/T
time = 2*T
dt = time/100000
n = int(round(time/dt))
mud = 0.2
mus = 0.4
# Variables
t = zeros(n,float)
x = zeros(n,float)
```

```
v = zeros(n,float)
f = zeros(n,float)
# Initial conditions
x[0] = 0.0
v[0] = 0.0
for i in range(n-1):
    x0 = A*sin(omega*t[i])
    FD = -k*(x[i]-x0)
    if v[i]==0.0: # Static friction
        f[i] = -FD
        if abs(f[i])>mus*N: # Slips
            f[i] = -sign(FD)*mud*N
    else:           # Dynamic friction
        f[i] = -sign(v[i])*mud*N
    Fnet = FD + f[i]
    a = Fnet/m
    v[i+1] = v[i] + a*dt
    if (v[i]!=0.0) and (sign(v[i+1])!=sign(v[i])):
        v[i+1] = 0.0 # The block has stopped
    x[i+1] = x[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
```

(Here, we have introduced parameters corresponding to a cubic asperity of of size 10 μm and density 2.0g/cm^3.) The resulting dynamics is shown in Fig. 9.11.

Notice that the block moves in slips: It follows the motion of the driving surface, but sticks and slips as it moves along. We call this stick-slip motion. You should try to change the parameters in the program to see if you can find other interesting regimes of behavior.

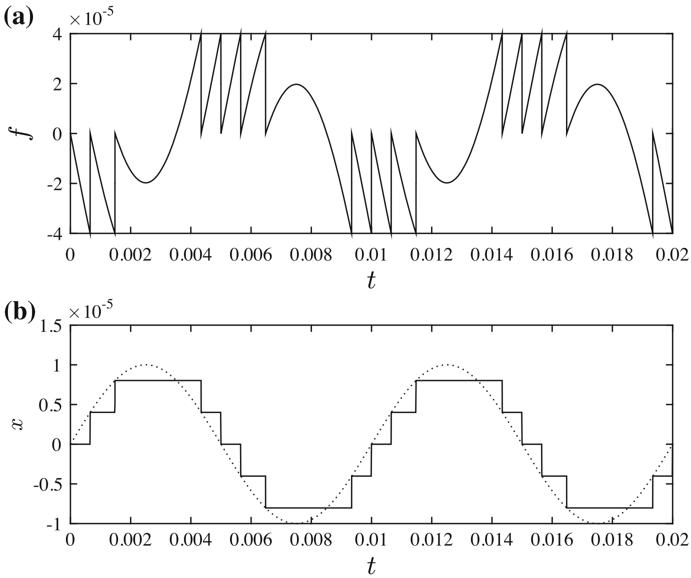

Fig. 9.11 **a** Plot of the friction force f. **b** Plot of the position x of the asperity (*solid*) and of the *top* surface, x_0 (*Dashed*)

9.3 Circular Motion

How large forces acts on a roller coaster cart as it passes through a vertical loop, or on a car as it drives through a shape turn on the road? These are questions we can answer without having a specific model for the forces acting, because the motion is constrained: A roller coaster cart driving around a loop is constrained to follow the loop, since the cart is attached to the rail.

We have already seen how we can use constraints to find forces for linear motion. Fortunately, we can use exactly the same technique to determine the forces acting on an object following a curve. However, for curved motion we know that the acceleration also has a component normal to the curve: The centripetal acceleration. For curved motion we must therefore remember that even though the object follows a particular path, there must be a net force acting on the object in order for it to follow the path.

Linear and Curved Motion

We can illustrate the difference between linear and curved motion by looking at a bead moving along a wire, as illustrated in Fig. 9.12.

If the wire is shaped as a line, the bead can be accelerated along the wire but not normal to the wire. Hence, if we analyze the forces acting on the bead and apply Newton's second law, we recognize that the net force in the x-direction along the wire corresponds to the acceleration along the wire:

$$\sum F_x = ma_x \ . \tag{9.55}$$

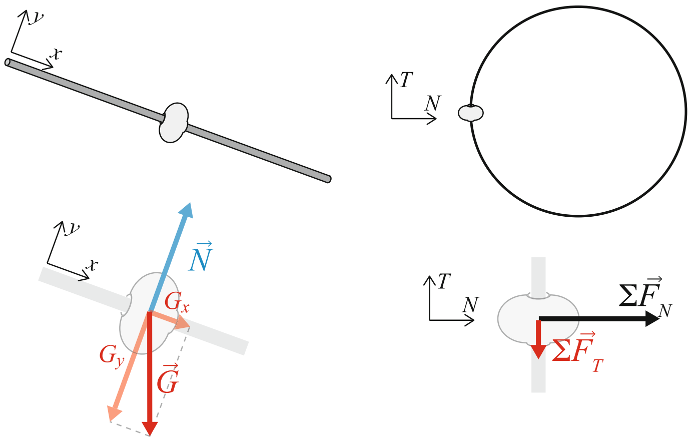

Fig. 9.12 *Left* Illustration of a bead moving along a straight (linear) wire *left* and a circular (*curved*) wire (*right*)

But in the direction normal to the wire (the y-direction), there is no motion and no acceleration. Therefore the net force is zero:

$$\sum F_y = ma_y = 0 \ . \tag{9.56}$$

If the wire is shaped as a circle, the analysis is different. Let us describe the motion of the bead using a coordinate system with one axis along the curve, the tangential axis, $\hat{u}_T$; and one axis normal to the curve pointing in towards the center of the circle, the normal axis, $\hat{u}_N$. Again, we analyze the forces acting on the bead and apply Newton's second law. The net force in the tangential direction is causing the tangential acceleration:

$$\sum F_T = ma_T \ , \tag{9.57}$$

But in this case, the motion is not linear. Therefore, even though the bead follows the wire, it is accelerated also in the direction normal to the wire because the wire is curved. The net force in the normal direction is therefore related to the centripetal acceleration:

$$\sum F_N = ma_N = m\frac{v^2}{R} \ . \tag{9.58}$$

This important difference between linear and curved motion is at the center of our analysis of curved constrained motion.

Structured Problem-Solving Approach

The method we apply when analyzing motion along a curved track consists of the following modification to the **Model** step in the structured problem-solving method:

- Find the forces acting on the object.
- Introduce models for the forces.
- In the normal direction the net force must produce the centripetal acceleration: $a_N = v^2/R$
- In the tangential direction the tangential acceleration is determined from the net tangential force.

You may then **Solve** to find the motion of the object and **Analyze** the solution to answer questions about the motion.

Types of Constraints

We can classify the constraints into two main types.

Non-conditional constraint: The constraint experienced by a bead moving along a wire. The bead follows the wire independently of the speed of the motion or any other property of the motion. Other similar cases are the motion of a weight attached to a rigid staff; the motion of a roller coaster car attached to a track; the motion of a person attached to a seat; or the motion of a train along a railway track. In the cases of a non-conditional constraint we can determine the forces and motion using the approach presented above.

Conditional constraint: The constraint experienced by a ball swung in a rope: As long as the rope is tight, the ball follows a circular path, because the rope pulls on the ball when stretched. But the rope does not exert any force when pushed. This means that the constraint in conditional: The ball is constrained to follow a circle only as long as the tensile force needed to make it follow a circle is positive. If the tensile force needed is negative, the rope cannot push the ball, and the constraint is no longer present. In this case the ball is only affected by gravity and air resistance until the rope is again tight. If you want to determine the motion of the ball when the rope is not tight, you apply Newton's laws of motion to find the acceleration and solve to find the position and velocity of the ball using methods you are now proficient in.

Examples of Constrained Systems

A car on a hill-top: Usually we assume that a car driving along a bumpy road follows the vertical motion of the road. This is really a constraint on the motion: We assume that the car follows the path given by the road. If the car drives along a flat part of the road, this gives us a way to estimate the normal force. However, the normal force can only be positive: The car is not glued to the surface. The constraint on the motion of the car is therefore conditional: It can only follow the shape of the surface if this requires a positive normal force. If a negative normal force is required in order to follow the surface, the car looses contact with the surface.

A car driving through a curve: Usually, we assume that a car driving around a turn does not slide. The car therefore follows a specific track—the track given by the road. We can therefore analyze the motion of the car around a turn as if it was constrained—following a given curved track. This requires a net force on the car to give it the necessary centripetal acceleration. An important component in the net force is the friction force from the ground on the tires. But the friction force is limited by the maximum static friction force. Hence, the motion of the car is constrained to follow the curve, but only as long as the friction force is not exceeding its maximum.

9.3.1 Example: A Car Driving Through a Curve

Problem: A car is driving through a circular curve at constant speed. The coefficient of static friction between the tires of the car and the ground is μ_s. The speed of the car is v and the radius of the circle is R. How fast can the car drive before slipping?

Approach: This is an example of a conditional constrained problem: The car is constrained to follow a circular path, but only as long as the static friction force needed to give the car the required centripetal acceleration does not exceed the maximum frictional force.

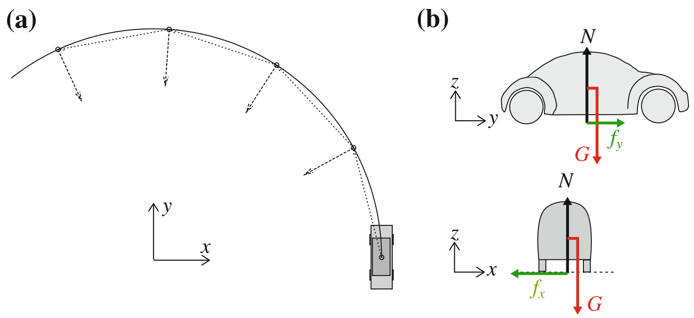

Fig. 9.13 **a** A car driving around a circular track. **b** Free-body diagram of the car

Identify: In this problem we address the motion of a car driving through a circular curve. We address the motion of the car as it passes the x-axis, as illustrated in Fig. 9.13. The motion of the car is in the xy-plane.

Model: The car is in contact with the road, giving rise to several contact forces: a normal force in the vertical direction, N, and a frictional force, f, from the ground on the car. The friction force has a component f_y along the road and a component f_x normal to the road (in the x-direction). In addition, the car is affected by the gravity, $G = mg$ in the z-direction. The free-body diagram of the car is shown in Fig. 9.13b.

We relate the forces acting on the car to its acceleration by applying Newton's second law. The car is not moving in the vertical (z) direction, hence the acceleration a_z is zero:

$$\sum F_z = N - G = N - mg = ma_z = 0 \,, \tag{9.59}$$

The normal force from the ground on the car is therefore $N = mg$.

Because the car drives at constant speed, the acceleration in the tangential direction along the road (the y-direction) is zero:

$$\sum F_y = f_y = ma_y = 0 \,, \tag{9.60}$$

Therefore the y-component of the friction force is zero.

Since the car follows a circular path, the car is accelerated in the normal direction, in toward the center of the circle. This corresponds to the $-x$ direction in Fig. 9.13. Since the car is driving with a speed v, the normal acceleration corresponds to the centripetal acceleration of an object moving in a circle of radius R:

$$\sum F_x = f_x = ma_x = m\left(-\frac{v^2}{R}\right) \,, \tag{9.61}$$

The friction force from the road on the car is therefore:

$$f = -m\frac{v^2}{R} . \tag{9.62}$$

However, we know that the maximum absolute value of the static friction force is

$$f_{\max} = \mu_s N , \tag{9.63}$$

(Notice that we use the static friction force model here because the car is not slipping). If the friction force f in (9.62) required to make the car follow the circular path exceeds the maximum static friction force, it means that our initial assumption that the motion is constrained fails and the car cannot follow a circular path. The car starts sliding at the velocity v_m when the static friction force f in (9.62) is equal to the maximum static friction force:

$$\left|-m\frac{v_m^2}{R}\right| = \mu_s N \Rightarrow v_m^2 = \mu_s g R \Rightarrow v_m = \sqrt{\mu_s g R} . \tag{9.64}$$

This means that as long as the friction force is smaller than $f_{\max}$ the car is able to follow the circular path. But if the friction force exceeds the static friction force, the car can no longer follow the circular path. Instead, it starts sliding and follows a non-circular path, which can also be found from Newton's second law. Sliding starts at a particular speed, v_m. For speeds below this, the car is able to follow the curve.

9.3.2 Example: Pendulum with Air Resistance

In this example we will address the motion of a pendulum with air resistance. The pendulum is a classic problem in mechanics, which can be solved analytically, but with the added complication of air resistance, we need to resort to numerical methods.

Problem: The pendulum consists of a ball of mass m attached to a massless rope of length L. The other end of the rope is attached to the ceiling as illustrated in Fig. 9.14. The ball starts at the bottom-most position with an initial velocity v_0. Find the motion of the pendulum with and without air resistance and discuss the differences. What initial velocity is needed for the ball to make a complete circle?

Approach: First, we will find the motion of the ball, assuming that the ball follows a circular pat, using Newton's second law. We will find the motion without air resistance analytically, and use this analytical solution to test a more general numerical method. The numerical model can be used to find the initial velocity needed to make a complete circle.

Identify: Our basic assumption is that the ball is constrained to follow a circular path. The position of the ball is measured by the distance $s(t)$ measured along the

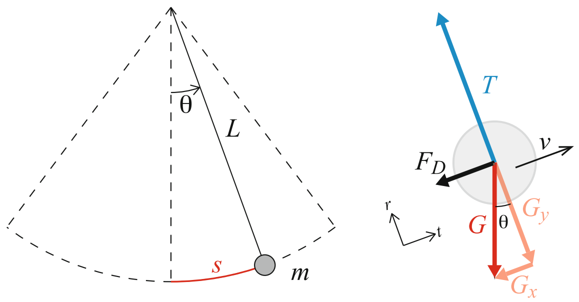

Fig. 9.14 *Left* Illustration of a pendulum. *Right* Free-body diagram of the pendulum ball

circle, starting at zero at the bottom of the circle. However, we will need to check the validity of this constraint to check that the conditions for the ball following the circular path are satisfied at all times. We can also describe $s(t)$ by the angle, $\theta(t)$, formed with the vertical as illustrated in Fig. 9.14. The arc length, $s(t)$, measured along the circle is relate to the angle, $\theta(t)$ measured in radians, by $s(t) = L\theta(t)$.

Model: The ball is in contact with the rope and the surrounding air, giving rise to the force from the rope on the ball, the rope tension, $\mathbf{T}$, and the air drag, $\mathbf{F}_D$. In addition, the ball is subject to the force from gravity, $\mathbf{G}$.

The forces are illustrated in Fig. 9.14 when the ball is at a position $s(t)$ corresponding to an angle $\theta(t)$. We decompose the forces using a local coordinate system with unit vectors $\hat{u}_T$ and $\hat{u}_N$ tangential and normal to the path respectively.

Trigonometrical decomposition: We may decompose the forces in two ways. Either we may examine the geometry in Fig. 9.14 and see that $G_x = G \sin\theta$ and $G_y = G \cos\theta$, and therefore

$$\mathbf{G} = -mg \sin\theta \hat{u}_T - mg \cos\theta \hat{u}_N \ . \tag{9.65}$$

Unit vector decomposition: Alternatively, we can find $\hat{u}_T$ and $\hat{u}_N$ expressed using θ, and use these to decompose the vectors using the dot product. From Fig. 9.14 we see $\hat{u}_N$, is directed towards the center of the circle, and that the tangential vector is normal to the normal vector:

$$\hat{u}_N = -\sin\theta \,\mathbf{i} + \cos\theta \,\mathbf{j} \ , \ \hat{u}_T = \cos\theta \,\mathbf{i} + \sin\theta \,\mathbf{j} \ . \tag{9.66}$$

From these expressions and $\mathbf{G} = -mg\,\mathbf{j}$ we find

$$G_T = \mathbf{G} \cdot \hat{u}_T = -mg \sin\theta \ , \ G_N = \mathbf{G} \cdot \hat{u}_N = -mg \cos\theta \ , \tag{9.67}$$

just as we found above. This method is more mathematical, less intuitive, but more robust—it always works if you are able to write down the unit vectors.

Air resistance model: We assume that we may use a quadratic law for the air resistance:

$$\mathbf{F}_D = -Dv\,\mathbf{v} = -D|v(t)|v(t)\hat{u}_T \ , \tag{9.68}$$

where we have used that $\mathbf{v} = v(t)\hat{u}_T$. Notice that $\hat{u}_T$ points in the positive rotational direction and that $v(t)$ may be positive and negative depending on the direction of motion.

String tension: The string tension acts in the normal direction: $\mathbf{T} = T\hat{u}_N$.

Newton's second law: Newton's second law for the ball gives:

$$\sum \mathbf{F} = \mathbf{T} + \mathbf{F}_D + \mathbf{G} = m\,\mathbf{a} \ . \tag{9.69}$$

We decompose the acceleration in a tangential and a normal component (from Chap. 8):

$$\mathbf{a} = \frac{dv}{dt}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N \ , \tag{9.70}$$

where $\rho = L$ is the radius of the circle. The net force on component form is:

$$\sum \mathbf{F} = T\hat{u}_N - Dv|v|\hat{u}_T - mg\sin\theta\hat{u}_T - mg\cos\theta\hat{u}_N = \frac{dv}{dt}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N \ . \tag{9.71}$$

The normal component of this equation is

$$T - mg\cos\theta = m\left(v^2/L\right) \ \Rightarrow \ T = mg\cos\theta + m\left(v^2/L\right) \ . \tag{9.72}$$

This allows us to calculate T given the velocity $v(t)$ and the angle $\theta(t)$. The condition for circular motion is that T is positive, since the rope can only pull and not push at the ball. If $T < 0$ our assumptions break down and the ball no longer follows a circular path.

In the tangential direction we get:

$$m\frac{dv}{dt} = -Dv|v| - mg\sin\theta = -Dv|v| - mg\sin\frac{s(t)}{L} \ , \tag{9.73}$$

where we have inserted $\theta = s(t)/L$. This gives us an equation of motion, just like we have found before, that we must now solve.

Solve: We want to find the position $s(t)$ as a function of time from (9.73). The initial conditions for the ball is that at the time $t_0 = 0$ s the ball is at $s(t_0) = 0$ m, and the initial velocity is $v(t_0) = v_0$.

Analytical solution: This problem can be solved when there is no air drag ($D = 0$) and $s(t) \ll L$. We can then approximate $\sin(s/L) \simeq (s/L)$,[3] giving the equation of motion

$$\frac{d^2 s}{dt^2} = -g \sin \frac{s(t)}{L} \simeq -\frac{g}{L} s \ , \tag{9.74}$$

which we recognize as the equation for harmonic motion, which we solved in Sect. 5.7. The general solution is

$$s(t) = A \cos \omega t + B \sin \omega t \ , \tag{9.75}$$

which is confined by the initial condition $s(0) = 0$ m/s, which gives that $A = 0$, and $v(0) = B\omega = v_0$, which gives $B = v_0/\omega$ The solution is therefore

$$s(t) = \frac{v_0}{\omega} \sin \omega t \ , \tag{9.76}$$

where we find ω by inserting $s(t)$ from (9.76) into (9.74), giving $\omega^2 = g/L$. The solution describes a periodic motion with period $T = 2\pi/\omega = 2\pi(L/g)^{1/2}$.

For small deviations and without air resistance, the pendulum will therefore oscillate back-and-forth with a period given by gravity and the length of the pendulum: A well know result from mechanics. We can use this simplified result to test our more general solution to the problem found below.

Numerical solution: The solution for small values of s is not relevant if we want to understand how far the ball can be swung. Instead, we must solve the equation of motion numerically. The differential equation of motion is:

$$\frac{d^2 s}{dt^2} = -\frac{D}{m} v(t)|v(t)| - g \sin \frac{s(t)}{L} \ . \tag{9.77}$$

with initial conditions $s(0) = 0$ and $v(0) = v_0$. In addition, we need to check that the rope tension remains positive, since our solution breaks down if this is not true. Using T from (9.72) the condition for rope tension is $T(t) = mv^2/L + mg \cos(s/L) \geq 0$.

Let us now address a specific pendulum with a ball of mass 0.5 kg, the rope has a length of $L = 1$ m, and the drag coefficient is $C_D = 0.04$ kg/m, and $v_0 = 2$ m/s. We solve the equations of motion numerically using Euler-Cromer's method in the following program:

```
from pylab import *
g = 9.8      # m/s^2
dt = 0.01    # s
time = 10.0  # s
v0 = 2.0     # s
D = 0.05     #
L = 1.0      # m
```

[3] The result $\sin u \simeq u$ when $u \ll 1$ is a result of a first order Taylor expansion.

```
m = 0.5      # kg
# Numerical initialization
n = int(round(time/dt))
t = zeros(n,float)
s = zeros(n,float)
v = zeros(n,float)
T = zeros(n,float)
# Initial conditions
v[0] = v0
s[0] = 0.0
# Simulation loop
i = 0
while (i<n AND T[i]>=0.0):
    t[i+1] = t[i] + dt
    a = -D/m*v[i]*abs(v[i])-g*sin(s[i]/L)
    v[i+1] = v[i] + a*dt
    s[i+1] = s[i] + v[i+1]*dt
    T[i+1] = m*v[i+1]^**/L + m*g*cos(s[i+1]/L)
    i = i + 1
```

Rope tension T is calculated at every time step, and the simulation is stopped if the tension becomes negative. This does not mean that the motion of the pendulum stops if tension becomes negative, but the motion is no longer described by the equation of motion in (9.77). Instead the ball will fall until the rope again becomes tight, and the ball bounces about in the rope until it comes to rest or starts swinging back and forth.

Results: The result of the simulation is shown in Fig. 9.15. The pendulum oscillates with a relatively small deflection, corresponding to small values of s. The damping due to air resistance is clearly evident. However, the damping becomes less dominating for smaller velocities, because the damping term depends on square of the velocity, $F_D \propto v^2$. If we turn air drag off completely, we can compare the numerical solution to the analytical solution in order to test our numerical model.

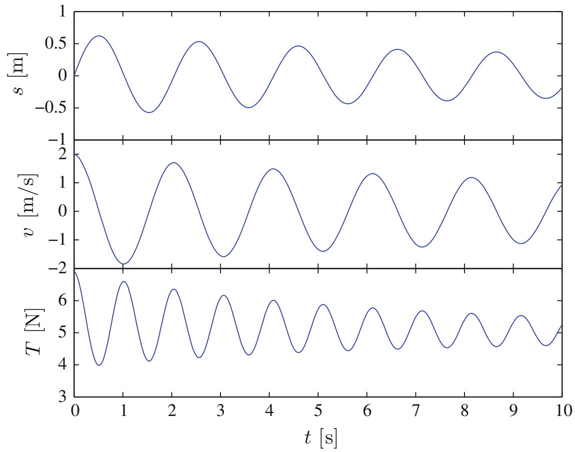

Fig. 9.15 Plot of the position, s, velocity, v, and rope tension, T for the damped pendulum with $v_0 = 2.0$ m/s

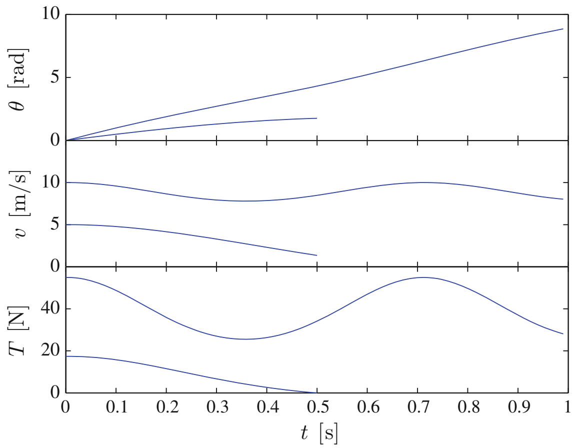

Fig. 9.16 Plot of the angle, θ, velocity, v, and rope tension, T for the undamped pendulum with $v_0 = 5.0\,\text{m/s}$ and for $v_0 = 10.0\,\text{m/s}$

Discussion: What happens if we increase the initial velocity, v_0? We expect the pendulum to continue all the way around the circle. However, we need to ensure that the rope tension, T, remains positive.

First, we turn air resistance off. We give the pendulum an initial angular velocity, and see if it goes all the way round. We start from $v_0 = 5\,\text{m/s}$. Here, the pendulum ball falls down at $s = 1.76\,\text{m}$, which corresponds to $\theta = 1.76\,\text{m}/1.0\,\text{m} \simeq 0.54\pi$. Increasing v_0 to $10\,\text{m/s}$ allows the pendulum to go all the way round without falling down, and the pendulum continues to rotate in the vertical plane. The two paths are illustrated in Fig. 9.16.

We may now use our program to find the maximum angle, θ^*, reached by the pendulum—that is the angle at which the pendulum falls down—as a function of the initial velocity: $\theta^*(v_0)$. When there is no air drag, we can solve this problem analytically using energy conservation methods, but with air drag, there is no simple analytical solution, and our numerical study would be the only practical solution.

Test your understanding: What happens when $\theta > 2\pi$ in Fig. 9.16?

Summary

Linearly constrained motion:

- An object is **linearly constrained** if it is forced to follow a straight line.
- Choose the T-axis along the motion, and N-axis normal to the motion.
- Decompose forces in the T and N directions.
- Set $a_N = 0$. Use Newton's second law to find the forces in the N-direction.
- Apply Newton's second law to find acceleration in the T-direction.

Friction force model:

- Friction forces are tangential forces at the contact between two objects.
- The friction force may depend on the normal forces at the contact.
- If the two objects in contact **are not moving** relative to each other, the friction force f is limited by $f_m = \mu_s N$, where μ_s is the coefficient of static friction.
- If the two objects in contact **are moving** relative to each other, the friction force f is $f = \mu_d N$, where μ_d is the coefficient of dynamic friction. The friction force acts against the relative movement of the objects.

Curved motion:

- Objects constrained to follow a curved path has an acceleration normal to the path given by the centripetal acceleration $a_N = v^2/R$.
- The **net force in the direction normal to the curve** is therefore always non-zero for curved motion!

Exercises

Discussion Questions

9.1 Walking on ice. Walking on ice is usually more tiring than walking on a dry road. Why?

9.2 High g. You want to perform experiments under higher g than at the surface of the Earth. How could you design a setup where you may vary g?

9.3 Spaceship loop. You want your spaceship to make a loop. How would you direct your thrusters?

9.4 Spaceship helix. You want your spaceship to make a helix-like motion (spinning around). How would you direct your thrusters?

9.5 Driving a pendulum. You hold an improvised pendulum in your hand: a rope tied to a small weight. How do you need to move your hand in order to keep the pendulum moving with an approximately constant maximum angle?

Problems

9.6 Rope with finite mass. A homogeneous rope with mass m hangs between two equally high poles. The angle between the rope and the horizontal at each of the attachment points is α.

(**a**) Find the tension at each end of the rope.
(**b**) Find the tension at the lowest point of the rope.
(**c**) Is it possible to tighten the rope so much that $\alpha = 0$?

9.7 Fireman on pole. A fireman of mass m is in hurry, and jumps onto a vertical pole leading into the garage. He holds the pole so tight that he slides downwards with constant velocity.
(**a**) Draw a free-body diagram of the fireman on the pole.
(**b**) Determine the friction force on the fireman.
(**c**) The dynamic coefficient of friction between the fireman and the pole is μ_d. Find the normal force from the fireman on the pole.

9.8 Pulling a box. You pull a box of mass m along the floor using a rope attached to a top corner of the box. The dynamic coefficient of friction between the box and the floor is μ. The rope makes an angle α with the horizontal. You pull at the rope with a force of magnitude T.
(**a**) Draw a free-body diagram of the box.
(**b**) Find the normal force from the floor on the box as a function of α and T.
(**c**) Find the acceleration of the box as a function of α and T.
(**d**) For what value of α is the acceleration of the box the maximum?

9.9 Hanging rope. A homogeneous rope of length L and mass m is lying on top of a table. A length x of the rope is hanging over the edge. The coefficient of static friction between the rope and the table is μ. How large part of the rope can hang over the edge before the rope starts to slide? (You may use the following questions as hints.)
(**a**) Divide the system into two parts: The part of the rope on the table and the part of the rope hanging over the table. Draw a free-body diagram for each of the parts.
(**b**) Find the rope tension, T, acting from one part of the rope on the other.
(**c**) Find the normal force on the part of the rope on the table.
(**d**) Find the maximum friction force, and find how large part of the rope hangs over the edge when the friction force is equal to the maximum friction force.

9.10 Pulling out a book. Two books are lying on top of each other on a table. The upper book has a mass m_1, and the lower book has a mass m_2. The coefficient of static friction between the books is μ_1. The coefficient of static friction between the book and the table is μ_2 and the coefficient of dynamic friction between the book and the table is μ_d. You pull on the lower book with a horizontal force F.
(**a**) How large must F be for you to start pulling both books along the table.
(**b**) How large must F be for you to pull out only the lower book?

9.11 Forces on a 200 m runner. A 200m runner with a mass of 70 kg has an approximately constant speed of 10 m/s through the first curve. The radius of curvature is $R = 25$ m.
(**a**) Find the friction force f on the sprinter through the curve.
(**b**) If we assume that the sprinter is not slipping, how large must the coefficient of static friction be for the sprinter to make the turn?

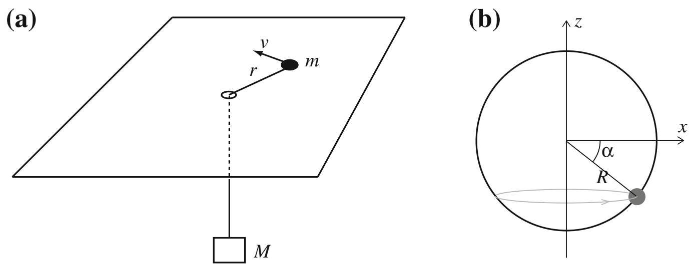

Fig. 9.17 **a** Rope through a hole. **b** Bead on a wire

9.12 Rope through a hole. A weight of mass M is hanging from one end of a massless rope. The rope passes through a small hole in the flat table. A block is attached to the other end of the rope. The block is sliding without friction on the table. The length of the rope from the hole to the block is R, and the mass of the block is m, as illustrated in Fig. 9.17a. For a particular velocity v the weight remains at a constant height. Find this velocity.

9.13 Bead on a wire. A circular bead of mass m is moving freely (without friction) along a circular wire of radius R. The wire is in an vertical plane, and the vertical plane is rotating around a vertical axis through the center of the circle with a constant period T, as illustrated in Fig. 9.17b. Find the angular position, α, of the bead on the wire.

9.14 Man in a wheel. You are trying out a new theme park attraction. You walk into a vertical cylinder of radius $R = 3$ m. You are all told to lean your backs towards the round inner walls of the cylinder. The cylinder starts spinning, slowly picking up speed. Suddenly, the floor drops down, but you do not fall down!

The coefficient of static friction between you and the wall is $\mu_s = 0.2$. How fast must the cylinder spin to ensure that you do not fall down?

9.15 Motorcycle in a loop. As a stunt motorcyclist you are trying to run through a vertical loop with radius 3m. What speed do you need to have at the top of the loop in order not to fall down?

Projects

9.16 Stick-slip friction. In this project we will study a phenomenon called stick-slip friction. If you pull a block along a flat table with a soft spring, you will find that the block does not move continuously with a constant velocity, instead it moves in small jumps. This intermittent motion is called stick-slip friction, and it is the origin of the high-frequency vibrating tone you often hear from wheels that are not well

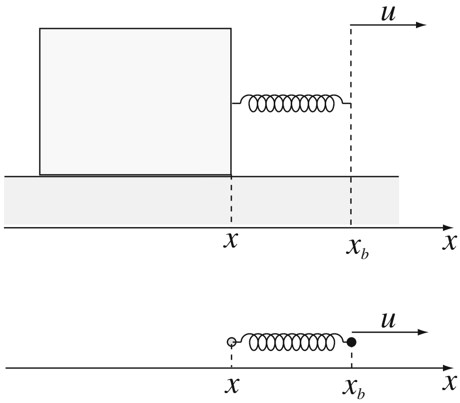

Fig. 9.18 Illustration of a block pulled by a spring

lubricated. It is also one of the basic mechanisms leading to the wide distribution of earthquake sizes.

Here, we will introduce and study a model for stick-slip friction for a block pulled by a spring sliding over a flat, horizontal surface, as illustrated in Fig. 9.18.

The block has mass m. A massless spring (with spring constant k and equilibrium length b) is attached to the block at the point x. The free end (the right-hand end in Fig. 9.18) of the spring is at the point x_b. We move x_b, the free end of the spring, with a constant velocity u. The static and dynamic coefficients of friction for the contact between the block and the bottom surface are μ_s and μ_d respectively. The acceleration of gravity is $g = 9.8\,\mathrm{m/s^2}$.

The block starts at the position $x(t_0) = 0$ at the time $t_0 = 0$. The position x_b of the free end of the spring is $x_b(t_0) = x(t_0) + b$ at t_0.

(**a**) Draw a free-body diagram for the block.
(**b**) Find the position of the spring attachment point $x_b(t)$ as a function of time.
(**c**) Show that the force, **F** on the block from the spring is $\mathbf{F} = k(x_b - x - b)\mathbf{i}$.
Stationary state: First, let us characterize the stationary state, where the block is moving at a *constant* velocity.
(**d**) Identify the forces acting on the block and draw a free-body diagram for the block in the stationary state.
(**e**) Introduce force models for all the forces acting on the block. Find the normal force, N, on the block.
(**f**) Find the acceleration of the block in the stationary state.
(**g**) Find the elongation ΔL of the spring in the stationary state.
(**h**) Find the position $x(t)$ of the block as a function of time in the stationary state.
Starting from rest: Let us now address the situation where the block starts from rest. That is, we assume that the block starts at $x(t_0) = 0\,\mathrm{m}$ with $v(t_0) = 0\,\mathrm{m/s}$ at the time $t_0 = 0\,\mathrm{s}$.
(**i**) Identify the force acting on the block and draw a free-body diagram of the block before the block starts moving. Introduce force models for all the forces.

(**j**) Find the elongation ΔL of the spring at the instant the block starts moving.
(**k**) Assume that the block starts at rest. Find the friction force on the block as a function of time in the period before the block starts moving. Sketch the friction force as a function of time until some time after the block has started moving.
(**l**) Show that the acceleration of the block immediately after it starts moving is $a = (k/m)\,(x_b - x - b) - \mu_d g$. Explain why you cannot use this relation for the acceleration to determine the subsequent motion of the block.
General motion: Now, we will develop a *general method* to find the motion, $x(t)$, of the block. First, we study the case when $u = 0$ m/s and the coefficients of friction are zero, $\mu_s = \mu_d = 0$.
(**m**) Find an expression for the horizontal acceleration of the block. Show that $x(t) = (v_0/\omega)\sin\omega t$, where $\omega = (k/m)^{1/2}$, when $v(0) = v_0$.
(**n**) Write a numerical algorithm to find the position and velocity of the block at a time $t_i + \Delta t$, $x(t_i + \Delta t)$ and $v(t_i + \Delta t)$, given the position and velocity of the block at a time t_i, $x(t_i)$ and $v(t_i)$.
(**o**) Implement the numerical algorithm in a program to find the position of the block as a function of time for $m = 0.1$ kg, $k = 100$ N/m, $b = 0.1$ m and $v_0 = 0.1$ m/s. Plot the behavior for a simulation of 2 s, and compare the result of your program with exact solution. Ensure that you choose a time-step Δt the reproduces the exact solution with sufficient accuracy. What happens if you choose a too large time-step Δt?

Let us now address the situation when the block is pulled at a finite velocity, u.
(**p**) Modify your program to find the position of the block when $u = 0.1$m/s and the block starts at rest. In this case, the exact solution is:

$$x(t) = ut - \frac{u}{\omega}\sin\omega t\ . \tag{9.78}$$

Compare your result with the exact solution by plotting both the simulated x and the exact x in the same plot.
General motion with friction: Finally, we address the full complexity of the situation, and introduce non-zero friction forces.
(**q**) Modify your program to include friction using $\mu_s = 0.6$, $\mu_d = 0.3$. Show a plot of $x(t)$ for $m = 0.1$ kg and for $m = 1.0$ kg.
(**r**) Plot the spring force F on the block as a function of time for both values of m and explain the differences.
(**s**) What happens if you instead decrease k to $k = 10$ N for $m = 0.1$ kg. Can you explain the behavior?

9.17 Feather in tornado. In this project you will learn to use Newton's laws and the force model for air resistance in a wind field to address the motion of a light object in strong winds. We start from a simple model without wind and gradually add complexity to the model, until we finally address the motion in a tornado.
Motion without wind: First, we address the motion of the feather without wind.
(**a**) Identify the forces acting on a feather while it is falling and draw a free-body diagram for the feather.

(**b**) Introduce quantitative force models for the forces, and find an expression for the acceleration of the feather. You may assume a quadratic law for air resistance.
(**c**) If you release the feather from rest, its velocity will tend asymptotically toward the terminal velocity, v_T. Show that the terminal velocity is $v_T = -(mg/D)^{1/2}$, where D is the constant in the air resistance model.
(**d**) We release the feather from a distance h above the floor and measure the time t until the feather hits the floor. You may assume that the feather falls with a constant velocity equal to the terminal velocity. Show how you can determine D/mg by measuring the time t. Estimate D/mg when you release the feather from a height of 2.4 m above the floor and it takes 4.8 s until it hits the floor.
(**e**) We will now develop a more precise model where we do not assume that the velocity is constant. You release the feather from the height h at the time $t = 0$ s. Find the equation you have to solve to find the position of the feather as a function of time. What are the initial conditions?
(**f**) Write a program that solves this equation to find the velocity and position as a function of time t. Use the parameters you determined above, and test the program by ensuring that it produces the correct terminal velocity.
(**g**) Fig. 9.19 shows the position and velocity calculated with the program using the parameters found above. Was the approximation in part (d) reasonable? Explain your answer.
Model with wind: We have now found a model that can be used to find the motion of the feather. We will now find the motion of the feather in three dimensions while it is blowing. The velocity of the wind varies in space, so that the wind velocity **w** is a function of the position **r**. We write this as **w** = **w**(**r**).
(**h**) Find an expression for the acceleration of the feather. The expression may include the wind velocity **w**(**r**). Let the z-axis correspond to the vertical direction.
(**i**) Assume that the feather is moving in an approximately horizontal plane—that is you may assume that the vertical acceleration is negligible. How does the wind have

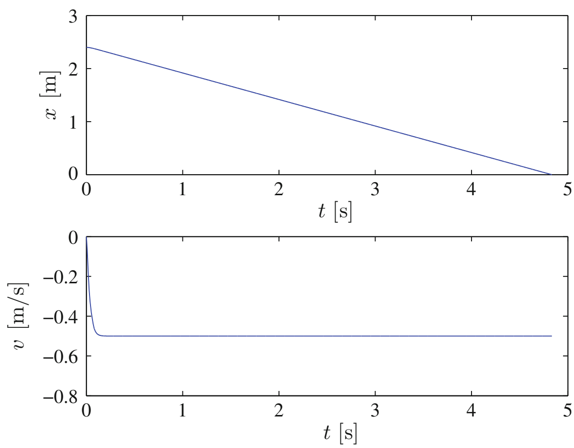

Fig. 9.19 Result of a simulation

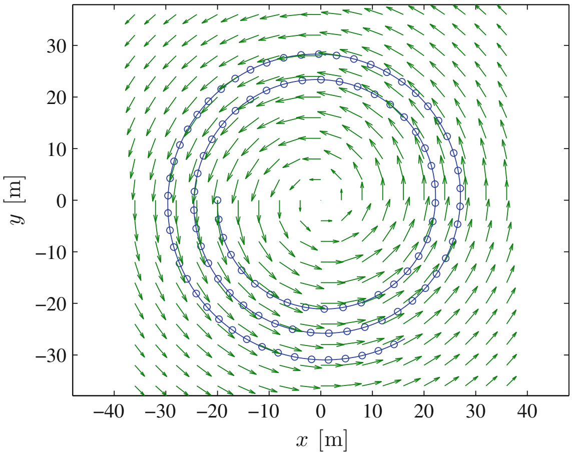

Fig. 9.20 Illustration of the velocity field of the tornado and a path for a feather released in a tornado

to blow in order for the feather to move in a circular orbit of radius r_0 with a constant speed v_0?

Motion in a tornado: For a tornado with a center at the origin, the wind velocity is expected to be approximately given by the model:

$$\mathbf{w}(\mathbf{r}) = u_0 r e^{-r/R} \hat{u}_\theta = u_0 \, (-y, x, 0) \, e^{-r/R} \ , \tag{9.79}$$

where u_0 is a characteristic velocity for the wind, R is the radius of the tornado, and $\hat{u}_\theta$ is a tangential unit vector in the horizontal plane ($\hat{u}_\theta$ is normal to $\mathbf{r}$). Here, $\mathbf{r} = (x, y, z)$, and $r = |\mathbf{r}|$. The velocity field is illustrated in Fig. 9.20.

(**j**) Is is possible to choose an appropriate set of initial conditions so that the feather moves in a circular path in the tornado? Explain your answer.

(**k**) Rewrite your program to find the velocity and position of the feather as a function of time. For the tornado you may use the values $u_0 = 100$ m/s and $R = 20$ m.

(**l**) Find the trajectory for the feather if it is released from rest from a height of 2.4m, and in a position corresponding to $\mathbf{r} = -R\,\mathbf{i} + h\mathbf{k}$.

(**m**) The trajectory of the feather is shown in Figs. 9.20 and 9.21. Compare the results with what happended when you dropped the feather without wind. Why does the feather now take longer to reach the ground?

9.18 Modelling Atomic Interactions. In this project we will create a model for an atom with mass m in the vicinity of two large molecules. We will assume that the molecules are massive and do not move. We will first look at a one-dimensional movement along the x-axis, and later expand to higher dimensions.

The interactions between the atom and the molecules can be described by the potential

$$U(x) = U_0 \left(\left[\left(\frac{x}{d} \right)^2 - 1 \right]^2 \right) = \frac{U_0}{d^4} (x^2 - d^2)^2 = \frac{U_0}{d^4} (x^4 - 2x^2 d^2 + d^4) \tag{9.80}$$

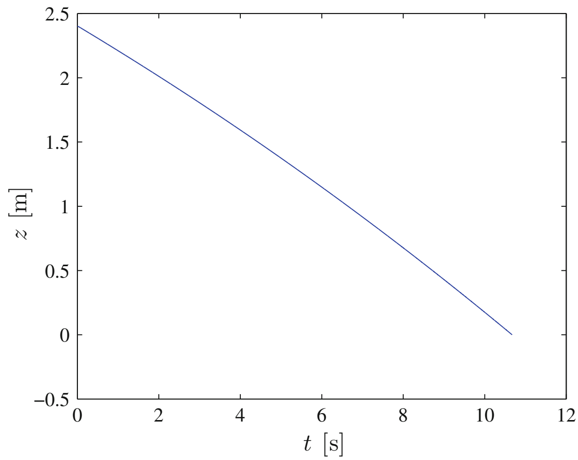

Fig. 9.21 Illustration of the (vertical) trajectory of a feather released from rest at $\mathbf{r} = -R\mathbf{i} + h\mathbf{k}$ at $t = 0$s

where U_0 is a known constant measured in Joules, x is the position of the atom, and d is a known length. We will assume all other forces on the atom are small in comparison, and neglect them in our model.
(**a**) Make an energy diagram. Find the equilibrium points, mark these on the diagram and characterize their stability.
(**b**) Choose two different energies that give two distinct types of motions, draw them into the diagram, and describe the motion in each case.
(**c**) If the atom starts at rest at $x = 2d$, what is the velocity of the atom at the point $x = d$?
(**d**) If the atom starts at rest at $x = d$ with the velocity v_0, how large must v_0 be for the atom to reach the point $x = -d$?
(**e**) Show that the acceleration of the atom is

$$a = -\frac{4U_0}{md^4}(x^3 - xd^2) \tag{9.81}$$

(**f**) Write a numerical algorithm to find the position and velocity of the atom at a time $t + \Delta t$, given the position and velocity of the atom at a time t.
(**g**) Implement the numerical algorithm in a program to find the position of the atom as a function of time from $t = 0$ns to $t = 10$ ns with a timestep of $\Delta t = 0.01$ ns for $m = 1$, $d = 0.1$ nm and $U_0 = 1$ nJ.
(**h**) Make a plot of two distinct x(t) from $t = 0$ ns to $t = 10$ ns, the first from a running of the program with the initial conditions $x_0 = d$ and $v_0 = 0.5$ m/s, and the other from a running of the program with the initial conditions $x_0 = d$ and $v_0 = 1.5$ m/s. Make another plot of v(t) for the same initial conditions.
(**i**) Describe the behavior of the atom in both simulations and sketch the motion in an energy-diagram.

We will now look at the same system, but in two dimensions. The Atom interacts with a surface in such a way that the potential of the atom is given as

$$U(r) = U_0\left(\left[\left(\frac{r}{d}\right)^2 - 1\right]^2\right) = \frac{U_0}{d^4}(r^2 - d^2)^2 = \frac{U_0}{d^4}(r^4 - 2r^2d^2 + d^4) \tag{9.82}$$

where $r = \sqrt{x^2 + y^2}$ is the distance to the origin. (We can no longer interpret this interaction as an effect from two molecules. Instead we interpret it as an approximation of a more complicated interaction from many surrounding atoms.)
(**j**) Show that the acceleration on the atom can be written

$$\mathbf{a} = -\frac{4U_0}{md^4}(r^3 - rd^2)\frac{\mathbf{r}}{r} \tag{9.83}$$

(**k**) Rewrite your program to find the velocity and position of the atom using the new expression for the force F. Use vectorized expressions in your code.
(**l**) Plot the motion of an atom starting in $\mathbf{r_0} = (d, 0)$ from $t = 0\,\text{ns}$ to $t = 20\,\text{ns}$ for the initial velocities $\mathbf{v} = (0, 0.5\,\text{m/s})$, $\mathbf{v} = (0, 1\,\text{m/s})$ and $\mathbf{v} = (0, 1.5\,\text{m/s})$.
(**m**) Can you choose initial conditions $\mathbf{r_0}$ and $\mathbf{v_0}$ in such a manner that the atom moves in a circular orbit with a constant radius? If so, what initial conditions are those? Plot the motion for these conditions.

Chapter 10
Work

How can you find the motion of an atom moving near a surface according to a complicated position-dependent force without solving Newton's equation?

Up to now, you have learned to use Newton's laws of motion to determine the motion of an object based on the forces acting on it. The methods you have learned are completely general and can always be applied to solve a problem. Unfortunately, in many cases we cannot find an exact solution to the equations of motion we get from Newton's second law.

Here we introduce a commonly used technique that allows us to find the velocity as a function of position without finding the position as a function of time—an alternate form of Newton's second law. The method is based on a simple principle: Instead of solving the equations of motion directly, we integrate the equations of motion. Such a method is called an *integration method*. You will learn two integration methods: In this chapter we integrate Newton's second law in space using the work-energy theorem to find the speed as a function of position; in Chap. 12 we integrate Newton's second law in time to get conservation of momentum. While these methods are simple from a mathematical point of view, they introduce very important physical concepts that you will rely on throughout your career. You should therefore pay more attention to the use of these methods than to their derivation.

In this chapter, we introduce the work-energy theorem as a method to find the velocity as a function of position for an object even in cases when we cannot solve the equations of motion. This introduces us to the concept of work and kinetic energy—an energy related to the motion of an object. Finally we also address the rate of work done by a force—the power.

10.1 Integration Methods

In principle, we can determine the motion of any object if we know the net force, $\mathbf{F}^{\text{net}}$ acting on the object, by applying Newton's second law:

$$m\,\mathbf{a} = \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \ . \tag{10.1}$$

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_10

and solve to the position at a time t, $\mathbf{r}(t)$, if we start from $\mathbf{r}(t_0)$. Since (10.1) is true, the integral of this equation must also be valid for the motion. Integral, you ask, what integral? Both the integral over time and the integral over the actual motion—the curve integral along the motion. The following two integrals of (10.1) also describe the motion:

$$\int_{t_0}^{t} m\,\mathbf{a}\,dt = \int_{t_0}^{t} \mathbf{F}^{\mathrm{net}}(\mathbf{r}, \mathbf{v}, t)\,dt \ , \tag{10.2}$$

$$\int_C \mathbf{a} \cdot d\mathbf{r} = \int_C \mathbf{F}^{\mathrm{net}}(\mathbf{r}, \mathbf{v}, t) \cdot d\mathbf{r} \ . \tag{10.3}$$

Ok. This may be true, but it seems entirely unmotivated. Why would we want to do this? Bear with us. It turns out that both these integrals are very useful and introduce powerful new concepts. In this chapter, we will focus on the integral in (10.3), while in the next chapter we focus on (10.2).

Path Integral

To understand the motivation, let us look at the integral in (10.3) in detail and calculate the left-hand side. What does the integral in (10.3) mean? It is the path integral along the curve, $\mathbf{r}(t)$. However, this integral may depend not only on the path, but also on how we move along the path—it may depend on the velocity $\mathbf{v}(t)$ along the curve. We should therefore replace the $d\mathbf{r}$ with $(d\mathbf{r}/dt)dt$ on both sides of the equation. The equation is still true since it is simply an integral of Newton's second law:

$$\int_{t_0}^{t} m\,\mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t_0}^{t} \mathbf{F}^{\mathrm{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \frac{d\mathbf{r}}{dt}\,dt \ . \tag{10.4}$$

Again, you may ask why this is useful. The short answer is that it is useful because we always can find the analytical solution to the left-hand side and we sometimes can find the solution to the right-hand side, even if we cannot find the analytical solution to the acceleration from Newton's second law.

What is the left-hand side of (10.4)? It can be solved using integration by parts:

$$\int_{t_0}^{t} m\,\mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t_0}^{t} m\,\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\,dt = \mathbf{v} \cdot \mathbf{v} - \int_{t_0}^{t} \mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\,dt \ , \tag{10.5}$$

which gives

$$\int_{t_0}^{t} m\,\mathbf{a} \cdot \frac{d\mathbf{r}}{dt}\,dt = \frac{1}{2} m\mathbf{v} \cdot \mathbf{v} \ . \tag{10.6}$$

The left-hand side of (10.4) therefore only depends on the magnitude of the velocity! We can therefore find the change in velocity, if we only can calculate the integral on the right-hand side of (10.4). The resulting integral equation is called the Work-energy theorem:

Work-energy theorem: For any motion $\mathbf{r}(t)$, we can find the change in velocities from the integral $W_{0,1}$:

$$W_{0,1} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \frac{d\mathbf{r}}{dt}\, dt = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 \,. \tag{10.7}$$

Application of the Work-Energy Theorem

The power of the work-energy theorem is best demonstrated by an example. For an atom moving along a surface, as shown in Fig. 10.1, the force from the surface on the atom can ba approximated as:

$$F(x) = -F_0 \sin \frac{2\pi x}{b} \,, \tag{10.8}$$

where x is the position of the atom and b is the distance between the atoms on the surface. If we apply Newton's second law to find the motion of the atom, we get

$$\sum F_x = F(x) = -F_0\, sin \frac{2\pi x}{b} = ma \;\Rightarrow\; a = -\frac{F_0}{m}\, sin \frac{2\pi x}{b} \,, \tag{10.9}$$

which we can solve numerically, but not analytically. However, we can use the Work-energy integral to find the velocity as a function of position for this motion. We calculate the work integral

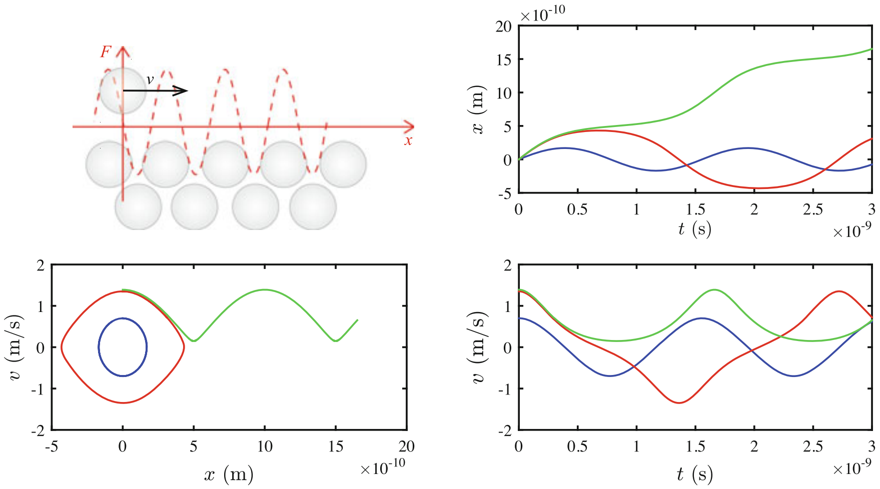

Fig. 10.1 Illustration of an atom moving along a period lattice of atoms, giving rise to a periodic force, $F(x) = -F_0 \sin kx$

$$\int_{t_0}^{t} F(x(t))\,\frac{dx}{dt}\,dt = \int_{x(t_0)}^{x(t)} F(x)\,dx = \int_{x_0}^{x} -F_0 \sin\frac{2\pi x}{b}\,dx \tag{10.10}$$

$$= \frac{F_0 b}{2\pi}\left(\cos\frac{2\pi x}{b} - \cos\frac{2\pi x_0}{b}\right) . \tag{10.11}$$

If the motion starts from $x_0 = 0$ with $v = v_0$, we get

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{t_0}^{t} F(x(t))\,\frac{dx}{dt}\,dt = \frac{F_0 b}{2\pi}\left(\cos\frac{2\pi x}{b} - 1\right) . \tag{10.12}$$

and for the velocity, we find

$$v(x) = \pm\sqrt{v_0^2 + \frac{F_0 b}{m\pi}\left(\cos\frac{2\pi x}{b} - 1\right)} . \tag{10.13}$$

where the sign depends on what direction the atom is moving in. This expression is a complete solution of the motion. Figure 10.1 illustrates the numerical solution for $x(t)$ and $v(t)$ for various initial velocities and positions, and the corresponding plot of $v(x)$. We have plotted the analytical solution on top using circles. Notice the interesting pattern in this figure. We will spend more time developing our understanding of this model further on.

10.2 Work-Energy Theorem

The work-energy theorem is an alternative form of Newton's second law and therefore has the same range of applicability. We call the path integral along the curve the **work of the net force**:

$$\boxed{W_{0,1}^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}, \mathbf{v}, t) \cdot \mathbf{v}\,dt\ .} \tag{10.14}$$

But this definitions seems to require that we know the both $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in order to solve the integral. Hmmmm. Was not the whole point that we did need to find the analytical solution?

The usefulness of the formulation first comes to its right when the net force depends on the position *only*—that is, when the net force does *not* depend on the velocity:

$$W_{0,1}^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot \mathbf{v}\,dt = \int_C \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot d\mathbf{r}\ . \tag{10.15}$$

In this case, we may be able to solve the integral, as we saw above, even if we cannot solve to find the motion. This gives the **work-energy theorem** for a position-dependent force:

$$W_{0,1}^{\text{net}} = \int_C \mathbf{F}^{\text{net}}(\mathbf{r}) \cdot d\mathbf{r} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \,. \tag{10.16}$$

This equation has an even simpler form in one dimension when $\mathbf{F} = F\mathbf{i}$ and $d\mathbf{r} = dx$, giving

$$\boxed{W_{0,1}^{\text{net}} = \int_{x_0}^{x} F(x)\,dx = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \,.} \tag{10.17}$$

It is usual to introduce the term **kinetic energy**, K, for the right-hand side in the work-energy theorem

$$\boxed{K = \frac{1}{2}mv^2 \,.} \tag{10.18}$$

This is the reason why we call the theorem the **work-energy theorem**. And we can now formulate it very compactly:

$$W_{0,1}^{\text{net}} = K_1 - K_0 \,. \tag{10.19}$$

Where it is usual to drop the subindex 0, 1 for the work.

Unit of work: The unit for work is *Joule* (J), which is defined as: $1\,\text{J}\,(\text{Joule}) = 1\,\text{Nm} = 1\,\text{kgm}^2/\text{s}^2$.

Comments on the Work-Energy Theorem

The work-energy theorem has several important features:

- The work-energy theorem is an alternative formulation of Newton's second law of motion, and is therefore valid as long as Newton's laws are valid. For example, it is only valid in an inertial system. It is not valid in an accelerated coordinate system.
- The work-energy is only true if you find the work of the *net force*. Do not forget or leave out any of the forces acting on the object.
- Notice that most microscopic laws of motion, including all interatomic interactions, only depend on position. The same is true for gravitational forces between astronomical objects. There is a large span of processes where the net force is only position dependent. The special formulation in (10.17) is therefore a very useful law.

For example, if you take a block and pull it back and forth a few times on the floor, you cannot use (10.17) to find the work because both the friction force from the floor on the block and the driving force (you pulling or pushing the block) varies not only with position, but also with time and velocity. After pushing the block back and forth you end up at the same place. If you used (10.17), the work would therefore be zero, which is incorrect. You have performed work on the block even if the block ends in the same place it started.

Superposition of Work

The work-energy theorem is only valid for the work done by the net force. If there are several forces acting on an object, we may number the forces $\mathbf{F}_j$ from $j = 1$ to $j = n$. The net force is the sum of the forces:

$$\mathbf{F}^{\text{net}} = \sum_j \mathbf{F}_j \ , \tag{10.20}$$

The work done by the net force can therefore be written as:

$$W^{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = \int_{t_0}^{t_1} \sum_j \mathbf{F}_j \cdot \mathbf{v}\, dt = \sum_j \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt = \sum_j W_j \ . \tag{10.21}$$

where W_j is the work done by force j. The work done by the net force is therefore the sum of the work done by each of the forces acting. For a one-dimensional motion with a position-dependent force, $\mathbf{F} = F(x)\,\mathbf{i}$ this simplifies to:

$$W_j = \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt = \int_{x_0}^{x} F(x)\, dx \ . \tag{10.22}$$

The Concept of Work

I am sure you have an intuition of what physical work is, but does that correspond to our definition of mechanical work? Intuitively, it requires work to push a box along the floor. The longer we push, the more work it requires. The heavier the box (and hence the larger the friction force), the more work is done. Here, our intuition is consistent with our definition.

However, if you lift a box from the floor, it requires work. Both in the ordinary use of the word and in our precise definition. But I am sure you know that just holding a heavy box in your arms requires effort, although it requires no work according to our definition. Trying to push a very heavy box without succeeding also requires no mechanical work, but still requires effort on your behalf. The reasons for this discrepancy are related to how we perceive and experience trying to move something, to how our muscles work inside our bodies, and to how we perceive movement: your body may still move somewhat while the box is kept approximately at a constant height.

However, a mechanical analysis of the work done when you move your body is useful, and does result in real effects that you can feel. It is, for example, possible to design a backpack that requires less effort to carry long distances based on our understanding of work (and energy conservation).

Most importantly, it is important to try to separate the very precise definition of mechanical work from the looser concept from everyday speech.

10.3 Work Done by One-Dimensional Force Models

We apply the work-energy theorem to various force models, such as a constant force, a spring force, and a given position-dependent force—all in one dimension.

Work of a Constant Force

The work done by a constant force, $F = F_0$, as a car accelerates/decelerates from x_0 to x_1 is

$$W = \int_{t_0}^{t_1} F_0 \, v \, dt = \int_{x_0}^{x_1} F_0 \, dx = F_0 \, (x_1 - x_0) = F_x \Delta x \; . \tag{10.23}$$

If the force F_0 is the only force (or the net force) on the car, the work W corresponds to the change in kinetic energy.

- Notice that if the force and the displacement, Δx, are in the same direction, the **work is positive**. If this is the net force, it means that the kinetic energy increases, and that the speed increases.
- If the force and the displacement are in opposite direction, for example if the car is moving in the positive x-direction while it is breaking with a constant force in the negative x-direction, the **work is negative**. If this is the net force on the car, the kinetic energy decreases and the speed decreases.

Work of a Spring Force

One of the most commonly used models for a contact force is the spring model. What is the work done by a spring force? Figure 10.2 illustrates the motion of a block on a frictionless horizontal surface. The block is attached to a spring with spring constant k. The other end of the spring is attached at the origin, $x = 0$, and the equilibrium length of the spring is L_0. The force, F_x, from the spring on the block is then:

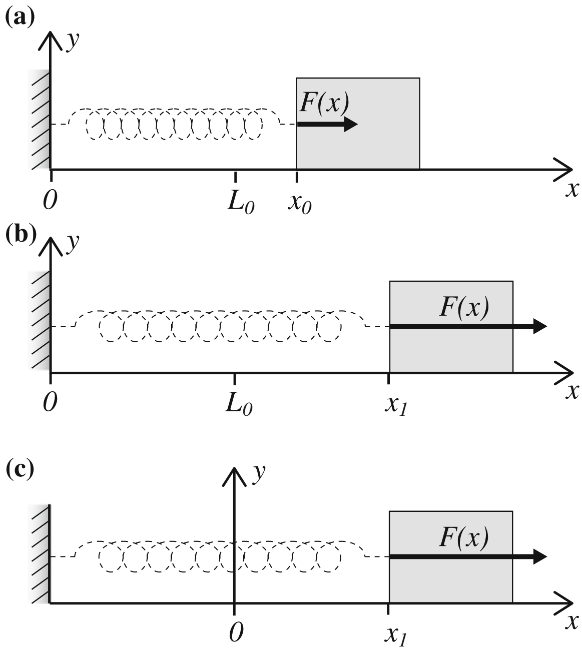

Fig. 10.2 A block attached to a spring on a frictionless surface. **a**, **b** The friction force F is illustrated at two different positions x_0 and x_1 of the block. **c** The origin is moved to the equilibrium position for the spring

$$F_x = -k\,(x - L_0) \ . \tag{10.24}$$

The position, x, where the spring force is zero is called the *equilibrium position*. Here, the equilibrium position is $x = L_0$.

The work done by the spring force on the block as the block moves from $x(t_0) = x_0$ to $x(t_1) = x_1$ is:

$$W_{0,1} = \int_{t_0}^{t_1} F_x v_x dt = \int_{x_0}^{x_1} F(x) dx = \int_{x_0}^{x_1} -k\,(x - L_0)\, dx = \ . \tag{10.25}$$

We change integration variable to $u = x - L_0$, $du = dx$, getting:

$$W = \int_{x_0 - L_0}^{x_1 - L_0} -ku\, du = \frac{1}{2}k(x_0 - L_0)^2 - \frac{1}{2}k(x_1 - L_0)^2 \ . \tag{10.26}$$

This result becomes simpler if we move the origin to the equilibrium position of the spring, as shown in Fig. 10.2c, so that $F(x) = -kx$. The work from x_0 to x_1 is then

$$W = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2 \ . \tag{10.27}$$

Work of a Position-Dependent Force

The work of a position-dependent force $F(x)$ is found through the integral

$$W = \int_{x_0}^{x} F(x)\,dx \ . \tag{10.28}$$

This force $F(x)$ may be a simple function, such as $F(x) = -kx$ or $F(x) = F_0 \sin 2\pi x/b$. In that case you can simply solve the integral analytically. But what if you cannot solve the integral analytically or the function $F(x)$ is not know exactly, but instead is measured in a discrete number of points, x_i. How can you then find the work?

Symbolic Integration of a Function $F(x)$

Even if *you* cannot (or you are too lazy to) solve the integral analytically, you can always check if Python can the indefinite integral symbolically. We demonstrate this for a force $F(x) = 1/(a + x^2)$ for $x > 0$. We integrate this function using the symbolic package in Python as follows:

```
>> from sympy import *
>> a = Symbol('a', real=True, positive=True)
>> x = Symbol('x')
>> integrate(1/(a+x**2),x)
atan(x/sqrt(a))/sqrt(a)
```

Numerical Integration of a Function $F(x)$

However, if we cannot find an analytical solution to the integral of $F(x)$, we can always calculate the definite integral numerically. The integral from x_0 to x_1 of $F(x)$ in Fig. 10.3 corresponds to the area under the curve from x_0 to x_1. We can calculate the area by first dividing the interval into smaller pieces, finding an approximate value for the area of each such piece, and summing the areas to find total area corresponding to the integral.

We divide the interval from x_0 to x_1 into n intervals of length $\Delta x = (x_1 - x_0)/n$ so that interval i spans from x_i to $x_{i+1} = x_i + \Delta$. (See Fig. 10.3). The area under the curve $F(x)$ over the interval from x_i to x_{i+1} is the integral:

$$W_{i,i+1} = \int_{x_i}^{x_{i+1}} F(x)\,dx \ . \tag{10.29}$$

When Δx is small, the area under the curve is approximately equal to the area of a rectangle of width Δx and a height given by the value of $F(x)$ at x_i, as illustrated in Fig. 10.3b. The area of this rectangle is $W_{i,i+1} = A_i \simeq \Delta x\, F(x_i)$, and the total area is the sum of all the areas A_i, $i = 1, \ldots, n$ as illustrated in Fig. 10.3b:

$$W_{0,1} = \sum_{i=1}^{n} W_{i,i+1} = \sum_{i=1}^{n} \Delta x\, F(x_i) \ . \tag{10.30}$$

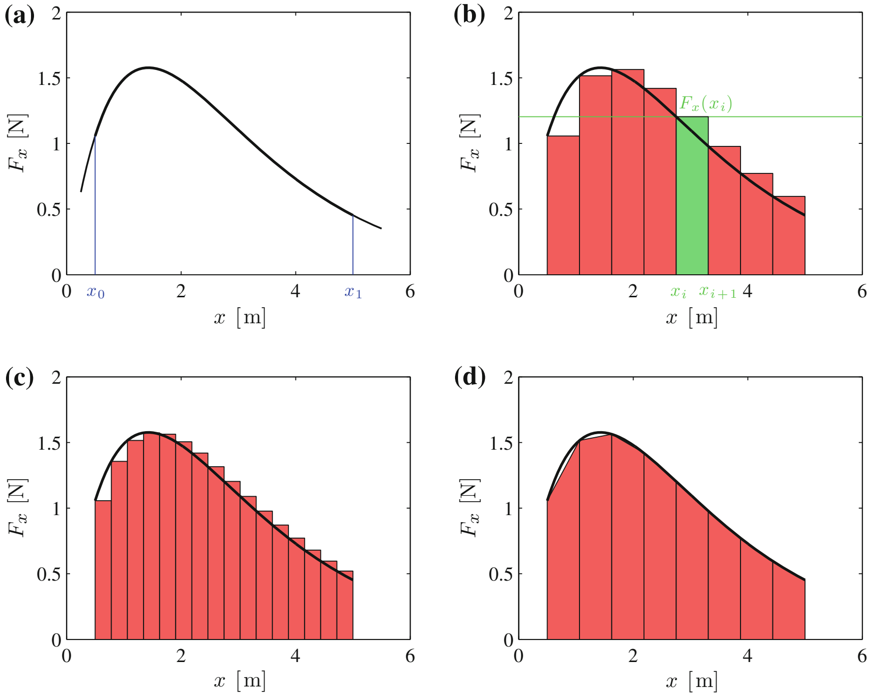

Fig. 10.3 Plot of the net force $F_x(x)$ on an object moving from x_0 to x_1. **a** Plot of $F_x(x)$. **b** Illustration of the numerical integral of $F_x(x)$ found as a sum of small rectangles of size Δx. **c** The numerical integral with half the box width, $\Delta x/2$. **d** The numerical integral as a sum of trapezoids

This approach is identical to Euler's method from Chap. 4. As we increase the number of intervals n, the numerical approximation becomes better and better. Figure 10.3c shows how the area of the rectangles becomes a better approximation to the area under the curve when n is doubled. We expect that our approximation approaches the exact value for the integral as n increases.[1] While Euler's method is simple to explain, we can obtain a better approximation by using trapeziods with corners $F(x_i)$ and $F(x_{i+1})$. The area of one such trapezoid is the baseline Δx multiplied by the average height h_i:

$$W_{i,i+1} = A_i \simeq \Delta x\, h_i = \Delta x\, \frac{1}{2}\left[F(x_i) + F(x_{i+1})\right] , \tag{10.31}$$

as illustrated by the green trapezoid in Fig. 10.3d. The total integral is then:

$$W_{0,1} = \sum_{i=1}^{n} \Delta x \frac{1}{2}\left[F_x(x_i) + F_x(x_{i+1})\right] . \tag{10.32}$$

[1] In the limit when n becomes infinitely large, this is indeed the definition of the integral.

This method is called the *trapezoidal rule* for numerical integration. The numerical implementation of this method is just as simple as Euler's method.

The trapezoidal rule is a standard numerical integration method that is built into Python through the function `trapz`. For example, we can integrate the work done by the function $F(x)$ shown in Fig. 10.3a:

$$F_x(x) = 3xe^{-0.7\,x} \ , \tag{10.33}$$

from $x_0 = 0.5\,\mathrm{m}$ to $x_1 = 5\,\mathrm{m}$ in $n = 1000$ steps by: first generating a set of n x_i values; then generating the corresponding set of $F(x_i)$ values; and finally calculating the integral using `trapz`:

```
xval = linspace(0.5,5.0,1000)
y = 3.0*xval*exp(-0.7*xval)
W = trapz(y,x=xval)
```

(Notice that the integral of this particular function $F(x)$ is analytically solvable. We have used it here to illustrate the principle of how to solve an integral numerically).

Numerical Integration of a Measured $F(x_i)$

In some case, the force $F(x)$ may be the result of a more complicated calculation or it may be an experimentally measured value. For example, you may calculate the net force on a basketball bouncing on the floor using an advanced model for the deformation of the surface of the ball; you may calculate the forces between two atoms as a function of their position based on an underlying microscopic picture such as quantum mechanics; or you may measure the force as a function of position as you pull on the string of your bow. In all these cases you know the force $F(x_i)$ for some values x_i, but you do not know the general function $F(x)$.

How can we estimate the work integral based on a few points? One approach would be to try to find a smooth function $F(x)$ that goes through all the points, and then use this function to calculate the work numerically. This is a powerful method, but beyond the scope of this text. Another method is to apply the trapezoidal rule on the discrete data as an approximation to the integral, as illustrated in Fig. 10.4. Fortunately, the trapezoidal rule is so robust that we may apply it also in cases when the data is not uniformly spaced—that is, when the intervals $\Delta x_i = x_{i+1} - x_i$ vary. In this case, we approximate the integral from x_0 to x_1 by the sum:

$$W_{0,1} = \int_{x_0}^{x_1} F(x)\,dx \simeq \sum_{i=1}^{n} \Delta x_i \, \frac{1}{2} \left[F(x_i) + F(x_{i+1}) \right] \ . \tag{10.34}$$

This method works for both calculated and measured values of $F(x_i)$, and has exactly the same implementation as above. For example, given a file forcedata.d[2] consisting of lines with values of x_i and $F(x_i)$, we can read the data and calculate the work integral by:

[2]http://folk.uio.no/malthe/mechbook/forcedata.d.

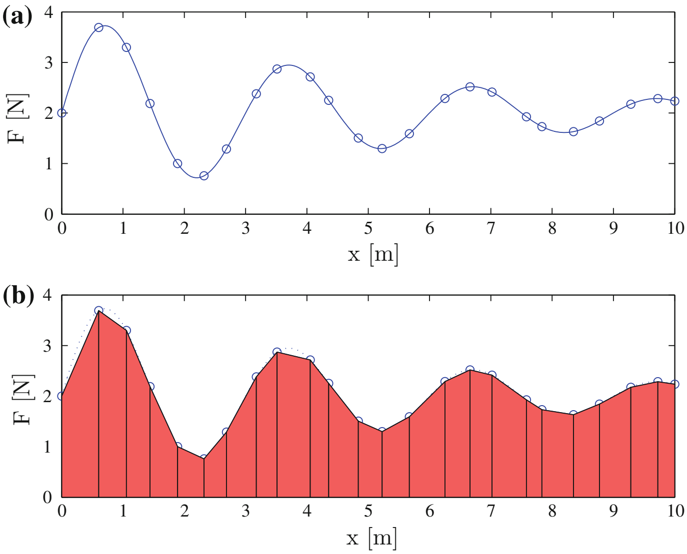

Fig. 10.4 **a** Illustration of a position-dependent force $F(x)$ showing a real underlying function, corresponding to the unknown $F(x)$ (which we here have supposed that we do not really know), and the values x_i where we have measured or calculated the value of the force, $F_i = F(x_i)$. **b** Illustration of the area under the curve, corresponding to the work integral, calculated using the trapezoidal rule using the discrete dataset only

```
xval, F = loadtxt('forcedata.d',usecols=[0,1],unpack=True)
W = trapz(F,x=xval)
```

The dataset is shown in Fig. 10.4 with an illustration of the trapezoidal approximation to the integral.

10.3.1 Example: Jumping from the Roof

In this example you will be introduced to how to apply the work-energy problem to solve actual problems, applying it to a constant force and a spring force.

You are standing on top of your house and are wondering how to jump down without getting hurt: You can jump into a thick snow cover, which exhibits a constant force, or onto a trampoline, which exhibits a spring force. What alternative would exert the smallest force?

Specify the problem: This problem is formulated loosely on purpose. You should be able to address such problems by adding the necessary components yourself. Let us make the problem more specific by adding details and assumptions: You have a mass m and your roof is a height h above the ground. You stop after a distance d. Also, we neglect air resistance.

Sketch: The problem addresses the motion of a person falling through air and then into various materials. We use $y(t)$ for the vertical position and place the origin at the top of the cushion with positive direction upward as shown in Fig. 10.5.

Model: We divide the motion into two phases, as illustrated in Fig. 10.5. Just like when we solve problems using Newton's second law, we start by analyzing the forces acting on the object.

Phase 1: The person falls through air from $y_0 = h$ to $y_1 = 0$. The only force acting is gravity, $G_y = -mg$.

Phase 2: The person is in contact with the surface from $y_1 = 0$ to $y_2 = -d$. There is a contact force from the surface, F acting upward in the positive y-direction and gravity, $G_y = -mg$.

Applying the work-energy theorem: We could now use Newton's second law to find the position $y(t)$ as a function of time. We know this would work, but it is a lot of work.

However, in this case, we do not care about the time it takes for the person to fall from the house and then brake during contact with the surface. We only care about the velocity immediately before contact with the surface, and the distance he needs to stop while in contact with the surface: We are only interested in questions relating the velocity v_y to the position y: First, what is the velocity when $y = 0$? Second, what is the position y when the velocity is zero, that is, when you have stopped? (Ok, this is not really the question. We actually want to find the force F necessary to ensure that the velocity is zero after a length d into the surface, but this is essentially the same as finding the distance d it moves before the velocity is zero when affected by a force F).

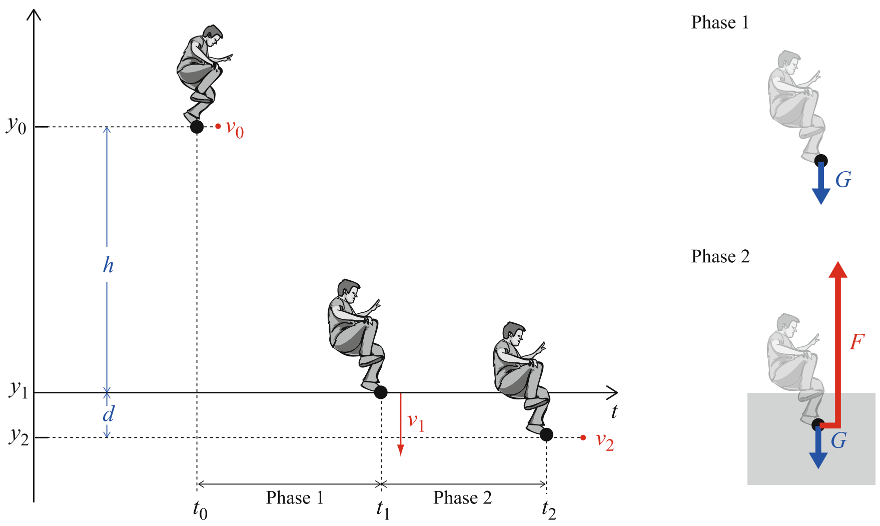

Fig. 10.5 A person jumping from the roof into various braking "devices"

Our plan is to use the work-energy theorem for the first phase to find the velocity immediately before you hit the surface, expressed as a function of the height, h, you jumped from. We will then use the work-energy theorem for the second phase to find the distance d you move before you stop, expressed in terms of the velocity you had when you hit the surface. Finally, we will relate the stopping distance d to the initial height h, and then find the maximum force during phase two.

Finding the velocity before impact: In *Phase 1* the net force is constant: $F_y^{\text{net}} = -mg$. We find the change in velocity from y_0 to y_1 using the work-energy theorem: $K_1 - K_0 = W_{0,1}$, where the work of the net force from y_0 to y_1 is:

$$W_{0,1} = \int_{t_0}^{t_1} F_y^{\text{net}} v dt = \int_{y_0}^{y_1} (-mg)\, dy = -mg\,(y_1 - y_0) = mgh\ , \tag{10.35}$$

where we have used that $y_1 = 0$ and $y_0 = h$. This is equal to the change in kinetic energy:

$$W_{0,1} = mgh = K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 = K_1\ . \tag{10.36}$$

where we have used that the person starts from rest, $v_0 = 0\,\text{m/s}$. This gives that $K_1 = mgh$, which is what we will need to determine the stop length, d.

Contact with the surface: In *Phase 2* the net force is:

$$F_y^{\text{net}} = F - mg\ , \tag{10.37}$$

where the force F from the surface depends on the nature of the surface. We will therefore treat the various surfaces independently.

Falling into snow—Constant force model: For snow, we assume that F is a constant force. We use the work-energy theorem $W_{1,2} = K_2 - K_1$ to relate the stopping distance d to the kinetic energy K_1 when the contact started. The work of the net force over the distance $\Delta y = y_2 - y_1 = 0 - d = -d$ is:

$$W_{1,2} = \int_{y_1}^{y_2} (F - mg)\ dy = (F - mg)\ (-d) = K_2 - K_1\ . \tag{10.38}$$

where $K_2 = 0$ sice the person stops at y_2. We insert K_1 from (10.36), getting:

$$-\,(F - mg)\,d = -K_1 = -mgh\ \Rightarrow\ F - mg = mg\frac{h}{d}\ \Rightarrow\ F = mg\left(1 + \frac{h}{d}\right)\ . \tag{10.39}$$

Falling into snow—Discussion: Notice the simplicity of this approach. We do not even have to calculate the velocity v_1 after the person has fallen a height h.

For a typical house of 6 m height and for a typical person of height 2 m and a stopping distance of 1 m, the force from the snow is $F = mg\,(1 + 6) = 7mg$.

The work integral during the free fall is illustrated as the blue area in Fig. 10.6, and the work integral during contact is illustrated as the red area. After the free fall, the person has a kinetic energy corresponding to the blue area. Similarly, the red area corresponds to the change in kinetic energy during contact. The person stops when the red area is equal to the blue area. Notice that the fall starts on the right hand side of Fig. 10.6, which corresponds to high y-values, and then progresses toward the left. This graph can be a useful tool to discuss the motion.

Test your understanding: Based on Fig. 10.6 and that you stop over a distance d, what do you think is the force model that gives the lowest maximum force during the brake?

Falling onto a trampoline—Spring force model: For a trampoline, the force F is a spring force $F(y) = -ky$ when $y < 0$ m. The net force on the person in *Phase 2* is therefore

$$F_y^{\text{net}} = F - mg = -ky - mg \; , \tag{10.40}$$

which is a force that depends *only* on the position. The work of the net force from y_1 to y_2 is therefore:

$$\begin{aligned} W_{1,2} &= \int_{t_1}^{t_2} F_y^{\text{net}} v dt = \int_{y_1}^{y_2} (-ky - mg)\, dy = \int_{y_1}^{y_2} -ky dy + \int_{y_1}^{y_2} -mg dy \\ &= -k \left(\frac{1}{2} y_2^2 - \frac{1}{2} y_1^2 \right) - mg\,(y_2 - y_1) = -\frac{1}{2} k d^2 + mgd \; , \end{aligned} \tag{10.41}$$

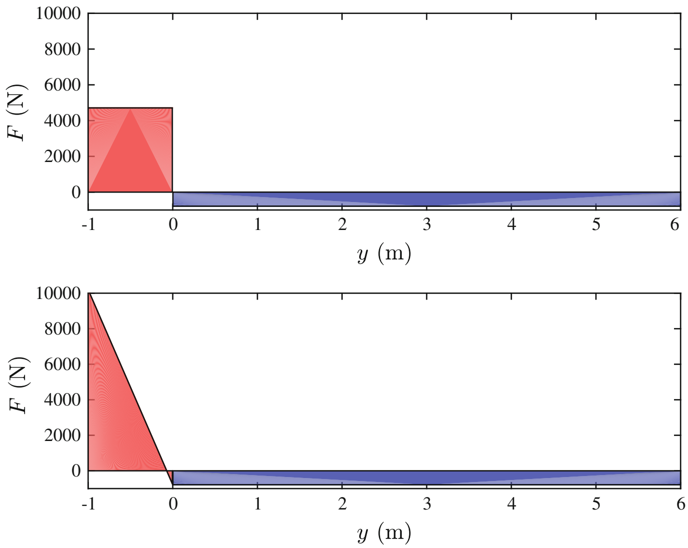

Fig. 10.6 Illustration of the work integrals. The red area corresponds to phase 1, when the person is falling through the air, and the blue area corresponds to phase 2, when the person is in contanct with the surface. *Top* Constant forces. *Bottom* Spring forces

where we have used that $y_2 = -d$ and $y_1 = 0\,\text{m}$. We insert this result in the work-energy theorem:

$$W_{1,2} = -\frac{1}{2}kd^2 + mgd = K_2 - K_1 = -K_1 = -mgh = \tag{10.42}$$

where we used that $K_2 = 0$ and that $K_1 = mgh$ from (10.36).

Falling onto a trampoline—Discussion: This result allows us to find k if we know d and h, or, alternatively, to find d if we know k and h. Here, we know d and h and are interested in finding k, since this allows us to calculate the force from the trampoline. From (10.42) we get:

$$\frac{1}{2}kd^2 = mg\,(d + h)\ . \tag{10.43}$$

The force from the trampoline on the person jumping increases with the deformation of the trampoline, and is at its maximum when the trampoline is maximally deformed, which occurs when $y = -d$. The force is then:

$$F_{\text{max}} = kd = \frac{2}{d}\frac{1}{2}kd^2 = \frac{2}{d}mg\,(d + h)\ , \tag{10.44}$$

where we have used the result from (10.43). Again, we assume that reasonable values are $h = 6\,\text{m}$ and $d = 1\text{m}$, giving:

$$F_{\text{max}} = kd = 2mg\,(1 + (6\,\text{m}/(1\,\text{m})) = 14mg\ , \tag{10.45}$$

which is double the value of the constant force $F = 7mg$ we found for the constant force above.

Again, the simplicity of the approach is striking. We do not have to calculate the velocity v_1 after the person has fallen a height h, we only need to know the kinetic energy at this point in order to carry out the rest of the calculation.

The work integrals are illustrated in Fig. 10.6. The blue area corresponds to the net work during free fall, and the red area corresponds to the net work during braking. Here, we have essentially selected the spring constant k, which is the slope of the curve, so that the red area over a length d corresponds to the blue area over the length h.

Test your understanding: How will d and F change if you double the spring constant k of the trampoline?

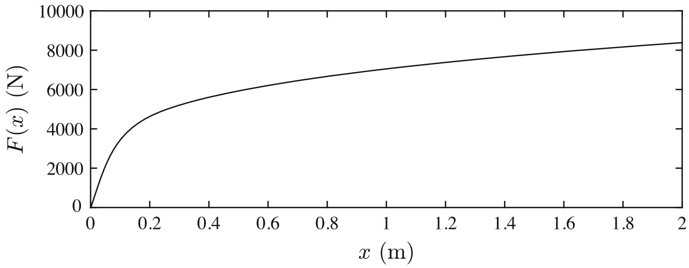

Fig. 10.7 The measured force $F(y)$ as a function of the position y of the top of the cushion. Notice that y is negative when the cushion is compressed

10.3.2 Example: Stopping in a Cushion

You have installed a new soft cushion from SoftCush technology to prevent falling damage. According to the producers, the cushion has been developed to produce a normal force that depends on the compression of the cushion only (and not on the speed of deformation). You do not know the functional form of the force response, $F(y)$, of the cushion, but you have measured the response in a controlled experiment where you pushed the cushion down to a position y and measured the corresponding force $F(y)$. The results are provided in the file cushionforce.d,[3] and shown in Fig. 10.7.

Problem: If you jump from a height h and onto the cushion, how far will it compress before you stop? What is the maximum of the force from the cushion on you?

Approach: The cushion reaches its maximum compression when you stop, that is, when your kinetic energy is zero. We will first use the work-energy theorem to find the kinetic energy you have after falling a distance h, and then we will use the work-energy theorem during the contact with the cushion, from you touch the cushion and until you stop.

Sketch and Identify: Figure 10.8 shows an illustration of the forces acting on the person during the jump and a free-body diagram of the person.

Model: We divide the motion into two phases, phase 1 from $x_0 = h$ to $x_1 = 0$, where the person is only affected by gravity, $G = mg$, and phase 2 from $x_1 = 0$ to $x_2 = -d$, where the person is affected by gravity, G, and the force from the cushion, F.

Phase 1: Free fall: In phase 1 the person starts from rest, $v_0 = 0$ and $K_0 = 0$, at a height $x_0 = h$, and reaches the height, $x_1 = 0$ with a velocity, v_1, and a kinetic energy, K_1.

[3] http://folk.uio.no/malthe/mechbook/cushionforce.d.

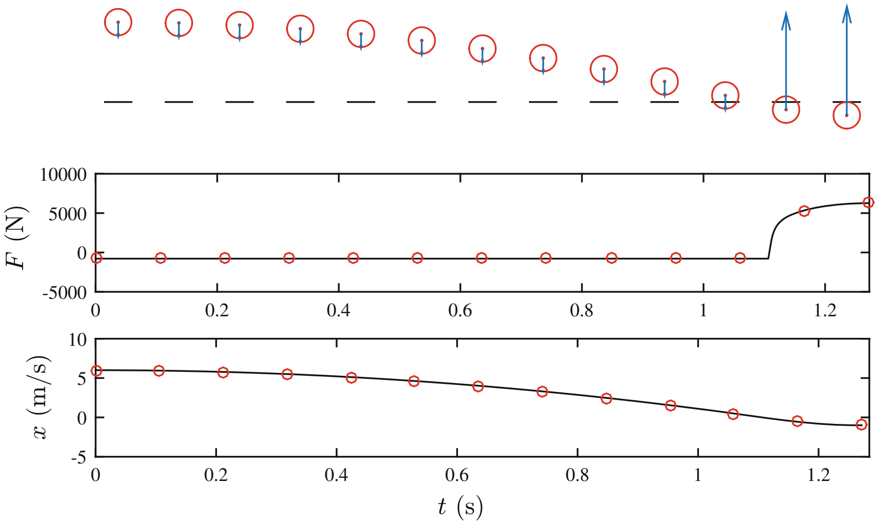

Fig. 10.8 The measured force $F(y)$ as a function of the position y of the top of the cushion. Notice that y is negative when the cushion is compressed

We find K_1 by applying the work-energy theorem from 0 to 1:

$$W_{0,1} = \int_{x_0}^{x_1} G\,dx = \int_h^0 -mg\,dx = mgh = K_1 - K_0 = K_1 \ , \tag{10.46}$$

where we have used that $K_0 = 0$. We see that $K_1 = (1/2)mv_1^2 = mgh$, so that we can determine v_1 if needed.

Phase 2: Contact with cushion: In phase 2 the person starts with velocity v_1 and kinetic energy K_1 at the height $y_1 = 0$ and stops with velocity $v_2 = 0$ and kinetic energy $K_2 = 0$ at height $y_2 = -d$. You are in contact with the SoftCusion, and the net force affecting you is therefore:

$$F^{\text{net}} = F(x) - mg \ , \tag{10.47}$$

We apply the work-energy theorem from 1 to 2 to determine d, the stopping distance:

$$W_{1,2} = \int_{x_1}^{x_2} F^{\text{net}}\,dx = \int_0^{-d} (F(x) - mg)\,dx = K_2 - K_1 = -mgh \ , \tag{10.48}$$

where we have used that $K_2 = 0$ and $K_1 = mgh$ from (10.46).

The idea is that the value for d that satisfies (10.48) gives us the maximum compression of the cushion. Unfortunately, (10.48) is not an explicit equation in d that we can solve since the unknown d is the upper limit in the integral. If we knew how to solve the integral analytically, we might have been able to solve the equation, but in this case we only know how to solve the integral numerically. How can we solve such an equation and find the d that satisfies (10.48)?

What we need to do, is to calculate the work:

$$W_{1,2} = \int_0^{x^*} F(x)dx - mgx^* \ , \tag{10.49}$$

as a function of the position x^* of the jumper. Our plan is to vary x^* until we find a value for x^* that satisfies the equation, which corresponds to $-d$. We need to vary x^* systematically. We start from $x^* = 0$, where the jumper comes in contact with the cushion, and then gradually decrease x^* (remember that the jumper is moving down during the contact with the cushion) until the equation is satisfied, that is, until we find the value for x^* for which $W_{1,2}$ is closest to $-mgh$. More precisely, we make a sequence of x^* values starting from 0, decreasing in small steps, Δx: $x_0^* = 0$, $x_1^* = x_0^* + \Delta x$, and we calculate the work for each of these numbers until the work is equal to $-mgh$.

Can we use any step size Δx in this sequence of x_i^* values? If we knew the force $F(x)$ for any position x, we could calculate the work at any resolution Δx. But in this case we only know the forces $F(x_i)$ for the values x_i where it has been measured. Therefore, the best possible resolution we can get is to use the measured x_i values as our sequence of x_i^*-values. This means that we start by calculating the work for $x_1^* = x_1$. Then we calculate the work for $x_2^* = x_2$, then for $x_3^* = y_3$ and so on, until, for some value i, the work is approximately equal to $-mgh$.

How do we calculate the work for $x^* = x_k$, where k is a number in the sequence of x_i values in the datafile? The integral in (10.49) can be calculated using the trapezoidal rule even if we do not know the underlying function—it works well also for a measured dataset:

$$\int_0^{x_k} F(x)\,dx \simeq \sum_{i=1}^{k} (x_{i+1} - x_i)\,\frac{1}{2}\left[F(x_i) + F(x_{i+1})\right] \ . \tag{10.50}$$

Numerically, this is done using the function `trapz`. Here, we only want to do the sum over the first k values of x_i, that is, we only want to do the sum from x_1 to x_k in the list of n such x-values. This corresponds to doing the sum over the first `[0:k-1]` elements in the position array `x`:

```
xval, F = loadtxt('cushionforce.d',usecols=[0,1],unpack=True)
k = 10
I = trapz(F[0:k-1],x=xval[0:k-1])
```

We calculate the work for all possible values of k, and store the results in an array `work`, which gives us the work as a function of k:

```
from pylab import *
m = 80.0 # kg
g = 9.8  # m/s^2
```

```
h = 6.0   # m
x, F = loadtxt('cushionforce.d',usecols=[0,1],unpack=True)
n = length(x)
work = zeros(n,float)
for k in range(1,n-1):
    I = trapz(F[0:k],x=-x[0:k])
    work[k] = I - m*g*x[k]
plot(y,work,'-b')
xlabel('x (m)')
ylabel('W(x) (J)')
```

Figure 10.9 shows the work $W_{1,2}$ as a function of the position x calculated using this program. In the same plot we have also shown the value for $-mgh$. We see that the work starts at zero and decreases gradually as the deformation increasesa and x becomes a large negative number because the work done is negative: The net work acts to reduce the kinetic energy. When the work reaches the value $-mgh$ the jumper stops: the kinetic energy is now zero. How can we find the x-value this occurs at from our numerical data? Based on the plot, we realize that the x value when $W_{1,2} = -mgh$ can be found as the first value in the sequence of x_i's for which $W_{1,2} < -mgh$. This value of x will correspond to a work that is slightly smaller than $-mgh$, but this is a good first approximation. How can we find this value numerically? We use the function `find`:

```
i2 = min(find(work<mgh))
x2 = x(i2)
```

The function `find(work<-mgh)` returns a list of all the indexes of `work` that satisfies the condition `work<-mgh`. We need to find the first of the indexes in this sequence, therefore we take the minimum index in this list, and find the corresponding y-value. We have found the maximum compression of the cushion!

What is the cushion force at this value? This is the corresponding cushion force: $F(y_2) = 7050\,\text{N}$ found by:

```
>>F2 = F[i2]
>>print F2
F2 =      7050
```

What would happen if the jumper started from $h = 9\,\text{m}$ instead? We plot the corresponding value for $-mgh$ in Fig. 10.9. You can read the maximum compression directly from this graph, or determine it numerically as we did above.

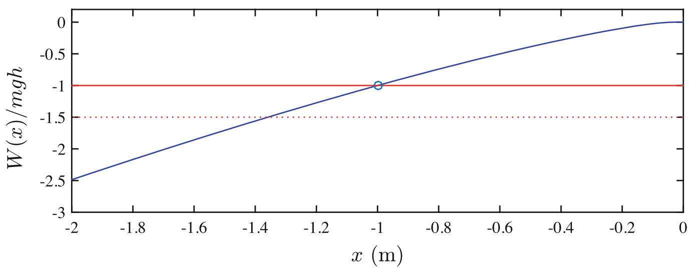

Fig. 10.9 Plot of the work $W_{1,2}$ as a function of the position x of the jumper

10.4 Work Done in Two- and Three-Dimensional Motions

The work-energy theorem was demonstrated for a three-dimensional motion. We have so far studied one-dimensional motions only. How does the application of the theorem change for three-dimensional motion?

Work of Tangential and Normal Forces

Figure 10.10 illustrates a general three-dimensional motion from 0 to 1. For this motion, the net work is

$$W = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = K_1 - K_0 \; . \tag{10.51}$$

Because $\mathbf{v}$ points in the tangential direction, we notice that *it is only the tangential component of the force* that does any work! We can demonstrate this by introducing a local coordinate system along the path of motion, with unit vectors $\hat{u}_T$ in the tangential direction and $\hat{u}_N$ in the normal direction. The velocity vector points along the tangential unit vector:

$$\mathbf{v} = v\, \hat{u}_T(t) \; . \tag{10.52}$$

If we decompose a force, $\mathbf{F}_j$, in the tangential and normal directions along the path of motion:

$$\mathbf{F}_j = F_{j,T}(t)\, \hat{u}_T(t) + F_{j,N}(t)\, \hat{u}_N(t) \; , \tag{10.53}$$

we see that the work done by the force $\mathbf{F}_j$ is:

$$W_j = \int_{t_0}^{t_1} \mathbf{F}_j \cdot \frac{d\mathbf{r}}{dt}\, dt \tag{10.54}$$

$$= \int_{t_0}^{t_1} \left(F_{j,T}(t)\hat{u}_T(t) + F_{j,N}(t)\hat{u}_N(t) \right) \cdot \frac{d\mathbf{r}}{dt}\, dt \tag{10.55}$$

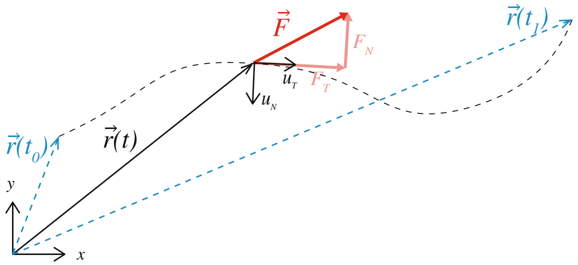

Fig. 10.10 Illustration of the path followed by an object moving from position 0 to position 1

$$= \int_{t_0}^{t_1} F_{j,T}(t)\hat{u}_T(t) \cdot \frac{d\mathbf{r}}{dt}\, dt + \int_{t_0}^{t_1} F_{j,N}(t)\, \underbrace{\hat{u}_N(t) \cdot \frac{d\mathbf{r}}{dt}}_{=0}\, dt \tag{10.56}$$

$$= \int_{t_0}^{t_1} F_{j,T}(t)\hat{u}_T(t) \cdot \frac{d\mathbf{r}}{dt}\, dt \ . \tag{10.57}$$

where we have used that the normal unit vector, $\hat{u}_N$, is normal to the velocity vector, $\mathbf{v}$. Consequently, it is only the tangential component of the force that contributes to the work.

This is also intuitive, since we know that normal forces only contribute to a change in the direction of the velocity vector and not to a change in the speed—it is only the tangential force that can cause a tangential acceleration which causes a change in speed.

Work of a Constant Force in Two and Three Dimensions

The work done by a constant force has a particularly simple solution. Let us address the work done on an object by a constant force $\mathbf{F}$ as the object is moved from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$ as illustrated in Fig. 10.11. The work done on the object is

$$W_F = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt = \int_0^1 \mathbf{F} \cdot d\mathbf{r} \ , \tag{10.58}$$

Since the force is constant, we move it outside the integration:

$$W_F = \mathbf{F} \cdot \underbrace{\int_0^1 d\mathbf{r}}_{=(\mathbf{r}(t_1)-\mathbf{r}(t_0))} = \mathbf{F} \cdot (\mathbf{r}(t_1) - \mathbf{r}(t_0)) \ . \tag{10.59}$$

We call $\mathbf{s} = \Delta\mathbf{r} = \mathbf{r}(t_1) - \mathbf{r}(t_0)$ the displacement. The **work of a constant force** is therefore:

$$\boxed{W = \mathbf{F} \cdot \Delta\mathbf{r} \ .} \tag{10.60}$$

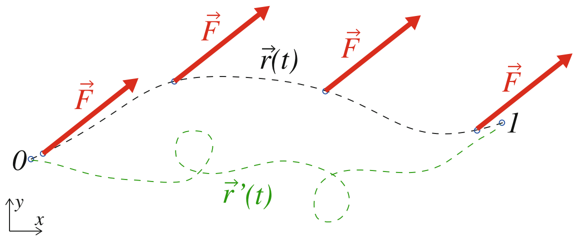

Fig. 10.11 Work done by a constant force $\mathbf{F}$ as the object is moved from point 0 at $\mathbf{r}(t_0)$ to point 1 at $\mathbf{r}(t_1)$

Notice that the work done by a constant force depends only on the displacement $\Delta\mathbf{r}$ and not on the path taken! The work done along the two paths $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ in Fig. 10.11 are therefore the same for a constant force. This is a very nice property of a force, since it makes it convenient to calculate the work done. A force with this property is called a conservative force, and we will see that such forces play a prominent role in our studies of mechanical energy.

10.4.1 Example: Work of Gravity

Problem: A projectile is moving through the air starting with an initial velocity $\mathbf{v}_0$ at the height y_0. Find the speed of the projectile at the height y_1. You can neglect air resistance.

Identify: In this problem we address the motion of the projectile, described by the position $\mathbf{r}(t)$. The projectile is at y_0 at t_0 with a velocity $\mathbf{v}_0$, and at a height h_1 at t_1 with a speed v_1.

Model: The projectile is only affected by gravity, which is constant, $\mathbf{G} = -mg\,\mathbf{j}$.

Solve: We can therefore use the work-energy theorem to find the kinetic energy of the projectile:

$$W = \int_{t_0}^{t_1} \mathbf{G}\cdot\mathbf{v}\,dt = \int_0^1 \mathbf{G}\cdot d\mathbf{r} = K_1 - K_0 \ . \tag{10.61}$$

Since the force is constant, the work only depends on the displacement:

$$W = \mathbf{G}\cdot\mathbf{s} = (-mg\,\mathbf{j})\cdot(\Delta x\,\mathbf{i} + \Delta y\,\mathbf{j}) = -mg\Delta y \ . \tag{10.62}$$

The motion in the x-direction has no impact on the work. It is only the displacement in the y-direction that contributes to a change in kinetic energy:

$$K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = -mg\Delta y \ , \tag{10.63}$$

The speed v_1 is therefore:

$$v_1^2 = v_0^2 - 2g\,(y_1 - y_0) \ , \tag{10.64}$$

The right-hand side must be positive for this equation to have a meaningful solution.

10.4.2 Example: Roller-Coaster Motion

Problem: A roller-coaster cart is rolling from the height h to the height 0 along a curving roller-coaster track. It starts with the speed v_0 at the top of the track. Find

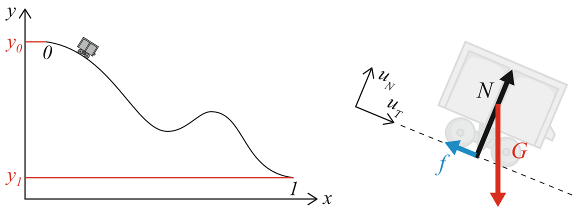

Fig. 10.12 A roller-coaster cart moving along a roller-coaster track

the speed v_1 of the roller-coaster cart at the bottom of the track. You can ignore air resistance and friction.

Identify: The cart follows the path $\mathbf{r}(t)$ from $\mathbf{r}(t_0) = h\,\mathbf{j}$ at $t = t_0$ to $\mathbf{r}(t_1) = 0\,\mathbf{j}$ at $t = t_1$, as illustrated in Fig. 10.12.

Model: The cart is affected by the normal force, $\mathbf{N}$, the friction force, $\mathbf{f}$, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. We assume that friction is negligible, $\mathbf{f} = 0$ throughout the motion. The normal force $\mathbf{N}$ varies throughout the motion. It is therefore not that simple to apply Newton's second law directly to determine the motion of the cart. However, because the normal force always is normal to the path of motion, it performs no work. We can therefore apply the work-energy theorem to determine the velocity of the cart at the end of the track.

Solve: The work energy-theorem relates the work of the net force to the change in kinetic energy:

$$W_{\text{net}} = \Delta K = K_1 - K_0 = \frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2\,. \tag{10.65}$$

We can therefore find the speed, v_1, if we know the net work:

$$W_{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}_{\text{net}} \cdot \mathbf{v}\,dt = \int_{t_0}^{t_1} (\mathbf{N} + \mathbf{G}) \cdot \mathbf{v}\,dt = \int_{t_0}^{t_1} \underbrace{\mathbf{N} \cdot \mathbf{v}}_{=0}\,dt + \int_{t_0}^{t_1} \mathbf{G} \cdot \mathbf{v}\,dt = W_G \tag{10.66}$$

As we argued above, the normal force does no work. The work done by gravity is the same as we have found previously:

$$W_G = \int_0^1 \mathbf{G} \cdot d\mathbf{r} = \mathbf{G} \cdot \int_0^1 d\mathbf{r} = \mathbf{G} \cdot (\mathbf{r}(t_1) - \mathbf{r}(t_0))\ . \tag{10.67}$$

Since $\mathbf{G} = -mg\,\mathbf{j}$, only the vertical component of the displacement is included in the work:

$$W_G = -mg\mathbf{j} \cdot (\mathbf{r}(t_1) - \mathbf{r}(t_0)) = -mg(y_1 - y_0) = -mg(0 - h) = mgh\ . \tag{10.68}$$

We use this result in the work-energy theorem to find v_1:

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = mgh \;\Rightarrow v_1^2 = v_0^2 + 2gh \; . \tag{10.69}$$

Analyze: We have learned that for motion with a normal force, the normal force does no work and we can therefore use the same analysis as for an object falling due to gravity. However, this is only true as long as there are no additional forces that depend on the normal force, such as a friction force, or a force that depends on for example the velocity of the cart, such as the air resistance. In these cases, we need to make a more detailed analysis.

10.4.3 Example: Work on a Block Sliding Down a Plane

Problem: A classical problem in mechanics is the motion of a block sliding down an inclined plane. If the block starts from rest, what is the velocity of the block when it has moved a vertical distance h? The plane forms an angle α with the horizontal, the mass of the block is m, the acceleration of gravity is g, and μ is the dynamic coefficient of friction between the block and the plane.

Approach: We use the work-energy theorem to find the change in velocity from the change in kinetic energy, which only depends on the vertical displacement.

Identify: We use $\mathbf{r}(t)$ to describe the motion of the block from $\mathbf{r}(t_0)$ at $t = t_0$ to $\mathbf{r}(t_1)$ at $t = t_1$, as illustrated in Fig. 10.13.

Model: The block is affected by the normal force from the plane, $\mathbf{N}$, the friction force, $\mathbf{f}$, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. Since the block is sliding relative to the surface, the we use a dynamic friction model:

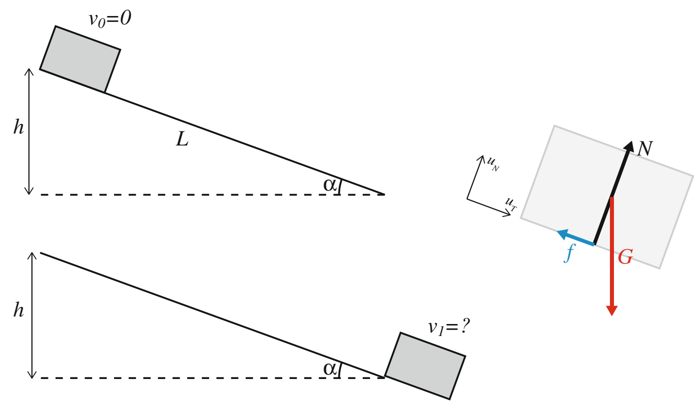

Fig. 10.13 A block sliding down along an inclined plane

$$\mathbf{f} = -\mu N \frac{\mathbf{v}}{v} . \tag{10.70}$$

where the normal force must be determined from the motion in the direction normal to the plane. Since we assume that there is no motion normal to the plane, Newton's second law gives:

$$\sum F_N = N + G_N = ma_N = 0 \Rightarrow N = -G_N . \tag{10.71}$$

where G_N is the component of gravity in the normal direction, given by the unit vector, $\hat{u}_N$. From Fig. 10.13 we see that:

$$\hat{u}_N = \sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{j} . \tag{10.72}$$

We find G_N by projecting $\mathbf{G}$ on $\hat{u}_N$:

$$G_N = \mathbf{G} \cdot \hat{u}_N = -mg\,\mathbf{j} \cdot (\sin\alpha\,\mathbf{i} + \cos\alpha\,\mathbf{j}) = -mg\cos\alpha , \tag{10.73}$$

which inserted in (10.71) gives: $N = mg\cos\alpha$.

Solve: The work-energy theorem for the motion of the block relates the work done by the net force on the block to the change in kinetic energy, $W_{\text{net}} = K_1 - K_0$, where

$$W_{\text{net}} = \sum_j W_j = W_N + W_f + W_G . \tag{10.74}$$

The work done by the normal force is zero, since it is always normal to the direction of motion: $W_N = 0$. Since gravity is a constant, the work done by gravity only depends on the displacement, $\Delta\mathbf{r}$:

$$W_G = \mathbf{G} \cdot \Delta\mathbf{r} = -mg\,(y_0 - y_1) = mgh . \tag{10.75}$$

In this particular case, we can find the work done by friction, because the friction force is constant during the motion:

$$W_f = \int_{t_0}^{t_1} \mathbf{f} \cdot \mathbf{v}\,dt = \mathbf{f} \cdot \Delta\mathbf{r} . \tag{10.76}$$

where both $\mathbf{f}$ and $\Delta\mathbf{r}$ are directed along the slope. The work done by friction is therefore

$$W_f = -f\,s , \tag{10.77}$$

where s is the distance along the slope. For a slope with an inclination α, the distance s is related to the height h by:

$$\frac{h}{s} = \sin\alpha \;\Rightarrow\; s = \frac{h}{\sin\alpha} \ . \tag{10.78}$$

This gives:

$$W_f = -f s = -\mu N \frac{h}{\sin\alpha} = -\mu m g h \frac{\cos\alpha}{\sin\alpha} \ . \tag{10.79}$$

We apply the work-energy theory to find the kinetic energy and the speed at the end of the slope:

$$W_{\text{net}} = W_N + W_f + W_G = mgh - \mu mgh \cot\alpha = K_1 - K_0 = \frac{1}{2} m v_1^2 \ , \tag{10.80}$$

where we have used that $K_0 = 0$. This gives

$$v_1^2 = 2gh \ (1 - \mu \cot\alpha) \ . \tag{10.81}$$

10.5 Power

You are pulling a crate along the floor with a force $\mathbf{F}$. In a small time interval Δt you have pulled the crate from the position $\mathbf{r}(t)$ to $\mathbf{r}(t + \Delta t)$, and the force $\mathbf{F}$ has performed the work

$$\Delta W = \int_t^{t+\Delta t} \mathbf{F} \cdot \mathbf{v}\, dt \ . \tag{10.82}$$

For a small time interval we can assume that the force $\mathbf{F}$ is constant, hence:

$$\Delta W = \mathbf{F} \cdot \int_t^{t+\Delta t} \mathbf{v}\, dt = \mathbf{F} \cdot (\mathbf{r}(t + \Delta t) - \mathbf{r}(t)) = \mathbf{F} \cdot \Delta \mathbf{r} \ . \tag{10.83}$$

We divide by Δt on each side and find that:

$$\frac{\Delta W}{\Delta t} = \mathbf{F} \cdot \frac{\Delta r}{\Delta t} \tag{10.84}$$

and in the limit when $\Delta t \to 0$ we have the definition of

Power: Power is the amount of work performed per unit time, that is, the rate at which we perform work:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \mathbf{F} \cdot \mathbf{v} \ . \tag{10.85}$$

The unit of power is *Watt* (W), which is defined as:

$$1\,\mathrm{W} = 1\,\mathrm{J/s} = 1\,\mathrm{Nm/s} \ . \tag{10.86}$$

We also use the unit of horsepower, hP or hK. The metric horsepower is defined as:

$$1\,\mathrm{hP} = 735.5\,\mathrm{W} \ . \tag{10.87}$$

10.5.1 Example: Power Exerted When Climbing the Stairs

Problem: You are climbing the stairs, moving upward at a constant vertical velocity v_0. What is the power exerted?

Solution: Since you are moving upward with a constant velocity, v_0, your acceleration upward is zero. Therefore, there is no net force in the vertical direction. The normal force N exerted from the ground on you therefore equals the gravitational force, $W = mg$, if we assume that other forces, such as air resistance, are negligible. The rate at which the normal force does work on you—the power of the normal force—is:

$$P = N v_0 = m g v_0 \ . \tag{10.88}$$

10.5.2 Example: Power of Small Bacterium

Problem: A small bacterium with approximately spherical shape is moving with a constant velocity $\mathbf{v}$ through water. Find the power exerted by the bacterium.

Model: The motion of the bacterium is determined by the forces acting on the bacterium. The bacterium is affected by two main forces: The fluid drag force, $\mathbf{D}$, and the propulsion force, $\mathbf{F}$, which is a force from the fluid on the bacterium due to the swirling motion of the bacterium's tail.

From Newton's second law we see that:

$$\mathbf{F} + \mathbf{D} = m\mathbf{a} = 0 \ , \tag{10.89}$$

where $\mathbf{a} = 0$ since the bacterium is moving with constant velocity. Consequently,

$$\mathbf{F} = -\mathbf{D} \, . \tag{10.90}$$

We have a good force model for the fluid drag. Because the bacterium is small and is moving at a tiny velocity, the fluid drag force is well approximated by the viscous force model:

$$\mathbf{D} = -k_v \mathbf{v} \, . \tag{10.91}$$

The power exerted by the bacterium corresponds to the power exerted by the force $\mathbf{F}$:

$$P = \mathbf{F} \cdot \mathbf{v} = k_v \mathbf{v} \cdot \mathbf{v} = k_v v^2 \, . \tag{10.92}$$

Summary

Work: The work done on an object by the force $\mathbf{F}$ moving the object along the path $\mathbf{r}(t)$ is:

$$W = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt \, .$$

We can write this as a line integral along the path C given by $\mathbf{r}(t)$:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} \, .$$

Kinetic energy: The kinetic energy of an object is defined as:

$$K = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m v^2 \, .$$

Work-energy theorem: The work done by the *net force* on an object along the path from 0 to 1 corresponds to the change in kinetic energy of the object:

$$W_{\text{net}} = \int_{t_0}^{t_1} \mathbf{F}_{\text{net}} \cdot \mathbf{v} dt = K_1 - K_0$$

Work of a constant force: The work of a constant force is $W = \mathbf{F} \cdot \mathbf{s}$.

Work of a position-dependent force in one dimension: If *a force only depends on the position*, $F_x = F_x(x)$, the work is an integral over x only:

$$W_F = \int_{x_0}^{x_1} F_x(x) dx \, .$$

It is often easier to solve the work integral for a complicated position-dependent force than to find the motion of the object. The work-energy theorem is therefore a useful tool to relate the velocity and position of an object without finding the complete motion $x(t)$.

Power: The power P exerted by a force $\mathbf{F}$ is the rate of work done by the force:

$$P = \mathbf{F} \cdot \mathbf{v}$$

Exercises

Discussion Questions

10.1 Accelerating a car. Compare the work required to accelerate a car from 50 to 70 km/h and from 70 to 90 km/h.

10.2 Leverage. Why is it easier to lift a car using a long lever than with your hands?

10.3 Friction. Can a frictional force increase the kinetic energy of a system?

10.4 Diving board. Divers often double-jump on a diving board to get higher. How does this work?

10.5 Trampoline. While you jump on a trampoline you are able to control your jump by either jumping higher or to almost stop at a single "jump". How?

10.6 Work in an elevator. You carry a heavy crate first from your car into your house, then into an elevator, and then you hold the crate in the elevator until you reach the top. What is the work done by your arm on the crate in the different cases?

Problems

10.7 Dragging a cart. You push a small cart 10 m along a flat floor with a force $F = 90$ N. The friction force on the cart is $f = 30$ N. The mass of the cart is 10 kg, and the cart starts at rest.
(**a**) What is the work done by you on the cart?
(**b**) What is the work done by the friction force?
(**c**) What is the velocity of the cart after 10 m?

10.8 Toboggan slide. A child is sliding down a hill in a toboggan. He starts from rest, and when he reaches the end of the slope, he has moved a vertical distance of 10 m and he has a speed of 13.5 m/s. The mass of the child is 40 kg.
(**a**) What is the work done by friction?

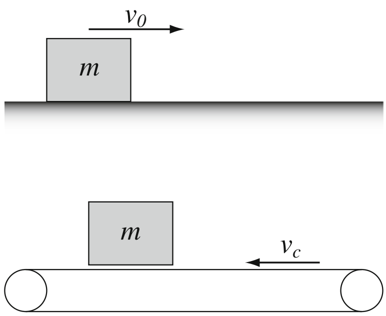

Fig. 10.14 *Top* Illustration of a crate sliding along the floor. *Bottom* Illustration of a crate released down onto a conveyor belt

10.9 Crate on conveyor belt. A crate is sliding along the floor with a horizontal velocity v_0, as illustrated in Fig. 10.14. The mass of the crate is m and the coefficient of dynamic friction between the floor and the crate is μ_d.
(**a**) Draw a free-body diagram for the crate.
(**b**) Find the distance, s, the crate slides before stopping.

Let us now study a slightly different situation. A conveyor belt is moving with a constant velocity, v_c, as illustrated in Fig. 10.14. A crate is released onto the conveyor belt, starting at rest relative to the ground. After a while the crate attains the same velocity as the belt. The dynamic coefficient of friction between the belt and the crate is μ_d. You can assume that the belt is long compared to the motion of the crate.
(**c**) Draw a free-body diagram for the crate on the belt.
(**d**) Find the work, W, done by the friction force.
(**e**) What distance, s, is the crate transported relative to the ground before it get the same velocity as the belt?

10.10 Volleyball smash. A volleyball player is smashing a ball. He gets the ball from behind with a velocity of 2 m/s towards the net. During the smash, he hits the ball with an approximately constant force of 300 N in the direction of the initial velocity. The mass of a volleyball is 270 g.
(**a**) Over how long distance must he be in contact with the ball in order to give the ball a velocity of 10 m/s?

10.11 A bouncing ball. A ball is lifted to a height h and released. When the ball hits the ground, the force from the ground on the ball has the form $F = k|\Delta y|^{3/2}$, where Δy is the indentation of the ball into the ground as illustrated in Fig. 10.15. This force-law is called the Hertz law for the deformation of a solid sphere. In this exercise we will find the maximum indentation of the ball—how long the surface of the ball has been deformed when the ball stops.
(**a**) Find the velocity v_1 of the ball as it just touches the ground.
(**b**) Draw a free-body diagram for the ball when it is in contact with the gound.
(**c**) Find the work done by gravity, W_G, and the deformation force, W_k, as a function of the vertical position y of the ball, when the ball is in contact with the ground.

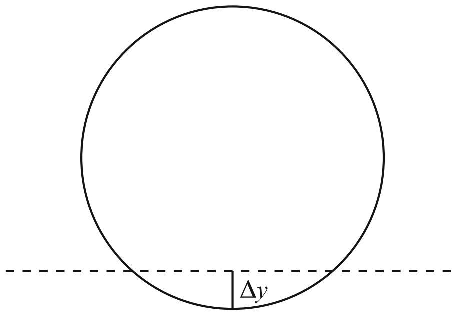

Fig. 10.15 Illustration of the deformation, Δy, of a ball in contact with the ground

(**d**) Find an equation for the maximum indentation, y_2, of the ball. (You do not have to solve the equation).
(**e**) What is the velocity of the ball when it loses contact with the ground?

10.12 Power of the heart. 7500 liters of blood is pumped through a typical human heart each day. Let us assume that the work done to each liter of blood by the heart corresponds to the work used to lift the blood to 1/2 of the height of a typical male. The average male height is 180 cm and the density of blood is $1065\,\mathrm{kg/m^3}$.
(**a**) How much work is done by the heart every day?
(**b**) What is the (average) power exerted by the heart?

10.13 Power station. At a small hydro-electrical facility operated on a small stream, water is released down a 10 m drop at a rate of 100 L/s.
(**a**) If you were able to convert all the work done by gravity on the water into energy, what would the power generated by the station be?

10.14 Accelerating car. A car engine produces a power of 250 hP and has a mass of 1200 kg.
(**a**) If we ignore the effects of air resistance and friction, how long time does the car need to accelerate to 100 km/h?

10.15 An accelerating motorbike. The engine of a motorbike produces a constant power P. The bike starts at rest and drives in a straight line. We neglect effects of friction and air resistance.
(**a**) Find the velocity of the bike as a function of time?
(**b**) Find the acceleration and show that it is not a constant.
(**c**) Find the position, $x(t)$, for the bike as a function of time.

Projects

10.16 Driving efficiently. In this project we address what energy dissipation mechanisms are dominating when driving a car: braking or air resistance. You will learn

how to use work and energy arguments, and hopefully also realize how to design energy-efficient cars. We start by studying a car driving along a horisontal surface.
(**a**) Identify the forces acting on the car, and draw a free-body diagram for the car.
(**b**) If the car is driving at constant velocity, in what direction does the friction force from the ground on the car act?

Let us first assume that the engine delivers a constant power, P_0, so that the force on the car acting in the direction of motion, F, is $F\,v = P_0 = \text{const.}$, where v is the velocity of the car. You may assume a square-law for the air resistance on the car $F_D = -Dv^2$.
(**c**) Find an expression for the acceleration of the car as a function of the velocity.
(**d**) Show that if the car drives with constant velocity, the velocity is $v = (P_0/D)^{1/3}$. What is the physical interpretation of this velocity?

The air resistance coefficient is $D = (1/2)\,\rho\,C_D\,A$ where $\rho = 1.208\,\text{kg/m}^3$ is the mass density of air (at 20 °C), C_D is the drag coefficient, and A is the cross-sectional area of the car. A few examples are given in the following table.

Car (hp)	Power (hp)	$C_D\,A$ (m^2)	m (kg)
Audi A2 2001 1.4 TDI	89	0.616	1030
Honda Civic 2001 1.4	115	0.682	1091
Hummer H2 2001 V8	316	2.46	3000

(**e**) Assume that 60 % of the engine power is used to propel the car forwards. Find the maximum velocities for each of the cars in the table.

Now, we want to study the motion of a car as it starts from rest, accelerates to a velocity v over a distance b, then travels at a constant velocity v, and finally brakes and comes to rest over a distance b. In total, the car has driven a length L.
(**f**) Sketch the velocity as a function of position for the car.
(**g**) Assume that we can ignore the effect of air resistance as the car accelerates. Find the work W_E done by the driving force on the car from the car starts at rest and until the car reaches the constant velocity v after a distance b.
(**h**) We introduce the effect of air resistance and assume that the car moves the whole distance L at a constant velocity v. (We ignore the brief acceleration and deceleration periods). Find the work W_D done by the driving force over the distance L.
(**i**) Compare the work W_E done in order to accelerate the car to the velocity v and the work W_D done in order to move the car a distance L through the air at constant velocity v. What term is the largest? At what distance L^* does the air resistance term become dominating? Calculate the distance L^* for the cars in the table above. Comment on your results.
(**j**) Based on your results, can you make recommendations for how to design cars for city traffic and for long distance travel?

Chapter 11
Energy

You have now learned to use the work-energy theorem as an alternative formulation of Newton's second law—as a "calculation" tool to determine the motion of an object. Using Newton's second law, we find the motion of an object described as the position as a function of time. The work-energy theorem allows us to find the velocity as a function of position without determining the whole motion, and we have learned that this may be useful, in particular when we are unable to find an exact solution to the equations of motion. While this was a practical application of the work-energy theorem, the real strength of the work-energy theorem is a conceptual change: We go from discussing motion and processes in terms of forces to instead discuss them in terms of *energy* and *energy conservation*.

Conservation laws: One of the most important consequences of Newton's second law and its alternative formulation through the work-energy theorem is the concept of a *conservation law*: That there are quantities in a system that are conserved throughout a process. We observe the system and record a particular quantity. Then we let the system develop in time, and we measure the same quantity again. If the quantity is conserved, it means that the quantity is unchanged, no matter what happens inside the system.

The two most important conservation laws you will learn are the conservation of (mechanical) energy and the conservation of momentum. The conservation laws in mechanics are consequences of Newton's laws of motion, but the conservation law for energy is much more general than that—it is one of the most general laws we know in nature. Throughout your studies of physics you will gradually learn to make energy considerations and energy conservation an integral part of your thought process, starting from this chapter.

In practice the conservation laws represent clever ways to solve physics problems. In many cases we cannot solve the equations of motion we get from Newton's second law directly, but in some cases we can solve the equations we get when we integrate Newton's second law. The conservation law for momentum comes from integrating Newton's second law in time and the conservation law for energy comes from in-

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
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tegrating along a path, integration along the x-axis in one dimension, as introduced through the work-energy theorem.

Overview: In this chapter, we introduce the concept of energy and energy conservation through two examples: A vertical bowshot and an atom moving along a surface. Based on the examples, we introduce the concept of potential energy for a position-dependent force, a positional energy, to complement the kinetic energy, an energy of motion. For objects subject only to position-dependent force, the sum of the potential and kinetic energy is constant. We can therefore interpret a motion as a transfer of energy between kinetic and potential energies.

We show how to calculate the potential energy for a constant force, a spring force, and a general position-dependent force, and how to use energy conservation to solve mechanics problems. We introduce the energy diagram as an alternative way to analyze and understand motion. We generalize the concept of potential energy to two- and three-dimensional motion. Finally, we introduce the general energy principle, the second law of thermodynamics, and relate external work to changes in the total energy of a system.

11.1 Motivating Examples

The work-energy theorem provides us with an alternative formulation of Newton's second law that is particularly suited to find the velocity as a function of position in the case when only position-dependent forces act on an object. But we can simplify the analysis even further by the introduction of a position-dependent energy to complement the velocity-dependent kinetic energy, allowing us to use the conservation of total energy as an every quicker way to resolve questions relating the velocity and the position of an object subject only to position-dependent forces. We demonstrate this method by addressing a vertical bowshot and motion along a surface.

Work-Energy as a Conservation Law

How can the work-energy integral be considered a conservation law? And what do we mean by conservation? We know that any motion must satisfy Newton's second law and its path integral, the work-energy theorem. Let us see how this behaves in a simplified case—for a one dimensional motion with a net force that only depends on the position, $F^{\text{net}} = F(x)$. The work-energy integral is then

$$W_{0,1} = \int_{t_0}^{t_1} F(x(t))v(t)\,dt = \int_{x_0}^{x_1} F(x)\,dx = \phi(x_1) - \phi(x_0) = K_1 - K_0 \ , \quad (11.1)$$

where we have introduced the function $\phi(x)$, which is the indefinite integral of $F(x)$, so that $dF/dx = \phi(x)$. This equation is valid for any two points x_0 and x_1 along the motion. We can rearrange this equation, so that all the quantities related to position 0 is on the left hand side and all the quantities related to position 1 is on the right hand side:

$$K_0 - \phi(x_0) = K_1 - \phi(x_1) \ . \tag{11.2}$$

The function, $K - \phi(x)$ is a *constant* along the motion—it is *conserved* for the motion. We have found an example of a conservation law! Let us examine this conservation law in two simple examples to gain more intuition:

Vertical Shot

If you shoot an arrow vertically upwards, and we neglect air resistance, the arrow is affected by gravity, $G = -mg$, alone. This is a one-dimensional motion with a net force that only depends on position. (The force is constant and does therefore not depend on anything else than the position). If the arrow starts from $y = y_0$ with a velocity v_0, we can apply the work-energy theorem to find the kinetic energy, K_1, at any other position, y_1. And from the kinetic energy we can find the velocity, v_1, if so we please. The work-energy theorem gives:

$$W_{0,1} = \int_{t_0}^{t_1} G v_y \, dt = \int_{y_0}^{y_1} -mg \, dy = mgy_0 - mgy_1 = K_1 - K_0 \ , \tag{11.3}$$

Let us rearrange this equation so that everything that refers to position 0 is on the left hand side, and everything that relates to position 1 is on the right hand side:

$$mgy_0 + K_0 = mgy_1 + K_1 \ . \tag{11.4}$$

What does this mean? The left hand side is a constant that depends on the initial position, y_0, and the initial velocity, v_0, of the arrow. But the position y_1 on the right hand side can be any position along the motion: This means that the sum $mgy + K$ is constant throughout the motion. How can we interpret the various terms? We have already introduced the notion "kinetic energy" for the velocity-dependent term, $K = (1/2)mv^2$. The mgy term must therefore also have units of energy, and we can interpret this as an energy too. However, this energy does not depend on the velocity of the arrow, but on its position. We call this a positional energy, or more commonly, it is called a *potential energy*, $U = mgy$. We see that U is related to the function ϕ we defined in (11.1). We can now write the work-energy theorem for the motion as:

$$U_0 + K_0 = U_1 + K_1 = E \ , \tag{11.5}$$

where we use the term *total energy* for the term E, which is the sum of the potential and kinetic energies of the arrow.

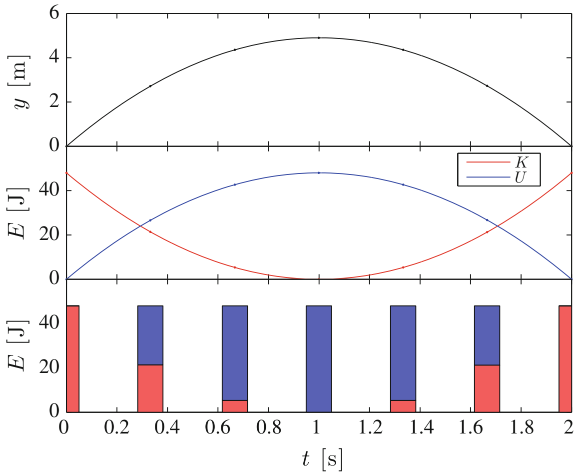

Fig. 11.1 Plot of the position, $y(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an *arrow* shot upward from $y_0 = 0$ m with initial velocity $v_0 = 9.8$ m/s

These new concepts provide insights into the motion of the arrow, as illustrated in Fig. 11.1. The plots show the position $y(t)$ and the kinetic and potential energies $K(t)$ and $U(t)$ for the motion. We see that initially, the arrow only has kinetic energy and no potential energy. This is also illustrated by the bar diagram at the bottom of Fig. 11.1. As the arrow moves upward, the kinetic energy reduces and the potential energy increases until the arrow reaches its maximum height, where the kinetic energy is zero and the potential energy is at its maximum, corresponding to the initial kinetic energy of the arrow. The kinetic energy cannot be negative, and therefore the arrow cannot go higher than this. As the arrow falls down again, the kinetic energy increases as the potential energy is reduced. The whole process is often illustrated by the bar chart diagram, which indicates that the total energy is conserved. It is only how the energy is distributed between the kinetic and potential energies that varies throughout the motion.

Motion Along a Periodic Surface

An atom moving along an atomic surface is subject to a periodic force $F_x(x)$:

$$F(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right) , \tag{11.6}$$

where b is the interatomic distance in the surface and F_0 gives the strength of the interaction. If this is the only force acting on the atom, we can use the work-energy theorem to determine the velocity of the atom as a function of position. If the atom moves from x_0 to x_1, the work-energy theorem gives us that:

$$W_{0,1} = \int_{x_0}^{x_1} F(x)dx = \int_{x_0}^{x_1} -F_0 \sin\left(\frac{2\pi x}{b}\right) dx \tag{11.7}$$

$$= \frac{F_0 b}{2\pi} \cos \frac{2\pi x_1}{b} - \frac{F_0 b}{2\pi} \cos \frac{2\pi x_0}{b} \tag{11.8}$$

$$= -U(x_1) + U(x_0) + K_1 - K_0 \ , \tag{11.9}$$

where we have gotten wiser and have introduced the function $U(x) = U^* \cos (2\pi x/b)$, and $U^* = F_0 b/2\pi$, as the *potential energy* of the atom. We can reorganize the terms, so that all the x_0 terms are on the left-hand side:

$$U(x_0) + K_0 = U(x_1) + K_1 \ . \tag{11.10}$$

We notice that the potential energy, $U(x)$, may be both positive and negative. Hmmm. Is that allowed? Yes. We have not said anything about the signs of the potential energy. Although, the way we have defined kinetic energy does not allow this to become negative. Still, you may be uncomfortable with an initial negative potential energy (although you should get used to this thought by the end of this chapter). There is a simple solution to this: We can add a constant to both sides in (11.10)—and we can add a constant to the function $U(x)$ since we only care that its derivative $dU/dx = -F(x)$ and the derivative of a constant will be zero. This means that you are free to determine the zero level of the potential energy. Here, we would like the potential energy to be zero for the initial state, where $x = x_0$, which we achieve with the potential energy:

$$U_1' = U_1 + U^* = U^* \left(1 - \cos \frac{2\pi x}{b}\right) \ . \tag{11.11}$$

This is nice and positive for all values of x_1 and it is zero at $x = 0$. Good. Let us use this expression for the potential energy, giving the conservation law:

$$U'(x) + K = U^* \left(1 - \cos \frac{2\pi x}{b}\right) + K = \text{const.} \ . \tag{11.12}$$

This conservation law gives useful insight into the motion illustrated in Fig. 11.2. The plots show the position $x(t)$ of the atom for a small initial velocity v_0, and corresponding plots of the kinetic, $K(t)$, and potential, $U'(t)$, energies. The atom start with an initial kinetic energy, but no potential energy (we designed the potential energy this way, remember). As it starts moving in the positive x-direction, the potential energy increases, and the kinetic energy decreases, and the atom slows down. Until, at a particular value of x, $x = x_a$, the kinetic energy becomes zero. Since the kinetic energy cannot become less than zero, the atom cannot progress further in the positive x-direction. All the initial kinetic energy is now potential energy. The atom then starts moving in the negative x-direction, increasing its kinetic energy (and therefore speed) until it reaches $x = 0$. Then the kinetic energy decreases again until

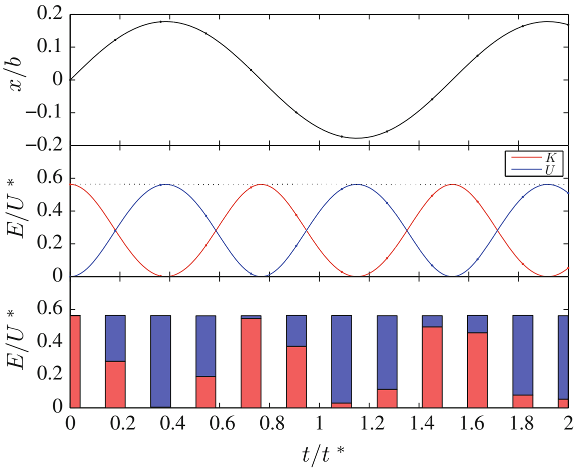

Fig. 11.2 Plot of the position, $x(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an atom moving along an atomic surface from $x_0 = 0$ with initial velocity $v_0 = 0.75\sqrt{2U^*/m}$

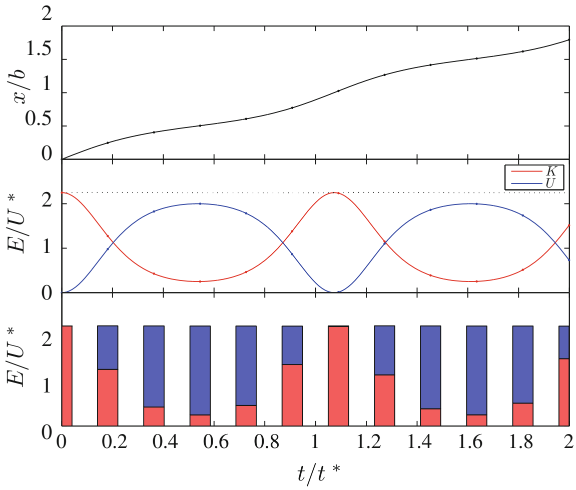

Fig. 11.3 Plot of the position, $x(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an atom moving along an atomic surface from $x_0 = 0$ with initial velocity $v_0 = 1.5\sqrt{2U^*/m}$

the atom stops at a position $x = x_b$ (because the potential energy, $U'(x)$ is symmetric around $x = 0$, we know that $x_b = -x_a$). Throughout this oscillation back and forth the total energy is distributed between kinetic and potential energy in such a way that the total remains a constant.

What happens if we start the atom from the same position, x_0, but with a larger initial velocity, v_0? Fig. 11.3 illustrates the motion, $x(t)$, and energies for a large initial velocity, v_0. Since the initial velocity is larger, the total energy, corresponding to the sum of the kinetic and potential energies, $E = K + U'$, is also larger than in Fig. 11.2. The total energy is illustrated by the dotted line in Figs. 11.2 and 11.3. Also in this case, we see that as the atom starts to move in the positive x-direction, the potential energy increases and the kinetic energy decreases. However, in this case, the kinetic energy never goes to zero. The atom therefore continues to move indefinitely in the positive x-direction. The velocity is therefore always positive, but it varies in magnitude according to the potential energy. You may wonder at what initial velocity v_0 the atom starts to be "free"—when does the kinetic energy not become zero? For now, we leave that for you to figure out, but we will revisit this situation later in the chapter.

These examples introduces the concepts of potential energy and energy conservation. In the rest of the chapter we will put these concepts in a more formal setting, and see many examples of the power of their use.

11.2 Potential Energy in One Dimension

Through these examples we have introduced the concept of potential energy, defined through the work-energy theorem. For a one-dimensional motion where the object is subject to a single force $F(x)$, which only depends on the position and not on the velocity or the time, we can introduce a potential energy for the force and a conservation law for the motion. The work-energy theorem for a motion from x_0 to x_1 gives

$$W_{0,1} = \int_{t_0}^{t_1} F(x(t)) \frac{dx}{dt}\, dt = \int_{x_0}^{x_1} F(x)\, dx = K_1 - K_0 \ , \tag{11.13}$$

We can now introduce the function $U(x)$, which we call the *potential energy*, so that the work integral becomes

$$W_{0,1} = \int_{x_0}^{x_1} F(x)\, dx = U(x_0) - U(x_1) = U_0 - U_1 \ . \tag{11.14}$$

Notice the choice of signs in this equation, introduced this way so that we could "pair" the energies at 0 and 1:

$$U_0 + K_0 = U_1 + K_1 \ , \tag{11.15}$$

In the examples above we found that for gravity:

$$U(x) = mgx \ , \tag{11.16}$$

and for atomic motion along a surface

$$U(x) = U^* \left(1 - cos \frac{2\pi x}{b}\right) \ . \tag{11.17}$$

We can find the *potential energy* at any x by calculating the integral of $-F(x)$ from some point x^*:

$$\boxed{U(x) = U(x^*) + \int_{x^*}^{x} -F(x)\,dx\ .} \tag{11.18}$$

Hmmm. What is the value of $U(x^*)$? We do not know. Actually, we are free to choose any value we like for $U(x^*)$, because the change in velocity from x_0 to x_1 only depends on the difference between the potential energy at $U(x_0)$ and $U(x_1)$. This may seem confusing at first—that the potential energy is not given as a particular number, but rather as a function with an undetermined zero level. But you may have some experience with this already based on your knowledge of calculus. If you look at (11.18) you may recognize that this defines the potential enegy $U(x)$ as the indefinite integral of $F(x)$:

$$U(x) = \int -F(x) + C\ , \tag{11.19}$$

that is, the function $U(x)$ is the anti-derivative of $F(x)$. We could put this the other way, the definition of the potential energy $U(x)$ in (11.18) implies that $-F(x)$ is the derivative of $U(x)$:

$$\boxed{F_x(x) = -\frac{dU}{dx}\ .} \tag{11.20}$$

We see that whatever constant we add to $U(x)$ does not affect the force we get from taking the derivative.

Conservation of Energy

These relations define the potential energy $U(x)$ and relate it to the force $F(x)$ on the object. Notice that these definitions are general, and are valid also in cases where we cannot find a closed expression for the integral of $F(x)$. We have therefore shown that (11.20) is general, showing that for an object subject to a single, position-dependent force, $F(x)$, the total energy is conserved throughout the motion:

$$E_0 = U(x_0) + K_0 = U(x_1) + K_1 = E_1\ . \tag{11.21}$$

We say that the *mechanical energy of the object is conserved*.

We call the corresponding force $F(x)$ a *conservative force*, if the mechanical energy is conserved. The mechanical energy is conserved if there exists a potential $U(x)$ for the force $F(x)$, or, to formulate it in an alternative way: If the work done by the force only depends on the end-points of the motion. Both statements may be used as equivalent definitions of a conservative force:

The work done by a **conservative force** only depends on the end-points of the motion and not on the path taken.

A force $F(x)$ is **conservative** if and only if it can be written as the derivative of the potential $U(x)$:

$$F_x(x) = -\frac{dU}{dx} \ . \tag{11.22}$$

We call $U(x)$ the potential for $F(x)$ or the potential energy of the object.

In one dimension, a force $F(x)$ that only depends on the position x (and not on the velocity or time) is conservative, since we can always find a potential $U(x)$, as demonstrated above. But in two- and three-dimension, a force that depends only on the position is generally not conservative. A conservative force needs to fulfil an additional condition that we return to later.

Potential Energy for a Constant Force

An object moves from x_A to x_B while subject to a constant net force, F_0, in the x-direction. What is the potential energy for the object?

We find the potential energy from the integral in (11.18), starting from $x = x_0$:

$$U(x) = U(x_0) + \int_{x_0}^{x} -F_0 \, dx = U(x_0) - F_0 \, (x - x_0) \ , \tag{11.23}$$

where we are free to choose the value for $U(x_0)$. For example, we may choose $U(x_0) = -F_0 x_0$, which gives:

$$U(x) = -F_0 x \ . \tag{11.24}$$

We recognize this from our previous calculation of the potential energy for the vertical bowshot, where the arrow is only subject to a (constant) gravity, $G = F_0 = -mg$, and therefore:

$$U(x) = -F_0 x = -(-mg)\, x = mgx \ . \tag{11.25}$$

Energy conservation for an object subject to a constant force can therefore be expressed as:

$$E_0 = U_0 + K_0 = U_1 + K_1 = E_1 \tag{11.26}$$

$$E_0 = -F_0 x_0 + \frac{1}{2} m v_0^2 = -F_0 x_1 + \frac{1}{2} m v_1^2 = E_1 \ , \tag{11.27}$$

and for motion in the gravity field, we recover the well-know result:

$$m g x_0 + \frac{1}{2} m v_0^2 = m g x_1 + \frac{1}{2} m v_1^2 \ . \tag{11.28}$$

Potential Energy in the Gravity Field with Normal Forces

The conservation of mechanical energy for motion subject only to gravitational forces is a useful tool to find simple solutions and make quick judgements about motions. However, the method is even more versatile. We can also use it when an object is subject to gravity and a normal force, because the normal force does no work on the object during the motion.

Let us demonstrate the use of potential energy methods for motion under gravity and normal forces by studying the motion of a bead moving along a friction-free wire shaped as a helix, as illustrated in Fig. 11.4. The forces acting on the bead are gravity, $\mathbf{G} = -mg\,\mathbf{j}$, and a normal force, $\mathbf{N}$, from the wire on the bead.

We apply the net work-energy theorem for motion from position 0 at a height y_0, to a point P at a height y. The work done on the bead by the net force $\mathbf{F}^{\text{net}} = \mathbf{G} + \mathbf{N}$ is then:

$$\begin{aligned} W^{\text{net}} &= \int_0^P \underbrace{\mathbf{F}_{\text{net}}}_{=\mathbf{G}+\mathbf{N}} \cdot d\mathbf{r} = \int_0^P (\mathbf{G}+\mathbf{N}) \cdot d\mathbf{r} = \int_0^P \mathbf{G} \cdot d\mathbf{r} + \int_0^P \mathbf{N} \cdot d\mathbf{r} \\ &= \mathbf{G} \cdot \int_0^P d\mathbf{r} + \int_{t_0}^t \underbrace{\mathbf{N} \cdot \mathbf{v}}_{=0} \, dt = -mg(y - y_0) + 0 \\ &= mgy_0 - mgy = K - K_0 \ , \end{aligned} \tag{11.29}$$

where we place all the terms for state 0 on the left side and for state 1 on the right side, getting:

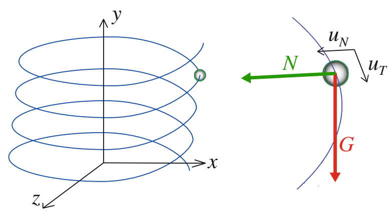

Fig. 11.4 *Left* A bead moves along a *vertical helix-shaped* wire. *Right* Free-body diagram for the bead

$$mgy_0 + K_0 = mgy + K \ , \tag{11.30}$$

which shows that mechanical energy is conserved for motion under gravity with a normal force, which is what we expected since the normal force does no work during the motion. The shape of the path does not matter: The energies depend only on the vertical position of the bead.

Potential Energy for a Spring Force

We have already discussed the potential for a constant force. The next step is to study the simlest possible (one dimensional) force that depends on position alone: the spring force. What is the potential energy of an object subject to a spring force $F(x)$?

Figure 11.5 illustrates a spring block system, where a block of mass m lies on a frictionless table. The block is attached to a spring with spring constant k, and the equilibrium position of the block-spring system is b. The force F on the block from the spring is modelled using a spring force model: $F(x) = -k(x - b)$, where x is the position of the block.

The force F depends on the position of the block only: It does not depend on the velocity of the block, and it does not depend directly on time. We can therefore find the potential energy by integrating the force $F(x)$ using the definition in (11.18):

$$U(x) - U(x_0) = \int_{x_0}^{x_1} -F(x)\,dx = \int_{x_0}^{x_1} k(x - b)\,dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2 \ . \tag{11.31}$$

We are free to choose $U(x_0)$, so we may choose $U(x_0)$ so that $U(x)$ is zero when the spring is in its equilibrium position. That is, we want $U(b) = 0$:

$$U(b) = U(x_0) + \frac{1}{2}kb^2 - \frac{1}{2}kx_0^2 = 0 \ \Rightarrow \ U(x_0) = \frac{1}{2}kx_0^2 \ . \tag{11.32}$$

The potential energy for a spring force $F(x) = -k(x - b)$ is therefore:

$$U(x) = \frac{1}{2}k(x - b)^2 \ . \tag{11.33}$$

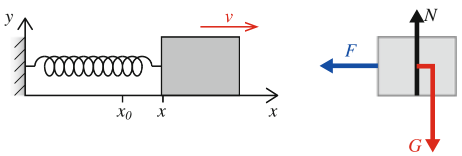

Fig. 11.5 *Left* Illustration of a block attached to a spring. *Right* A free-body diagram for the block

Potential Energy of Many Individual Forces

In generatl, objects are affected by several forces simultaneously. For example, a person jumping on a trampoline is affected both by gravity and by the contact force from the trampoline. How can we extend our energy conservation approach to such cases?

The net force on an object affected by several forces F_j, where $j = 1, 2, \ldots,$ is

$$F^{\text{net}} = \sum_j F_j(x) \; . \tag{11.34}$$

Let us assume that each of the forces $F_j(x)$ are conservative, that is, they depend on the position x only (and not on velocity or time). The potential energy for force $F_j(x)$ is $U_j(x)$. The work-energy theorem for a motion from x_0 to x_1 gives:

$$\begin{aligned} W_{0,1}^{\text{net}} &= \int_{x_0}^{x^1} F^{\text{net}} dx = \int_{x_0}^{x_1} \sum_j F_j(x) dx = \sum_j \int_{x_0}^{x_1} F_j(x) dx \\ &= \sum_j \big(U_j(x_0) - U_j(x_1) \big) = K_1 - K_0 \; . \end{aligned} \tag{11.35}$$

We move all the terms relating to position 0 to the right-hand side of the equation:

$$\underbrace{\sum_j U_j(x_1)}_{U(x_1)} + K_1 = \underbrace{\sum_j U_j(x_0)}_{U(x_0)} + K_0 \; . \tag{11.36}$$

where the potential energy $U(x)$ is the sum of the potential energies for each of the forces $F_j(x)$. This gives a general law for energy conservation:

$$U(x_1) + K_1 = U(x_0) + K_0 \; . \tag{11.37}$$

We can therefore generalize the energy conservation methods introduced above by introducing a potential energy for each of the forces affecting the object. Energy conservation still holds, as long as we include the sum of all the potential energies for all the forces affecting the object.

11.2.1 Example: Falling Faster

Problem: You throw an apple directly downward from a high cliff, giving it an initial velocity v_0. How far does the apple need to fall before it doubles its velocity? You can neglect the effects of air resistance.

Identify: The position of the apple is given by $y(t)$. At t_0, the apple is at $y = y_0$ with velocity $v = -v_0$. At t_1, the apple is at $y = y_1$ with velocity $v_1 = -2v_0$.

Model: The only force acting on the apple is gravity. We know that for motion under gravity, the total mechanical energy of the object is conserved throughout the motion:

$$E = U + K = \text{constant} \tag{11.38}$$

We use this to find the relation between the position and velocity of the apple.

Solve: The mechanical energy of the apple when at y_0 is:

$$E_0 = U_0 + K_0 = mgy_0 + \frac{1}{2}mv_0^2 \ , \tag{11.39}$$

and at y_1 it is:

$$E_1 = U_1 + K_1 = mgy_1 + \frac{1}{2}mv_1^2 \ . \tag{11.40}$$

Since energy is conserved, we know that:

$$\begin{aligned} E_0 &= E_1 \\ mgy_0 + \frac{1}{2}mv_0^2 &= mgy_1 + \frac{1}{2}mv_1^2 \end{aligned} \tag{11.41}$$

We want to find y_1 when we know that $v_1 = -2v_0$. We insert $v_1 = -2v_0$, and find:

$$mgy_0 + \frac{1}{2}mv_0^2 = mgy_1 + \frac{1}{2}m\left(-2v_0\right)^2 \ . \tag{11.42}$$

We solve for y_1, finding the position where the velocity is doubled:

$$mgy_0 + \frac{1}{2}mv_0^2 - 4\frac{1}{2}mv_0^2 = mgy_1 \ \Rightarrow \ y_0 - \frac{3}{2}\frac{v_0^2}{g} = y_1 \ . \tag{11.43}$$

11.2.2 Example: Roller-Coaster Motion

Problem: Let us revisit the roller-coaster problem using an energy conservation approach: A roller-coaster cart is rolling from the height h to the height 0 along a curving roller-coaster track. It starts with the speed v_0 at the top of the track. Find the speed of the roller-coaster cart at the bottom of the track. You can ignore air resistance and friction.

Identify: The cart moves from the position 0, $x(t_0) = x_0$, $y(t_0) = y_0$, to position 1, $x(t_1) = x_1$, $y(t_1) = y_1$ as illustrated in Fig. 11.6.

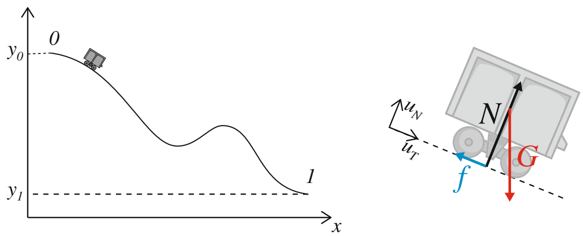

Fig. 11.6 A roller-coaster cart moving along a roller-coaster track

Model: The cart is affected by the normal force, **N**, the friction force, **f**, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. We assume that friction is negligible, $\mathbf{f} = 0$ throughout the motion.
Solve: Since the normal force, **N**, does no work on the cart during the motion, we can use the conservation of energy to determine the relation between position and velocity of the cart. The conservation of mechanical energy of the cart gives:

$$E = K_0 + U_0 = K_1 + U_1 \ , \tag{11.44}$$

where we use that $U(y) = mgy$:

$$\frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_1^2 + mgy_1 \ , \tag{11.45}$$

From this equation, we find the velocity v_1 as a function of the height difference, $h = y_0 - y_1$ and the initial velocity v_0 of

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2 + mg\underbrace{(y_0 - y_1)}_{=h} \ \Rightarrow \ v_1^2 = v_0^2 + 2gh \ . \tag{11.46}$$

Analyze: This is, of course, exactly the same result as we found above when we used the net work-energy theorem. However, the use of energy considerations is much simpler and allows us to use more of our intuition about the process than the application of the net work-energy theorem.

11.2.3 Example: Pendulum

Problem: A classical problem is that of the motion of a pendulum. A pendulum consists of sphere of mass m hanging in a massless rope of the length L. The pendulum moves in a vertical plane with a maximum angle θ_0 with the vertical. Find the velocity v of the sphere as a function of the angle θ. We neglect air resistance.

Identify: We describe the motion of the pendulum with the angle θ between the rope and the vertical. We assume that the sphere follows a circular path. The system is illustrated in Fig. 11.7.

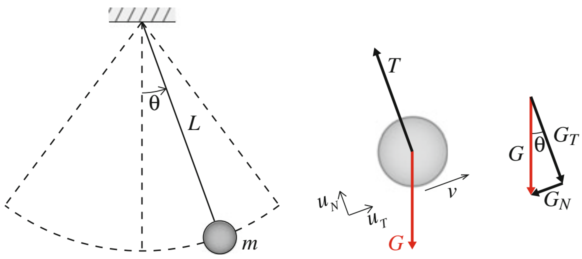

Fig. 11.7 *Left* Illustration of a pendulum. *Right* Free-body diagram for the sphere

Model: The sphere is affected by the tension, **T**, from the rope and gravity, **G**. We already know that for motion under gravity we can use energy considerations to relate position and velocity. What about the tension **T**? It is always normal to the direction of motion of the sphere. The rope tension **T** is a *normal force*, which does no work during the motion. We can therefore use energy conservation to solve the problem.

We examine two positions, as illustrated in Fig. 11.8, and use energy conservation to relate the two positions. In position 0 the sphere is at the top of its path. This is where the motion changes direction. The velocity is zero at this position. In position 1 the sphere is at an angle θ and the velocity of the sphere is v.

Energy conservation gives:

$$E = U_0 + K_0 = U + K \ , \tag{11.47}$$

$$E = mgy_0 + K_0 = mgy + K \ , \tag{11.48}$$

In order to find the velocities as a function of θ, we need to relate the vertical positions y_0 and y to the angle θ. From Fig. 11.8 we see that the vertical position of the sphere is

$$y_0 = -L\cos\theta_0 \ \text{ and } \ y = -L\cos\theta \ , \tag{11.49}$$

where we have chosen the origin of the coordinate system to at the attachment point of the pendulum. We insert this into (11.48):

$$-mgL\cos\theta_0 + \frac{1}{2}mv_0^2 = -mgL\cos\theta + \frac{1}{2}mv^2 \ , \tag{11.50}$$

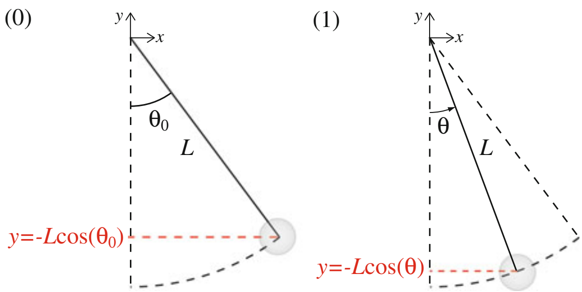

Fig. 11.8 Illustration of the positions. In postion 0 (*left*) the sphere is at the *top* of its path. In position 1 (*right*) the sphere is at the angle θ

and we solve to find the velocity v:

$$v^2 = 2gL\,(\cos\theta - \cos\theta_0) \;\Rightarrow\; v = \sqrt{2gL\,(\cos\theta - \cos\theta_0)}\;, \tag{11.51}$$

where we have used that $v_0 = 0$.

Analyze: For this problem, energy considerations are particularly powerful, since it is difficult, if not impossible, to find an analytical solution to the motion of the pendulum. But using energy conservation we find the exact, analytical solution to the posed question, without solving for the path $\theta(t)$.

11.2.4 Example: Spring Cannon

This problem is a classic in mechanics

Problem: A block of mass m is placed on top of a vertical, massless spring with spring constant k. The spring is contracted a distance Δy from its equilibrium position (when there is no block on top of it). The spring is released, and the block is shot up through the air. How high up does the block go? You may neglect friction and air resistance.

Identify: The motion of the block is one dimensional, and we characterize the position of the block by its vertical position y. The block starts at a vertical position y_0 at the time t_0 with no vertical velocity $v_0 = 0$. The block is in contact with the spring until the time t_1 when the vertical position is y_1. From there on, the block is moving through the air without being in contact with the spring. The block reaches its maximum height y_2 at the time t_2, and the velocity of the block at this point is $v_2 = 0$. The process is illustrated in Fig. 11.9.

Model: The block is affected by two forces: The spring force $\mathbf{F}$ and the force from gravity, $\mathbf{G} = -mg\,\mathbf{j}$. Both gravity and the spring force are conservative forces. We can therefore apply energy conservation to relate the position and velocity of the block. However, the spring force is only acting when $y < y_1$.

Solve: We use conservation of mechanical energy to solve this problem. Energy is conserved because all forces only depend on the position of the block. They do not depend on the velocity of the block or directly on time. Mechanical energy is therefore conserved throughout the motion. In particular, it is the same at the points 0, 1, and 2.

$$E_0 = U_0 + K_0 = U_1 + K_1 = E_1 = U_2 + K_2 = E_2\;. \tag{11.52}$$

Here, we realize that the total potential energy U consists of the potential energy from the spring force, U_F and the potential energy from gravity, U_G:

$$U(y) = U_G(y) + U_F(y)\;. \tag{11.53}$$

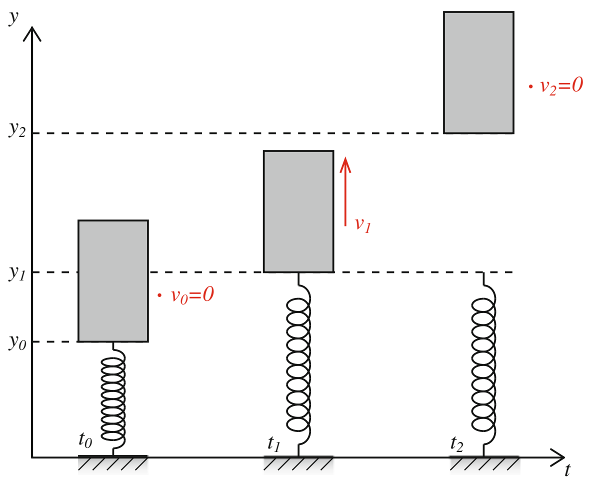

Fig. 11.9 Illustration of the motion of a block ejected by a spring cannon. At the point t_0, the spring is compressed to the position y_0, and the block is at rest. At t_1, the spring reaches its equilibrium length, and the block looses contact with the spring. At t_2 the block reaches it maximum height when $v_2 = 0$

$$U_G(y) = mg(y - y^*) \ , \tag{11.54}$$

where we are free to chose where the potential energy is zero. That is, we are free to choose the value y^*. Here, we simply choose $y^* = 0$.

Also, we know that the potential energy of the interaction between the spring and the block is

$$U_F(y) = \frac{1}{2}k(y - y_1)^2 \ , \tag{11.55}$$

where we have chosen the potential energy of the spring-block interaction to be zero when the spring is at its equilibrium position, that is, for $y = y_1$. We find the energies at the three positions 0, 1, and 2:

$$E_0 = U_G(y_0) + U_F(y_0) + K_0 = mgy_0 + \frac{1}{2}k(y_0 - y_1)^2 + \underbrace{\frac{1}{2}mv_0^2}_{=0\ (v_0=0)} \tag{11.56}$$

$$E_1 = U_G(y_1) + U_F(y_1) + K_1 = mgy_1 + \frac{1}{2}k(y_1 - y_1)^2 + \frac{1}{2}mv_1^2 \tag{11.57}$$

$$E_2 = U_G(y_2) + U_F(y_2) + K_2 = mgy_2 + 0 + \frac{1}{2}m\underbrace{v_2^2}_{=0} \ . \tag{11.58}$$

In position y_2, the block is not in contact with the spring. The force from the spring on the block is therefore zero from y_1 to y_2. The potential energy of the spring-block interaction is thererfore the same for y_1 and for y_2, that is, the potential energy of the spring-block interaction is also zero at y_2.

Notice that even though we include the position y_1 in this calculation, we do not need this in order to determine y_2. Because energy is conserved throughout the whole motion, we only need to relate position 0 to position 2: $E_0 = E_2$. This is a particularly nice feature of using energy considerations when solving a problem!

We can now solve to find the height y_2:

$$E_0 = E_2 \Rightarrow mgy_0 + \frac{1}{2}k \underbrace{(y_0 - y_1)^2}_{=\Delta y^2} = mgy_2 \Rightarrow y_0 + \frac{\frac{1}{2}k\Delta y^2}{mg}\Delta y^2 = y_2 \, , \tag{11.59}$$

Analyze: This example illustrates the power and ease of using energy considerations for such a problem. If we instead opted to use Newton's laws of motion, the calculations would be significantly longer. But remember that we can only use energy considerations when all the forces are conservative: If there was friction, air resistance, or other velocity-dependent force, we could not have used energy considerations directly, and in most cases, we would need to determine the motion in order to determine the work done by the velocity-dependent forces, and we would need to solve for the motion anyway.

Comment: Notice how we chose the potential energy for the spring-block interaction. We do not need to choose the potetial energy $U_F(y)$ to be zero at y_1, we could for example also have chosen it to be:

$$U_F(y) = \frac{1}{2}k(y - y_1)^2 + C \tag{11.60}$$

where C is a constant. However, in this case, we need to ensure that we include this constant both in the energy at position y_1 *and* at the position y_2. Why also at position y_2—there is no spring force acting here? It is correct that we are free to choose the constant C, but we are not free to change this choice between the various phases. If we have chosen a particular C, we must ensure that our choise is consistent when we go from y_1 to y_2.

11.3 Energy Diagrams

If you launch a small cart in a hilly terrain, as illustrated in Fig. 11.10, how is the subsequent motion of the cart? For simplicity, we assume that we can neglect the air drag and friction. What is the velocity as a function of position, and how far will the cart travel?

We can visualize the motion directly on the figure. Several points have been highlighted in Fig. 11.10. The cart starts at the point 0. The cart rolls downhill, increasing its velocity. It reaches the bottom of the hill at 1, and starts rolling uphill. Will it make it up the hill?

Energy conservation: We can answer such questions using energy conservation. Because the only forces acting on the cart are gravity and the normal force, and the

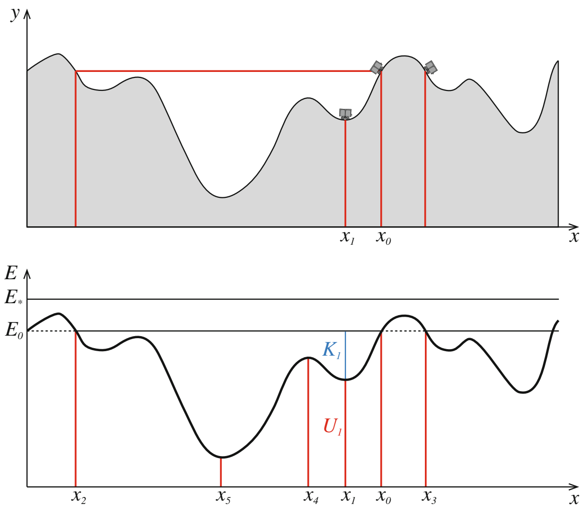

Fig. 11.10 *Top* Illustration of a cart rolling on a frictionless surface. *Bottom* Energy diagram for the motion of the cart

normal force does not do any work, the mechanical energy of the cart is conserved throughout the motion. This means that the sum of the potential and the kinetic energy is constant:

$$E = U + K = U(x) + K(x) \ , \tag{11.61}$$

where the potential energy for the cart is the potential energy in a (constant) gravity field:

$$U(x) = mgy(x) \ . \tag{11.62}$$

The vertical position, $y(x)$, is simply the vertical position of the cart as a function of the horizontal position x. This corresponds to the vertical position of the hill at that horizontal position.[1]

Potential energy landscape: We can therefore construct a plot of the potential energy of the cart as a function of the horizontal position x. What will this drawing look like? We know that the potential energy is $U(x) = mgy(x)$. The potential energy therefore looks exactly like the landscape, as illustrated in the bottom plot in Fig. 11.10.

The cart starts in point 0 at the position x_0, y_0 on the terrain. The same point is illustrated in the plot of the potential energy. At this point, the cart has no kinetic energy since it starts with zero velocity. At the point 0, the total internal energy is:

$$E = U_0 + K_0 = U_0 \ , \tag{11.63}$$

but since the total energy is conserved throughout the motion, the sum of the kinetic and potential energy is constant and equal to this value everywhere. We illustrate the

[1]If we assume that the cart has no height, that is, that the cart is only a point mass moving along the surface.

total energy by drawing a horizontal line in the potential energy plot. From the figure we can now read off both the total energy, this is the horizontal line, and the potential energy as a function of position—this is the curve that looks like the terrain.

Interpreting kinetic energy: How can we use this figure to understand the motion? At the point 1, the potential energy is smaller than the total energy:

$$E_1 = U_1 + K_1 = E_0 \ , \tag{11.64}$$

This means that the amount of energy above the potential energy curve is the kinetic energy, illustrated by the blue line in the figure. We can therefore read both the potential energy and the kinetic energy directly from the figure: The kinetic energy is the distance from the potential energy curve up to the horizontal line that gives the total energy. We call this diagram an **energy diagram**. The energy diagram is a useful tool for discussing the motion of an object.

Limits of possible motion: The kinetic energy is zero where the horizontal line intersects the potential energy curve (for $E = E_0$ at points x_0, x_2, and x_3.) The cart cannot cross this line. At this point the speed is zero, and the cart changes direction. We can therefore use the energy diagram to determine *the limits of the possible motion*. The cart cannot have negative kinetic energy—for a given total energy, this limits the possible positions of the cart. Consequently, when the total energy is $E = E_0$, the cart cannot reach the regions indicanted by the dashed line.

Notice that for a given total energy, the cart can be locked into separate domains. If the cart starts at position 0 with zero kinetic energy, it will never reach point 3. Similarly, if the cart starts at point 3 with zero kinetic energy, it cannot reach point 0.

However, if we increase the total energy, for example by increasing the speed of the cart, to the value E_* on the figure, the cart can move all over the terrain.

Equilibrium points: How do we interpret a flat part of the terrain? A flat part of the terrain corresponds to a flat part of the potential energy curve. At a flat part, there are no horizontal forces. A flat part is therefore *an equilibrium point*. If the cart starts at rest at an equilibrium point, it will not start moving, because there are no horizontal forces.

Stable and unstable equilibrium points: The top of a hill and the bottom of a valley are also flat points. There are no horizontal forces acting on the cart at these points. But there is a difference between the hilltop and the valleybottom. If you start from the hilltop (e.g. at x_4), and give the cart a tiny push, it accelerates away from the hilltop. We therefore call such a point an *unstable equilibrium point*. However, if you start at the bottom of a valley (e.g. at x_5), and give the cart a tiny push, the cart accelerates back towards the equilibrium point—we therefore call such a point a *stable equilibrium point*.

Limitation of this analogy: While the terrain analogy is visually striking, it has its limitations. In particular, it obscures the relation between the horizontal force and the potential energy curve, since the normal force from the ground on the cart also contributes to the horizontal force. We will therefore continue our discussion of energy diagrams with other examples, where we can use further properties of the potential energy curve in our interpretation of the motion.

Energy Diagram of a Spring Force

We can use an energy diagram to address the motion of a block in a spring. A block of mass m is attached to a spring with spring constant k and equilibrium position b. The block is lying on a horizontal, frictionless table, as illustrated in Fig. 11.11a.

Model: The only horizontal force on the block is the spring force:

$$F(x) = -k(x - b) \; , \tag{11.65}$$

where x is the position of the block. The net horizontal force on the block is $F(x)$, which only depends on the position x of the block. It does not depend on the velocity of the block. The total mechanical energy of the spring-block system is therefore conserved throughout the motion:

$$E = U(x) + K = \text{constant} \; . \tag{11.66}$$

We found in (11.33) that the potential energy $U(x)$ of the spring is:

$$U(x) = \frac{1}{2}k(x - b)^2 \; , \tag{11.67}$$

and this potential energy is illustrated in the energy diagram in Fig. 11.11b.

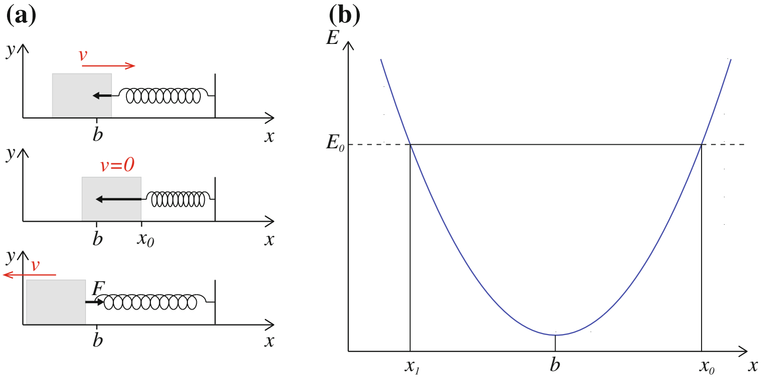

Fig. 11.11 **a** A block on a frictionless, *horizontal* plane is attached to a spring. **b** The energy diagram for the block

Finding motion from the energy diagram: Let us use the energy diagram to determine the motion in a few situations. First, we pull the block from the equilibrium position $x = b$ to a position $x_0 > b$, and let it go. That is, the block starts at the position $x = x_0$ with zero initial velocity. How do we illustrate this in the energy diagram? When we release the block at the position x_0, the block only has potential energy. That is, at $x = x_0$, the total energy is:

$$E_0 = U_0 + K_0 = \frac{1}{2}k(x_0 - b)^2 \ . \tag{11.68}$$

The total energy is given by the intersection between the potential energy curve and the point $x = x_0$. We draw the total energy as a horizontal line through this point as illustrated in Fig. 11.11b.

How does the block move? We know that we can read off the kinetic energy as the distance from the line $E = E_0$ down to the potential energy:

$$K(x) = E_0 - U(x) \ . \tag{11.69}$$

In addition, we know that the spring force on the spring also can be found from the potential energy curve, because we recall that:

$$F(x) = -\frac{dU}{dx} \ . \tag{11.70}$$

When the motion starts, we know that $dU/dx > 0$, which means that the force acts in the negative direction, which we of course can verify from the force law in (11.65), $F(x) = -k\,(x - b)$, which is negative when $x = x_0 > b$. We see from the energy diagram that as the block accelerates towards smaller values of x, the slope of the potential energy curve decreases, and therefore the spring force also decreases, until the block reaches the point $x = b$, where $dU/dx = 0$. Motion continues with $x < b$, but now the force is positive, slowing the block down, until it reaches the point x_1, where the kinetic energy is zero. We see that the force at this point is positive, since the slope of $U(x)$ is negative, hence the net force acts in the positive direction: The spring block is accelerated back towards larger x-values. From this point we realize that the motion repeats itself indefinitely. The block is oscillating back and forth between x_1 and x_0.

The point $x = b$ is a minimum of the potential energy. At this point, the net force is zero, because:

$$F(b) = -\frac{dU}{dx} = 0 \ , \tag{11.71}$$

We call all extremal points, where $dU/dx = 0$, *equilibrium points* because if the block starts from this point, there is no force acting on it. It will therefore stay at this point, in equilibrium.

Equilibrium Points

The simple spring-block system has only one local minimum in the potential energy and no local maximum. We need a more complex system to discuss the general behavior of a system near an equilibrium point. This is found in the motion of an ion trapped between two charged atoms, as illustrated in Fig. 11.12. In this case, the energy diagram contains two local minima and a local maximum.

Extremal points of the potential energy: What characterizes a local extremum of the potential energy function? It is where the potential energy is horizontal, that is, where the slope of the potential energy is horizontal. This occurs when:

$$\frac{dU}{dx} = 0 \, , \tag{11.72}$$

For an energy diagram, we also know that:

$$F(x) = -\frac{dU}{dx} \, . \tag{11.73}$$

The force acting on the system is therefore zero at the extrema of the potential energy function. This is why we call these points *equilibrium points*. The equilibrium points in Fig. 11.12 occur at x_1, x_2, and x_3.

Local minima: What characterizes the behavior around the local minimum at x_2? If we place the particle exactly at this point with zero velocity, the force acting on the particle is zero, and the particle will not move. What happens if the atom did not start exactly at x_2, but very close, $x = x_2 + dx$.

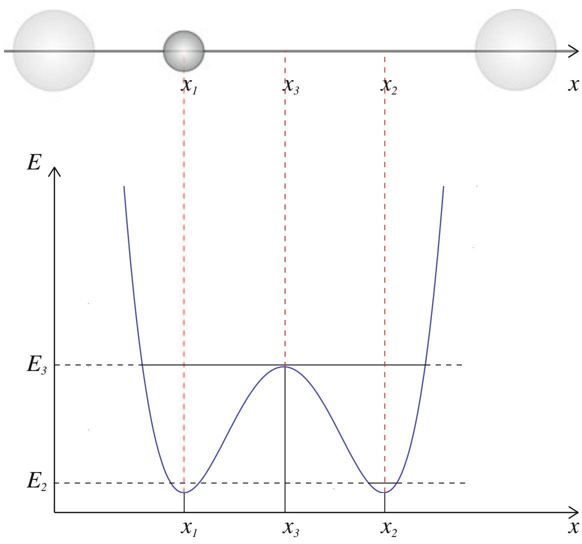

Fig. 11.12 *Top* Illustration of an ion trapped between two charged atoms. *Bottom* Energy diagram for the ion

From the energy diagram, we see that the energy at x must be slightly higher than the energy at x_2, because x_2 is a local minimum. Figure 11.12 shows that if the atom starts at $x = x_2 + dx$, with an energy E_2, the particle will start moving around x_2, but it will not stray far from the equilibrium point. This is also evident directly from the properties of $U(x)$. You may recall from your experience with calculus, that a local *minimum* in $U(x)$ is characterized by

$$\frac{dU}{dx} = 0 \text{ and } \frac{d^2U}{dx^2} > 0 \; . \tag{11.74}$$

The first derivative is zero and the second derivative is positive. What is the physical interpretation of a positive second derivative? Since the force is related to the first derivative, this means that:

$$\frac{dF}{dx} < 0 \; ; \, , \tag{11.75}$$

at this point. Therefore, if the particle is given a small positive velocity, it starts moving in the positive direction. But since $dF/dx < 0$, the force on the particle therefore becomes negative (since it starts from zero at the equilibrium point). Consequently, the force acting on the particle tends to return the particle back toward the equilibrium point. You can make a similar argument when the velocity is negative.

Because the particle will be localized close to the equilibrium point when a small perturbation is applied, we call this equilibrium point a *stable equilibrium point*.

Local maxima: What about the local maximum in $x = x_3$? Also in this case, the force acting on the particle when at this point is zero. This is therefore also an equilibrium point. But what happens if we give the particle a small perturbation, a small increment ΔK to E_3 in the total energy? We see from Fig. 11.12 that in this case the particle will not stay close to the equilibrium point. For any perturbation, the particle will move far away from the equilibrium point—this point is called an *unstable equilibrium point*.

This argument can also in this case be made from a discussion of the behavior of the force $F(x) = -dU/dx$. From calculus, you know that for a local maximum, the second derivative of $U(x)$ is negative:

$$\frac{dU}{dx} = 0 \text{ and } \frac{d^2U}{dx^2} < 0 \; . \tag{11.76}$$

The derivative of the force is therefore positive:

$$\frac{dF}{dx} < 0 \; , \tag{11.77}$$

at the unstable equilibrium point. What does this result in? If we give the particle a small positive velocity starting from the equilibrium point, the particle will move in the positive x-direction, and the force acting on the particle will increase—starting from zero at the equilibrium point. The force will be positive, giving the particle a positive acceleration which will bring the particle even further from the equilibrium point.

11.3.1 Example: Energy Diagram for the Vertical Bow-Shot

Problem: Let us revisit the vertical bow-shot using energy diagrams: Sketch the energy diagram for a vertical bow-shot, find equilibrium points for the arrow, and discuss the motion of the arrow for various total energies.

Potential energy of the arrow: We have previously (see Sect. 11.1) found that the potential energy for an arrow affected by only gravity is:

$$U(y) = mgy \; , \tag{11.78}$$

where y is the vertical position of the arrow. The total mechanical energy of the arrow is conserved throughout the motion:

$$U(y) + K(y) = E_0 = \text{const} \; , \tag{11.79}$$

Energy diagram for the arrow: The plot of $U(x)$ in Fig. 11.13 constitutes an energy diagram. The total energy is conserved and corresponds to a horizontal line, a line of constant energy, in the energy diagram. The potential energy varies with position, $U(y) = mgy$. The kinetic energy can therefore be read from the diagram as the distance (in energy) from the potential energy and up to the total energy.

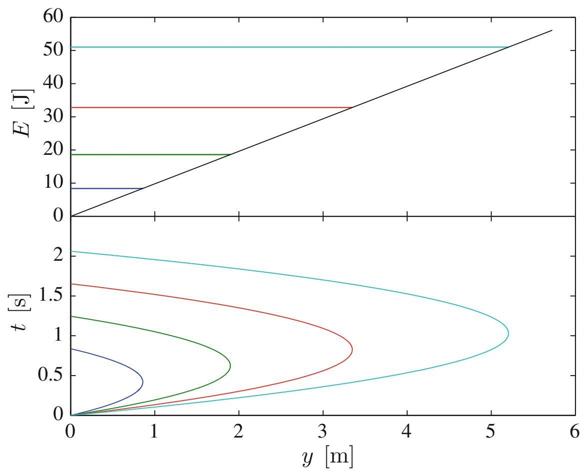

Fig. 11.13 *Top* Energy diagram for a vertical bow-shot. The various *horizontal lines* illustrate different total energies. *Bottom* Illustration of motions ($y(t)$) for each of the total energies shown at the *top*

Initial conditions: The total energy is determined by the initial conditions. For example, we could start the motion from $y = 0\,\mathrm{m}$ with an initial velocity v_0. The total energy would then be given by

$$E_0 = \frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_0^2 \; . \tag{11.80}$$

Each horizontal line, E_0, illustrated in Fig. 11.13 corresponds to a different set of initial conditions for the arrow: Each line corresponds to a different motion.

Equilibrium points: What can we say about the motion of the arrow for a given total energy? First, we see if the energy diagram has any equilibrium points—points where the derivative of the potential is zero. There are no such points: The derivative of the potential is a constant which is not zero. Second, we analyze the motion for a given E_0. Notice that we can only discuss general features of the motion based on the energy diagram alone. From this analysis we do not know at what time the arrow is at a particular position, but we may relate the position and the velocity of the arrow.

Turning points: Since the kinetic energy of the arrow cannot be negative, the arrow cannot propagate into the region where the potential energy is larger than the total energy. At the point where the potential energy is equal to the total energy, the kinetic energy is zero, which means that the velocity is zero. This is the *turning point* of the arrow: The arrow reverses direction and start moving downwards. The position with zero velocity is therefore the maximum height of the arrow.

Numerical solution and animations: Notice that we can tell several stories based on the energy diagram. But for a given set of initial conditions, there is still only one resulting motion. To illustrate the relation between the motion and the energy diagram, we have illustrated the motion $y(t)$ for an arrow fired from $y = 0\,\mathrm{m}$ with various initial velocities v_0, corresponding to the energies shown in the energy diagram. The motions and the corresponding energy diagrams are shown in Fig. 11.13. To further illustrate the relation between the motion and the energy diagram, we can generate an animated plot that shows the position of the arrow as a function of time, and the time development of position of the arrow in the energy diagram as a function of time for a given motion. This is implemented by the following program:

```
from pylab import *
m = 1.0     # kg
g = 9.8     # m/s^2
v0 = 10.0   # m/s
y0 = 0.0    # m
time = 2.0*v0/g
ntime = 1000
t = linspace(0.0,time,ntime)
y = y0 + v0*t - 0.5*g*t**2
v = v0 - g*t
U = m*g*y
K = 0.5*m*v**2
E = U + K
for i in range(ntime):
    subplot(2,1,1)
```

```
plot(y,E,'-k',y,U,'-r',y[i],E[i],'o')
ylabel('E [J]'), xlabel('y [m]')
subplot(2,1,2)
plot(y,t,'-',y[i],t[i],'o')
ylabel('t [s]'), xlabel('y [m]')
```

which is illustrated by the animation[2] for $y_0 = 0\,\text{m}$ and a given $v_0 = 10\,\text{m/s}$.

Test your understanding: Using this program, you can explore the effect of changing the initial conditions. What happens if you double the initial velocity to $v_0 = 20\,\text{m/s}$? What happens if you instead start the arrow from the height $y_0 = 10\,\text{m/s}$, but with zero velocity?

11.3.2 Example: Atomic Motion Along a Surface

Problem: Let us also revisit the motion of an atom along a surface using energy diagrams: An atom is moving along the x-axis under influence by a surface force. The interaction between the atom and the surface is described by the potential energy of the atom:

$$U(x) = U^* \left(1 - \cos \frac{2\pi x}{b} \right) . \tag{11.81}$$

Sketch the energy diagram for the atom, find and characterize equilibrium points, and discuss the motion of the atom for various energies.

Energy diagram: The potential energy is shown in Fig. 11.14. The potential energy is a periodic function with period b, which we would interpret as a lattice distance in the atomic arrangement of the surface.

Equilibrium points: Equilibrium points corresponds to positions where the force on the atom is zero:

$$F(x) = -\frac{dU}{dx} = 0 , \tag{11.82}$$

which also are local extrema of the potential energy function. Inserting the potential energy, we find:

$$F(x) = -\frac{d}{dx} U^* \left(1 - \cos \frac{2\pi x}{b} \right) = U^* \frac{2\pi}{b} \sin \frac{2\pi x}{b} = 0 , \tag{11.83}$$

which occurs when $2\pi x/b = \pi n$, where odd numbers n correspond to local maxima and even numbers n, including zero, corresponds to local minima. The local maxima are unstable equilibrium points and the local minima are stable equilibrium points.

Motion and turning points: If the potential energy from the surface represents the only force on the atom, the total mechanical energy of the atom is conserved. The total mechanical energy is illustrated by a horizontal line in the energy diagram in

[2] http://folk.uio.no/malthe/mechbook/enebowenediag01.gif.

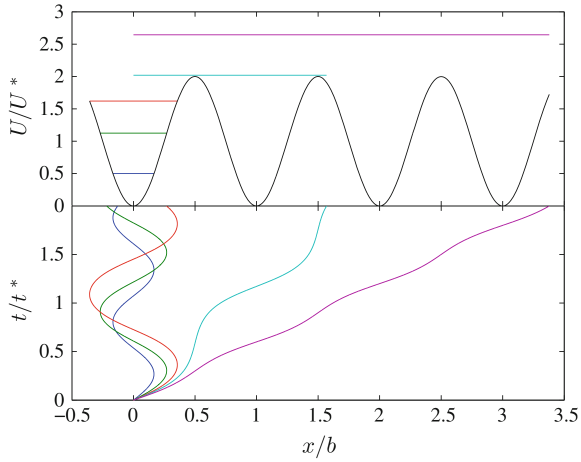

Fig. 11.14 *Top* Energy diagram for an atom moving along a surface. The various *horizontal lines* illustrate different total energies. *Bottom* Illustration of motions ($x(t)$) for each of the total energies shown at the *top*

Fig. 11.14. Two different regimes are evident. If the total energy is smaller than $2U^*$, the total energy becomes smaller than the potential energy for some values of x. Since the total energy is the sum of the potential and kinetic energies:

$$E = U(x) + K(x) \ , \tag{11.84}$$

and the kinetic energy cannot be negative, the atom cannot move into regions where the potential energy is larger than the total energy. These regions therefore become unaccessible to the atom. For example, if the atom starts near $x = 0$ and the total energy is less than $2U^*$, the atom cannot reach $x = b$. What happens instead? If the atom starts at $x = 0$ with an initial velocity v_0, the velocity gradually decreases as the atom moves to the right, until the velocity becomes zero at $x = x_1$, and the atom reverses direction. The atom the accelerates until it reaches $x = 0$, where the atom again starts to slow down, until it reaches zero velocity at $x = x_1$. We can find the position where the velocity of the atom is zero from the total mechanical energy, if the atom starts at $x = 0$ with velocity v_0, the total mechanical energy is:

$$E_0 = U(0) + K_0 = 0 + \frac{1}{2}mv_0^2 \ , \tag{11.85}$$

where $U(0) = 0$. The atom reaches its maximum position when the velocity is zero, $v_1 = 0$:

$$E_1 = U(x_1) + K_1 = U^* \left(1 - \cos\frac{2\pi x_1}{b}\right) + 0 \ . \tag{11.86}$$

We find the position x_1 from energy conservation, $E_0 = E_1$:

$$K_0 = U^* \left(1 - \cos\frac{2\pi x_1}{b}\right) + 0 \ \Rightarrow \ 1 - \frac{K_0}{U^*} = \cos\frac{2\pi x_1}{b} \ . \tag{11.87}$$

which gives:

$$x_1 = \frac{b}{2\pi} \arccos\left(1 - \frac{K_0}{U^*}\right) . \tag{11.88}$$

We notice that if K_0/U^* becomes larger than 2 there is no solution for x_1. This means that the atom is free to move along the surface. Although the velocity of the atom still varies as the atom moves along the surface, the velocity now never reaches zero. This also means that the atom does not turn: If it starts moving in the positive direction it will continue to move in the positive direction indefinitely.

Numerical solution and animation: We can relate the motion of the atom to the energy diagram by finding the motion of the atom for various initial positions and velocities. Since the only force acting on the atom comes from the interaction with the surface, Newton's second law gives:

$$ma = F(x) = -\frac{dU}{dx} , \tag{11.89}$$

inserting $F(x)$ from (11.83), we find:

$$a = \frac{d^2x}{dt^2} = -\frac{2\pi U^*}{mb} \sin\frac{2\pi x}{b} = 0 . \tag{11.90}$$

We solve this equation using Euler-Cromer's method, through the following program that generates an animated plot of the motion as $x(t)$ and as $U(x(t))$ in the energy diagram:

```
from pylab import *
# Initialize
Uast = 1.0
b = 1.0
v0 = 1.5 # 1.0 1.5 1.8 2.01 2.3
time = 2.0
m = 1.0
n = 1000
dt = time/n
t = zeros(n,float)
x = zeros(n,float)
v = zeros(n,float)
x[0] = 0.0
t[0] = 0.0
v[0] = v0
# Calculate
for i in range(n-1):
  a = -2*pi*Uast/b*sin(2*pi*x[i]/b)/m
  v[i+1] = v[i] + dt*a
  x[i+1] = x[i] + dt*v[i+1]
  t[i+1] = t[i] + dt
end
# Plot
Etot = 0.5*m*v0**2
xx = linspace(x.min(),x.max(),1000)
Ux = Uast*(1.0-cos(2*pi*xx/b))
for i in [0:n-1:10]:
  subplot(2,1,1)
  plot(xx,Ux,'-b',xx,Etot,'-k',x[i],Etot,'o')
```

```
xlabel('U/U^{\ast}'), ylabel('x/b')
subplot(2,1,2);
plot(x,t,'-b',x[i],t[i],'o');
ylabel('t/t^{\ast}'), xlabel('x/b')
```

11.4 The Energy Principle

So far we have addressed processes where the work is done by conservative forces only. Let us now use what we have learned about potential energy to also discuss processes with non-conservative forces. Non-conservative forces are typically forces that not only depend on position, but for example also on velocity, such as air resistance or friction. Let us examine the motion of an object subject to both conservative and non-conservative forces.

We decompose the net force acting on an object into *conservative forces*, $\mathbf{F}_j$, and a *non-conservative force*, $\mathbf{f}$:

$$\mathbf{F}^{\text{net}} = \sum_j \mathbf{F}_j + \mathbf{f} \; , \tag{11.91}$$

If the object moves from $\mathbf{r}(t_0) = \mathbf{r}_0$ at t_0 to $\mathbf{r}(t_1) = \mathbf{r}_1$ at t_1, the net work is:

$$\begin{aligned} W_{0,1}^{\text{net}} &= \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = \int_{t_0}^{t_1} \left[\sum_j \mathbf{F}_j + \mathbf{f} \right] \cdot \mathbf{v}\, dt = \sum_j \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt + \int_{t_0}^{t_1} \mathbf{f} \cdot \mathbf{v}\, dt \\ &= \sum_j \left[U(\mathbf{r_0}) - U(\mathbf{r_1}) \right] + W_{0,1}^f = U_0 - U_1 + W_{0,1}^f \; . \end{aligned} \tag{11.92}$$

The work-energy theorem relates the net work to the change in kinetic energy for the object:

$$W_{0,1}^{\text{net}} = U_0 - U_1 + W_{0,1}^f = K_1 - K_0 \; . \tag{11.93}$$

We group terms related to position 1 on the left hand side:

$$U_1 + K_1 = U_0 + K_0 + W_{0,1}^f \; . \tag{11.94}$$

If we introduce the term $E_1 = U_1 + K_1$ for the total (mechanical) energy of the object, we get:

$$E_1 = E_0 + W_{0,1}^f \; , \tag{11.95}$$

or

$$E_1 - E_0 = \Delta E = W_{0,1}^f \; . \tag{11.96}$$

The change in the (mechanical) energy of the object is equal to the work done by the non-conservative force. This law is a completely general law of nature usually called the *second law of thermodynamics* or the *energy principle*:

Energy principle:

$$\Delta E = W \ . \tag{11.97}$$

The change in (mechanical) energy is equal to the work done by non-conservative forces (W).

We will later see how we can relate this energy principle for a single object to a more general principle for a system of several objects interacting with its environment through work and heat (thermal energy transfer).

11.4.1 Example: Lift and Release

Problem: If you lift a book from the floor to a height h using a constant force F, and release it, you have increased the total energy of the book, but the book was affected by a constant force. How can we explain this using the energy principle?

Solution: When you are not holding the book, and the book is falling, it is affected by the gravitational force alone. The total mechanical energy of the book (when falling) is therefore:

$$E = U + K = mgy + \frac{1}{2}mv^2 \ . \tag{11.98}$$

Initally, the book was lying on the floor with zero velocity. In this case, the total mechanical energy is:

$$E_0 = mgy_0 + \frac{1}{2}mv_0^2 = 0\,\text{J} \ . \tag{11.99}$$

If we choose the potential energy to be zero when the book lies on the floor, that is, when $y = 0$ m and $v_0 = 0$ m/s. When the book is at rest ($v_1 = 0$ m/s) at $y_1 = h$, the total mechanical energy is:

$$E_1 = mgh + \frac{1}{2}mv_1^2 = mgh \ . \tag{11.100}$$

The change in mechanical energy is therefore:

$$\Delta E = E_1 - E_0 = mgh \ . \tag{11.101}$$

where did this energy come from? It came from the work done by the constant force F lifting the book to the height h.

$$W_{0,1}^F = F(y_1 - y_0) = Fh \ . \tag{11.102}$$

Because F is a non-conservative force, the energy principle gives that:

$$\Delta E = W^F_{0,1} = Fh = mgh \ . \tag{11.103}$$

The work done by F corresponds to the work done by a constant force mg. But, you protest, the force F is a conservative force, since it is a constant. Something must be wrong with the energy principle!

Not so fast. The force F was constant only when you lifted the book. Afterwards, you released the book, and this applied force therefore became zero. The force F is therefore not that constant after all. This is indeed how we should regard such a force: The force F lifting the book is a time-dependent force. It only acted when the book was lifted. It does not act when the book is released afterwards. Then it is only affected by gravity. This is why we call it a non-conservative force. We could look at it in a different way: The force F does not only depend on the position y of the book, because when we lift the book, it is affected by the lifting force, but afterwards, when the book is released, it may be falling through the same position without being affected by the lifting force. Thus the lifting force is not a conservative force, and we must use the energy principle to understand the change in mechanical energy as caused by the work done by non-conservative forces.

11.4.2 Example: Sliding Block

Problem: A block is sliding down an inclined plane forming an angle α with the horizontal. The kinetic coefficient of friction for the contact between the block and the plane is μ. What is the velocity of the block as a function of the length L it has slid?

Identify: We use a coordinate system oriented along the plane, as illustrated in Fig. 11.15. The block slides from position x_0, with initial velocity $v_0 = 0$, to x_1. Our task is to find the velocity v_1 as a function of x_1.

Model: Figure 11.15 shows that the block is affected by the normal force N, friction, f, and by gravity, G. Because the block is affected by a non-conservative force, f, we cannot use energy conservation to find the velocity as a function of position, but we can still use the energy principle: The change in energy is equal to the work done by the non-conservative forces.

$$\Delta E = W^f \ , \tag{11.104}$$

where the change in energy is

$$\Delta E = (K_1 + U_1) - (K_0 + U_0) = W^f_{0,1} \ . \tag{11.105}$$

The work done by the constant friction force, $f = -\mu N$, is

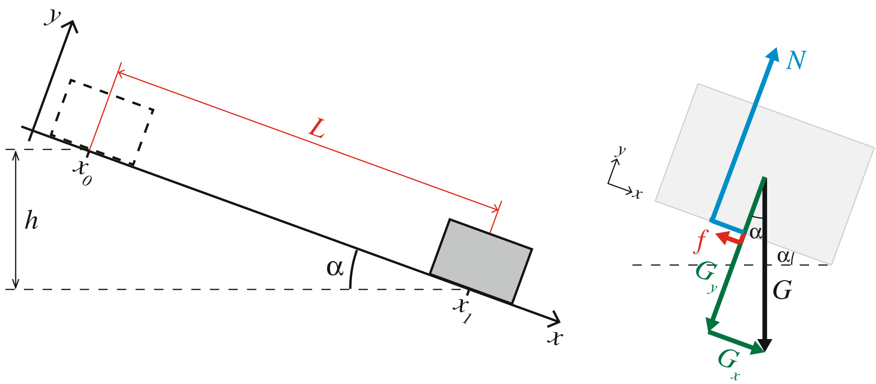

Fig. 11.15 *Left* Illustration of a block sliding from x_0 to x_1 along an inclined plane. *Right* Free-body diagram for the block

$$W_{0,1}^{f} = \int_{x_0}^{x_1} f\,dx = -\mu N\,(x_1 - x_0)\ , \tag{11.106}$$

where $L = x_1 - x_0$ is the distance moved along the plane. Because the block is not moving in the y-direction we find the normal force from Newton's second law:

$$\sum F_y = N - G_y = N - mg\cos\alpha = ma_y = 0\ , \tag{11.107}$$

where Fig. 11.15 shows that $G_y = mg\cos\alpha$. The work done by friction is therefore:

$$W_{0,1}^{f} = -\mu mgL\cos\alpha\ . \tag{11.108}$$

To find the change in energy, ΔE, we need the change in potential energy of the block, $U_1 - U_0$. Since the normal force does no work, the potential energy is the same as for motion under gravity alone: $U = mgh$, where h is the vertical position of the block. Figure 11.15 shows that $\Delta h = L\sin\alpha$, and

$$U_1 - U_0 = mg\Delta h = -mgL\sin\alpha\ . \tag{11.109}$$

We insert these results in $\Delta E = W^f$:

$$\begin{aligned}\Delta E &= E_1 - E_0 = U_1 + K_1 - (U_0 + K_0) = K_1 - K_0 + U_1 - U_0\\ &= \frac{1}{2}mv_1^2 - 0 - mg\,L\,\sin\alpha = W_{0,1}^{f} = -mg\,L\,\cos\alpha\ .\end{aligned} \tag{11.110}$$

We rearrange the equation to find:

$$v_1^2 = mgL\,(\sin\alpha - \mu\cos\alpha)\ \Rightarrow\ v_1 = \sqrt{mgL\,(\sin\alpha - \mu\cos\alpha)}\ . \tag{11.111}$$

Comment: This demonstrates how to use the energy principle for practical calculations. We use the same procedure as for energy conservation, but include the work of non-conservative forces. In most cases, we need to know the path, $\mathbf{r}(t)$ taken by the object in order to calculate the work done by the non-conservative forces. This method is therefore often not that practical.

11.5 Potential Energy in Three Dimensions

So far we have restricted ourselves to one-dimensional motions. Let us now introduce the potential energy for a three-dimensional motion and a three-dimensional force.

We call a force $\mathbf{F}(\mathbf{r})$ conservative if the work done by the force from a point 0 to a point 1 is *independent of the path taken*. That is, we call a force $\mathbf{F}(\mathbf{r})$ conservative, if (and only if) there is a function $U(\mathbf{r})$ so that:

$$W_{0\text{ to }1} = \int_0^1 \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = U(\mathbf{r}_0) - U(\mathbf{r}_1) \ , \tag{11.112}$$

for all (possible) paths between $\mathbf{r}_0$ and $\mathbf{r}_1$.

It is trivial to show that the work is independent of the path if a function $U(\mathbf{r})$ with the property of (11.112) exists. What about the other way—if the work along all paths are the same, can we prove that there must exist a function $U(\mathbf{r})$? This is also not difficult, since the function is simply defined as the work integral between the two end-points.

How is the force $\mathbf{F}(\mathbf{r})$ and the function $U(\mathbf{r})$ related? Let us look at the work done between 0 and 1:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \underbrace{F_x\,dx + F_y\,dy + F_z\,dz}_{=-dU} = -\int_0^1 dU = U(\mathbf{r}_0) - U(\mathbf{r}_1) \ , \tag{11.113}$$

Because $U(\mathbf{r}) = U(x, y, z)$, we can write:

$$dU(x, y, z) = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz = -F_x\,dx - F_y\,dx - F_z\,dz \ , \tag{11.114}$$

consequently,

$$F_x = -\frac{\partial U}{\partial x} \ , \ F_y = -\frac{\partial U}{\partial y} \ , \ F_z = -\frac{\partial U}{\partial z} \ , \tag{11.115}$$

that is:

If the force $\mathbf{F}$ is conservative, we can find a function U so that

$$\mathbf{F} = -\nabla U \ , \tag{11.116}$$

We call the function $U(\mathbf{r})$ *the potential energy* for the force (field) $\mathbf{F}(\mathbf{r})$.

Criterion for a conservative force: This means that the criterion for a force to be conservative is a bit stronger in two- and three-dimensions than in one dimension. In one dimension, a force $F(x)$ is conservative if $F(x)$ is a function of position alone.

In two- and three-dimensions, it is a *necessary* condition that the force $\mathbf{F}(\mathbf{r})$ is only a function of the position, for the force to be conservative. But it is not a sufficient condition. There are forces $\mathbf{F}(\mathbf{r})$ that are only functions of position, but that still are not conservative. In order for the force to be conservative, it must be the gradient of a potential:

$$\mathbf{F} = -\nabla U \ . \tag{11.117}$$

From calculus, we know that a force $\mathbf{F}(\mathbf{r})$ can be written as a gradient of a function, if and only if, the curl of $\mathbf{F}(\mathbf{r})$ is zero everywhere (for all $\mathbf{r}$):

$$\nabla \times \mathbf{F} = 0 \text{ (for all } \mathbf{r}) \ , \tag{11.118}$$

because in this case, the integral of all closed curves is zero.

11.5.1 Example: Constant Gravity in Three Dimensions

Problem: Find the potential energy $U(\mathbf{r})$ for the gravitational force $\mathbf{G} = -mg\,\mathbf{j}$.
Solution: We want to find a function $U(\mathbf{r})$ that satisfies:

$$\mathbf{G} = -mg\,\mathbf{j} = -\nabla U \ , \tag{11.119}$$

Let us try the solution we already know from one dimension:

$$U(x, y, z) = mgy \ , \tag{11.120}$$

We find the gradient of U:

$$\nabla U = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right)(mgy) \tag{11.121}$$

$$= \frac{\partial}{\partial x} mgy\,\mathbf{i} + \frac{\partial}{\partial y} mgy\,\mathbf{j} + \frac{\partial}{\partial z} mgy\,\mathbf{k} = 0\,\mathbf{i} + mg\,\mathbf{j} + 0\,\mathbf{k}\ . \tag{11.122}$$

This means that the graviational force $\mathbf{G}$ is the gradient of the potential $U = mgy$, that $U = mgy$ is a potential energy (function) also in three dimensions, and that the constant gravitational force is conservative.

11.5.2 Example: Gravity in Three Dimensions

Problem: The gravitational force on an object of mass m at position $\mathbf{r}$ from an object of mass M in the origin is:

$$\mathbf{F} = -\frac{GmM}{r^2}\frac{\mathbf{r}}{r}\ . \tag{11.123}$$

Is this force conservative, and can you find the potential energy for this force?
Approach: We know that the force is conservative if the work on the object (of mass m) does not depend on the path. Let us find the work done along a path, and demonstrate that it is only dependent on the displacement and not on the path.
Solution: The work done on an object when it is moved along a path $\mathbf{r}(t)$ from $\mathbf{r}(t_0) = \mathbf{r}_0$ to $\mathbf{r}(t_1) = \mathbf{r}_1$ is:

$$W_{0,1} = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -\frac{GmM}{r^3}\mathbf{r} \cdot d\mathbf{r} \tag{11.124}$$

We introduce a common trick for such integrals: We use that $d(\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot d\mathbf{r}$, and therefore that $\mathbf{r} \cdot d\mathbf{r} = d\left((1/2)\mathbf{r} \cdot \mathbf{r}\right)$, and write the integral as:

$$W_{0,1} = \int_0^1 -\frac{GmM}{r^3} d\left(\frac{1}{2}\mathbf{r} \cdot \mathbf{r}\right) = -GmM \int_{r_0}^{r_1} \frac{d(\frac{1}{2}r^2)}{r^3} = -GmM \int_{r_0}^{r_1} \frac{r}{r^3}\,dr \tag{11.125}$$

This integral does not depend on the path, only on the end-points. The gravitational force is therefore conservative.

We solve the integral to find the potential energy function:

$$W_{0,1} = -GmM \int_{r_0}^{r_1} \frac{dr}{r^2} = -\frac{GmM}{r_0} + \frac{GmM}{r_1} = U(r_0) - U(r_1)\ . \tag{11.126}$$

The potential energy function is therefore:

$$U(\mathbf{r}) = -\frac{GmM}{r}\ . \tag{11.127}$$

Analyze: We can check this result by calculating the gradient of the potential energy function:

$$\nabla U(\mathbf{r}) = -GmM\,\nabla\frac{1}{r} = -GmM\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\frac{1}{r} \tag{11.128}$$

Let us calculate the x-component, using that $r = \left(x^2+y^2+z^2\right)^{1/2}$:

$$\begin{aligned}\nabla U\cdot\mathbf{i} &= -GmM\frac{\partial}{\partial x}\frac{1}{r} = -GmM\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) = \frac{GmM}{r^2}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2} \\ &= \frac{GmM}{r^2}\frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}2x = \frac{GmM}{r^3}x\ .\end{aligned} \tag{11.129}$$

And similarly for the y and the z components. This shows that:

$$\nabla U = \frac{GmM}{r^3}\mathbf{r} = -\mathbf{F}\ , \tag{11.130}$$

which proves that $U(\mathbf{r})$ is the potential energy function for the gravitational energy.

11.5.3 Example: Non-conservative Force Field

Problem: Show that the force:

$$\mathbf{F} = -y\,\mathbf{i}+x\,\mathbf{j}\ , \tag{11.131}$$

is not conservative, even though it depends on the position $\mathbf{r}$ alone.
Approach: We can prove that the force is not conservative in several ways. First, we recall the definition of a conservative force: A force is conservative if the work done by the force from point 0 to a point 1 is independent of the path. Our plan is to calculate the work along two different paths. If they are not the same we can conclude that the force is not conservative.
Solution: We study two different paths, A and B, from $\mathbf{r}_0 = x_0\,\mathbf{i}+y_0\,\mathbf{j} = 0$ to $\mathbf{r}_1 = x_1\,\mathbf{i}+y_1\,\mathbf{j}$ as illustrated in Fig. 11.16. Let us calculate the work done along each path:
Path A: Path A goes as a straight line first from $(0,0)$ to $(x_1,0)$ (path A1), and then from $(x_1,0)$ to (x_1,y_1) (path A2). The work alongthese two subpaths are:

$$W_{A1} = \int_{A1}\mathbf{F}\cdot d\mathbf{r} = \int_{x_0}^{x_1}(-y\,\mathbf{i}+x\,\mathbf{j})\cdot\mathbf{i}\,dx = \int_0^{x_1} -\underbrace{y_0}_{=0}\,dx = 0\ . \tag{11.132}$$

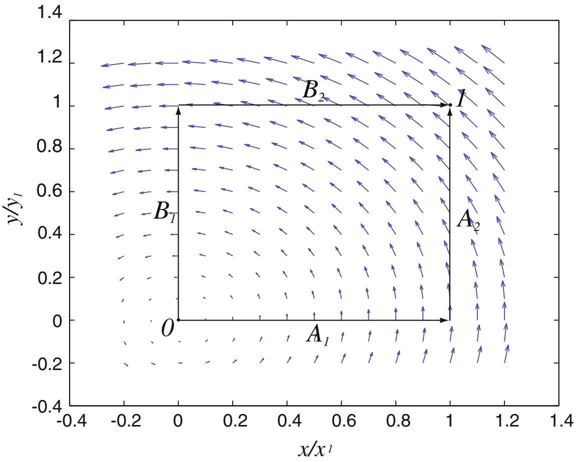

Fig. 11.16 Illustration of the two paths A and B between the points 0 and 1 are shown. The *arrows* indicate the force field, $\mathbf{F}(\mathbf{r})$

$$W_{A2} = \int_{A2} \mathbf{F} \cdot d\mathbf{r} = \int_{y_0}^{y_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{j}\,dy = \int_0^{y_1} x_1\,dy = x_1 y_1 \; . \tag{11.133}$$

The work done by $\mathbf{F}$ along path A is therefore:

$$W_A = W_{A1} + W_{A2} = 0 + x_1 y_1 = x_1 y_1 \; . \tag{11.134}$$

Path B: Path B goes as a straight line first from $(0, 0)$ to $(0, y_1)$ (path B1), and then from $(0, y_1)$ to (x_1, y_1) (path B2). The work along these two subpaths are:

$$W_{B1} = \int_{B1} \mathbf{F} \cdot d\mathbf{r} = \int_{y_0}^{y_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{j}\,dy = \int_0^{y_1} \underbrace{x_0}_{=0}\,dy = 0 \; . \tag{11.135}$$

$$W_{B2} = \int_{B2} \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{i}\,dx = \int_0^{x_1} -y_1\,dx = -y_1 x_1 \; . \tag{11.136}$$

The work done by $\mathbf{F}$ along path B is therefore:

$$W_B = W_{B1} + W_{B2} = 0 - x_1 y_1 = -x_1 y_1 \; . \tag{11.137}$$

We see that the work done by $\mathbf{F}$ along the two paths A and B between the two points 0 and 1 are not the same. The work therefore depends on the path, and the force is not conservative!

Analyze: The force is conservative if and only if it can be written as the gradient of a potential. A necessary condition for that, is that the curl of $\mathbf{F}$ is zero. We can therefore alos determine is the force is conservative by calculating the curl of $\mathbf{F}$:

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial x}{\partial x} - \frac{\partial - y}{\partial y}\right)\mathbf{k} = 2\mathbf{k} \ .
\end{aligned}
\tag{11.138}
$$

The force is therefore not conservative.

11.6 Energy Conservation as a Test of Numerical Solutions

When we are solving the equations of motion based on Newton's second law, we can use our general knowledge of physics to check the results. This is what we do when we first look at the result of a calculation: If it is obviously wrong, we dismiss it immediately. For example, if you study a falling raindrop with air resistance, you know something is wrong if you calculation tells you the drop is accelerating upwards. Our physical intuition is therefore important both for the interpretation of the results, but also for evaluating their correctness. However, we can leverage our knowledge of physics further. For example, we can use conservation laws to test the accuracy of our solutions and of our solution methods. This is common practice for working physicists, and should be a part of your regular professional routine: Always check for energy (and later momentuum) conservation!

Spring-block system: Let us demonstrate this through a simple example: A spring-block system with a block ($m = 0.1$ kg) attached to a spring ($k = 1000$ N/m). We assume that the block is moving along the x-axis and that it is in equilibrium when $x = 0$. If the force from the spring is the only force acting on the block Newton's second law gives:

$$
\sum F_x = -kx = ma \ \Rightarrow \ a = -kx \ . \tag{11.139}
$$

Let us calculate the motion starting from $x(t_0) = 0$ m with a velocity $v(t_0) = v_0$.

Euler-method solution: In this case we know the exact solution of this equation. However, let us now rather analyze our numerical solution methods. We start by using Euler's method. We find the velocity and position at time $t_{i+1} = t_i + \Delta t$ from the velocity and position at time t_i using:

$$
\begin{aligned}
v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\
x(t_i + \Delta t) &= x(t_i) + v(t_i)\,\Delta t
\end{aligned}
\tag{11.140}
$$

This method is implemented in the following program:

```
from pylab import *
m = 0.1      # kg
k = 1000     # N/m
x0 = 0.0     # m
v0 = 1.0     # m/s
time = 1.0   # s
```

```
n = 100000
dt = time/n
t = zeros(n,float)
x = zeros(n,float)
v = zeros(n,float)
x[0] = x0
t[0] = 0.0
v[0] = v0
for i in range(n-1):
    F = -k*x[i]
    a = F/m
    v[i+1] = v[i] + dt*a
    x[i+1] = x[i] + dt*v[i]
    t[i+1] = t[i] + dt
plot(t,x); xlabel('t (s)'), ylabel('x (m)')
```

Validation: The resulting motion is shown in Fig. 11.17. However, if we did not know the exact solution for the motion, it would be difficult to evaluate if this solution is correct. Let us therefore check the solution by calculating the total energy of the block. Because the block is affected by only one force, which depends on the position of the block alone, the total energy of the block is conserved. In this case we know the potential energy of the block:

$$U(x) = \frac{1}{2}kx^2 \ . \tag{11.141}$$

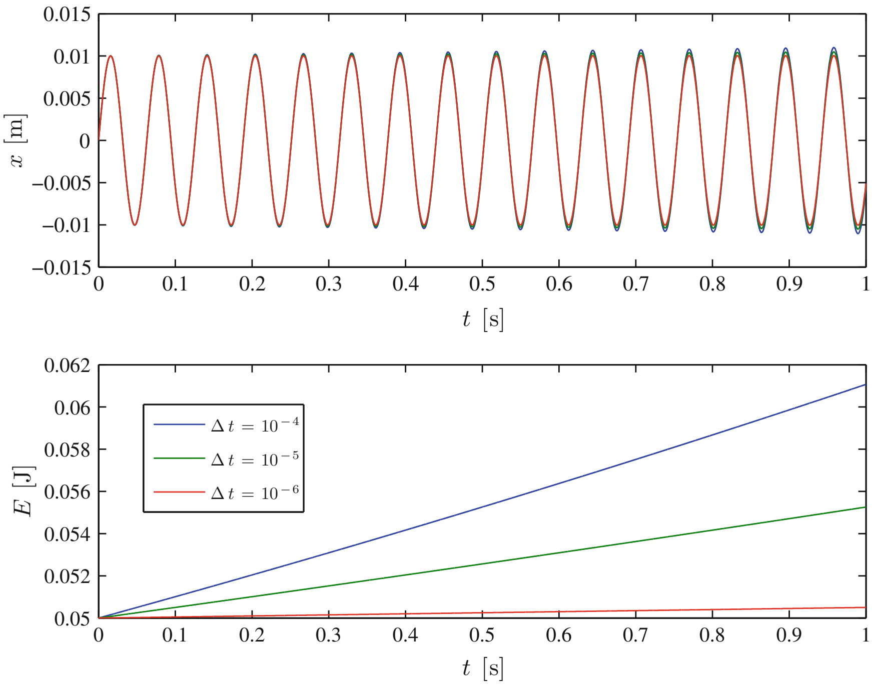

Fig. 11.17 Plot of the motion, $x(t)$, and the energy, $E(t)$, for a block in a spring calculated using Euler's method

The total energy is therefore:

$$E = U(x) + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \, . \tag{11.142}$$

This quantity should be conserved. We calculate and plot the energy in the program by adding the following lines:

```
U = 0.5*k*x**2
K = 0.5*m*v**2
E = U + K
plot(t,E), xlabel('t (s)'), ylabel('E (J)')
```

The result in Fig. 11.17 shows that the energy is not conserved. Something may be wrong. We know that we always make errors in our numerical integration due to numerical rounding. Could this be a result of numerical round-off errors? In addition, we know that the numerical scheme is not exact itself. It is only valid in the limit when Δt goes to zero and in this limit the effect of round-off error will be important.

Discussion: First, we notice that the errors are rather large. We would expect round-off errors to be related to the smallest numbers represented on the machine, and not on the order of 1 % of the value of the energy. However, it is often easier to check if the error is related to the numerical algorithm—we simply reduce the value of Δt and see what happens. Figure 11.17 shows that when Δt is reduced, the error is also reduced. We therefore conclude that the main contribution to the error we are observing comes from the numerical method. Our numerical scheme is not conserving energy for this equation. This is, unfortunately, a well-known problem with Euler's method: The error in the numerical solution to even a simple problem like a block in a spring increases with time.

Euler-Cromer method solution: We can fix this by introducing a higher-order method such as Euler-Cromer's method. This is done by a simple modification of the numerical scheme:

$$\begin{aligned} v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i + \Delta t)\,\Delta t \end{aligned} \tag{11.143}$$

This method is implemented by exchanging a single line in the program:

```
x[i+1] = x[i] + dt*v[i]
```

with

```
x[i+1] = x[i] + dt*v[i+1]
```

and the resulting plot of $E(t)$ is shown in Fig. 11.18. Now, the energy does not diverge, but instead oscillates around the theoretical, constant value. The amplitude of the oscillations decrease with Δt. This indicates that the solution method we have used now is sound.

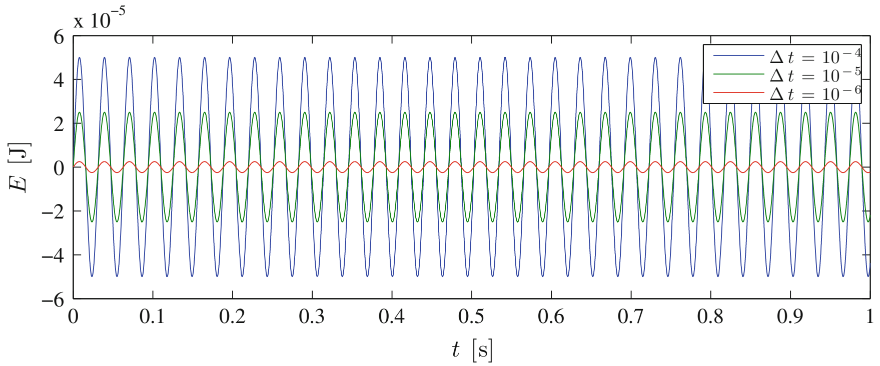

Fig. 11.18 Plot of the energy, $E(t)$, for a block in a spring calculated using Euler-Cromer's method

Summary

Conservative forces: A force $\mathbf{F}$ is conservative if the work done by the force only depends on the end-points of the motion. That is, if the work is independent of the path.

Potential energy: For a conservative force $\mathbf{F}$ we can write the work from $\mathbf{r}_0$ to $\mathbf{r}_1$ as a function of the end points:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = U(\mathbf{r}_0) - U(\mathbf{r}_1)$$

The quantity $U(\mathbf{r})$ is called the potential energy and is a positional energy.

Potential energy function in one dimension: For a one-dimensional force, the force is conservative if and only if it only depends on the position. The force can be written as the derivative of the potential energy $U(x)$:

$$F = -\frac{dU}{dx}$$

Potential energy function in three dimension: For a three-dimensional force, the force is conservative if and only if the force can be written as the gradient of a field U:

$$\mathbf{F} = -\nabla U \ ,$$

That is $\mathbf{F}$ must depend on the position $\mathbf{r}$ only, and $\nabla \times \mathbf{F} = 0$ as well.

Kinetic energy: The kinetic energy of an object is defined as:

$$K = \frac{1}{2} m v^2 \ .$$

Mechanical energy: The mechanical energy of a system is the sum of the kinetic and the potential energy:

$$E = K + U$$

Conservation of mechanical energy: If an object is only subject to conservative forces, the mechanical energy is conserved throughout the motion:

$$K(\mathbf{r}_0) + U(\mathbf{r}_0) = K(\mathbf{r}_1) + U(\mathbf{r}_1)$$

Energy diagrams: An energy diagram illustrates the potential and kinetic energy for an object as a function of position based on a plot of the potential energy $U(x)$. The potential energy can be interpreted as an energy landscape for the motion. A motion with constant mechanical energy, $E = U + K$, is illustrated by a horizontal line in this plot.

Equilibrium points: An *equilibrium point* occurs where the force is zero. This corresponds to a local extremum of the potential energy. An equilibrium point is *stable* in a local minimum and *unstable* in a local maximum.

Kinetic energy in energy diagrams: The kinetic energy can be found from the plot as the distance from the potential energy up to the line giving the total energy, E. An object cannot have negative kinetic energy and therefore cannot enter a region where the total energy is less than the potential energy.

The energy principle: The energy principle states that

$$\Delta E = W_{NC} \ .$$

A change in (mechanical) energy of an object is equal to the work done by non-conservative forces, W_{NC}.

Non-conservative forces: The energy principle can be used in reverse: The work done by non-conservative forces is equal to the change in mechanical energy:

$$W_{NC} = \int_{t_0}^{t_1} \mathbf{F}_{NC} \cdot \mathbf{v} dt = (K(\mathbf{r}_1) + U(\mathbf{r}_1)) - (K(\mathbf{r}_0) + U(\mathbf{r}_0)) \ .$$

Exercises

Discussion Questions

11.1 Potential energy. An object is subject to a force that is zero everywhere, exept between $x = 1\,\text{m}$ and $x = 2\,\text{m}$. Is the force conservative? Can you find the corresponding potential energy?

11.2 Potential energy of electic field. The force on an electron from an electric field is $F_x = -F_0 \sin \omega t \ \sin kx$. Is the force conservative? Can you find the potential energy for this force?

11.3 Zero level for potential energy. Can you choose the zero level for the potential energy of a spring force, so that the potential energy is zero when the spring is extended a distance Δx?

11.4 Losing energy. You drop a book onto a table, where it comes to rest. Is the mechanical energy conserved in this process?

11.5 Energy diagram for a bouncing ball. Draw an energy diagram for a ball bouncing on the floor. You can assume all forces are conservative.

11.6 Potential energy of bow-shot. You fire an arrow vertically using a bow. When is the total potential energy of the arrow at its maximum?

11.7 Potential energy from Earth and Moon. You fly from the Earth to the Moon. Sketch your total potential energy as a function of position. Include only interactions with the Earth and the Moon.

Problems

11.8 The loop. A block is sliding along the track illustrated in Fig. 11.19. First, the block slides down the ramp, then around the loop, and onto a rough, flat table. The block starts from rest at point A at a height h above the ground. The track is frictionless from A to B and around the loop. But from B to D the dynamic coefficient of friction between the block and the track is μ.
(**a**) Find the speed v_B of the block when at point B before going around the loop.
(**b**) Find the speed v_C of the block at the point C.

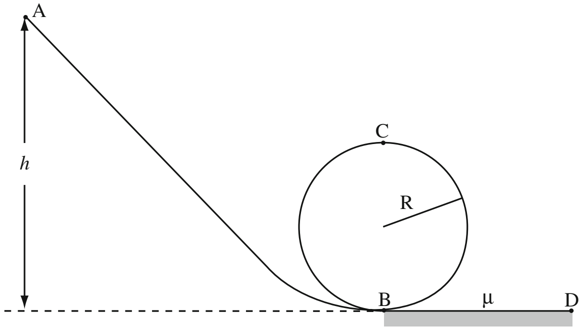

Fig. 11.19 Illustration of a track

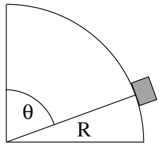

Fig. 11.20 A block sliding on a cylinder surface

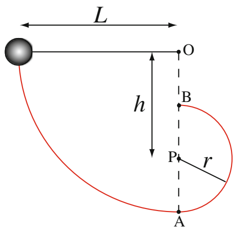

Fig. 11.21 A two-point pendulum

(**c**) What is the condition on the speed v_C for the block to keep in contact with the track? Explain what happens if this condition is not fulfilled.
(**d**) How high, h, above the ground must the block start to make it around the loop?
(**e**) After the block is leaving the loop at point B, it slides onto the rough table. How far does the block slide before stopping?

11.9 Sliding on a cylinder. A block is sliding on the outside surface of a frictionless cylinder, as illustrated in Fig. 11.20. The block starts on the top of the cylinder with zero velocity (or a very small velocity, so that it just starts to move). The mass of the block is m and the radius of the cylinder is R.
(**a**) If the block is in contact with the surface, find the speed of the block as a function of the angle θ.
(**b**) For what value of θ does the block lose contact with the cylinder?
(**c**) Can you explain how your answers would change if there was a friction force between the block and the surface?

11.10 Vertical pendulum. A pendulum consists of a sphere of mass m attached to a massless rope of length L. You start with the rope in your hand and the sphere hanging down. You hit the sphere, giving it a horizontal velocity v_0.
(**a**) What is the speed of the sphere at the top of the path?
(**b**) How large must v_0 be for the sphere to be able to make a complete revolution with the rope tight at all times?

11.11 Two-point pendulum. A pendulum consists of a sphere of mass m attached to a massless rope of length L. You release the sphere a horizontal position, as illustrated in Fig. 11.21. The rope hits a stick (P) at a distance h below the attachment point, O, of the pendulum, and continues rotating around the point P. The length of the pendulum is therefore reduced.
(**a**) What is the speed of the sphere at the point A at the bottom of the path?
(**b**) What is the speed of the sphere when it is at point B directly above the stick?
(**c**) How large must h at least be for the sphere to reach this point with a tight rope?

11.12 Lennard-Jones Potential. The Lennard-Jones potential is often used to describe the interaction between two atoms in a diatomic molecule. The potential energy of the molecule is:

$$U(r) = U_0 \left(\left(\frac{a}{r}\right)^{12} - \left(\frac{b}{r}\right)^{6} \right) , \tag{11.144}$$

where r is the distance between the atoms.
(**a**) What is the force acting on one of the atoms from the other atom?
(**b**) Sketch the potential $U(r)$.
(**c**) Find and classify the equilibrium points.

11.13 A bouncing ball—part 1. You lift a ball of radius R and mass m vertically up, until its center is a height h above the ground, and let it go. The ball starts from rest. You may model the collision between the ball and the ground using a spring force, where all the deformation occurs in the ball. The spring constant is k.
(**a**) What is the height above ground of the center of the ball as the ball comes in contact with the ground?
(**b**) What is the speed of the ball when it comes in contact with the ground?
(**c**) What is the maximum deformation, δy, of the ball during the collision with the ground?

11.14 A bouncing ball—part 2. You lift a ball of radius R and mass m vertically up, until its center is a height h above the ground. Here, you hit the ball, giving it an initial horizontal velocity v_0. You may model the collision between the ball and the ground using a spring force acting in the direction normal to the ground. You can assume that all the deformation occurs in the ball and that the spring constant is k. There is no friction between the ball and the ground.
(**a**) What is the (vector) velocity of the ball when it comes in contact with the ground?
(**b**) What is the (vector) velocity of the ball when it reaches its maximum compression?
(**c**) What is the maximum deformation, δy, of the ball during the collision with the ground?
(**d**) What is the (vector) velocity of the ball as it loses contact with the ground?
(**e**) Based on your results, can you propose a law for how the velocity of a ball changes during a collision with a (frictionless) wall?

Projects

11.15 Shooting Ions. In this project you will apply you knowledge of potential energy to find the force law for the collision between an ion and a molecule, and apply this to address the motion of the ions through the collision, integrating the equations of motion in two dimensions.

We will study the motion of a small ion with mass m that is shot toward a massive molecule located in the origin. The molecule is so large that you can assume that it does not move throughout the whole process. We start by a simplified case, and address a one-dimensional motion along the x-axis. The interaction between the ion and the molecule is described by the potential:

$$U(x) = \frac{C}{x} , \tag{11.145}$$

where C is a know constant, and x is the position of the ion. The ion starts at the position $x = b$ with the velocity v_0, where $b > 0$ and $v_0 < 0$. You can ignore all other forces acting on the ion.
(**a**) Sketch the potential, find equilibrium points and characterize these. Show the motion of the ion in the energy diagram, and describe the motion of the ion.
(**b**) How close to the origin does the ion get?
(**c**) What is the velocity of the ion when it is infinitely far away from the origin?

Let us now address the same process, but in two dimensions. We assume that the ion starts in the position $\mathbf{r}_0 = (b, d)$, where b and d are two given lengths. You may assume that d is less than b, as illustrated in Fig. 11.22. The ion starts with the velocity $\mathbf{v}_0 = (v_0, 0)$, where $v_0 < 0$. The potential energy for the ion due to the interaction with the molecule is now:

$$U(\mathbf{r}) = \frac{C}{r} , \tag{11.146}$$

where $r = |\mathbf{r}|$. You can neglect all other interactions.
(**d**) Show that the force on the ion is:

$$\mathbf{F}(\mathbf{r}) = \frac{C}{r^3}\mathbf{r} . \tag{11.147}$$

(**e**) Find an expression for the acceleration of the ion. What are the initial conditions for the motion?
(**f**) Write a program to find the motion of the ion, that is, to find the velocity and position of the ion as a function of time.
(**g**) Write a program to find the motion of the ion, that is, to find the velocity and position of the ion as a function of time for $m = 1$, $b = 1$, $d = 0.2$, $C = 1$, and $v_0 = 2.5$.

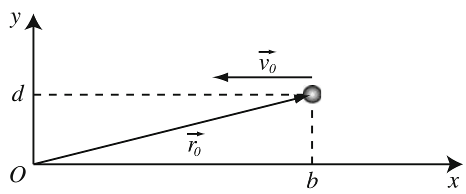

Fig. 11.22 Initial condition for two-dimensional case

(**h**) Use your script to study the behavior of the ion as you vary v_0 from 0.5 to 10.0. Provide a simple description of what is happening.
(**i**) Is possible to choose a value for v_0 to make the ion move radially outward after the collision? (The ion moves radially outward if the velocity is parallel with the position-vector **r**).

Chapter 12
Momentum, Impulse, and Collisions

Two galaxies collide, leading to millions and millions of stars interacting. Can we say anything general about such a collision? For example, if you know the galaxies velocities after the collision, what can you learn about their velocities before the collision?

We have started to study the consequences of Newton's laws of motion. Our first discovery is the conservation of mechanical energy. For an object subject only to conservative forces, the mechanical energy is conserved. Energy conservation provides a useful tool when solving problems in mechanics: we can relate the velocity and position of an object without having to find the position as a function of time. This is particularly useful when the interactions are complicated, and we have limited knowledge about the forces between objects. Actually, it allows us to reason about the behavior of a system without having force models, as long as we know that the forces are conservative.

Conservation of mechanical energy: The conservation of mechanical energy is an example of a conservation law, which we found by integrating Newton's second law along the path:

$$\int_{t_0}^{t_1} \mathbf{F}^{\mathrm{net}} \cdot \mathbf{v} dt = \Delta K \ . \tag{12.1}$$

For a one-dimensional motion where the net force only depends on the position, x, we get:

$$\int_{x_0}^{x_1} F_x^{\mathrm{net}} dx = \Delta K = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 \ . \tag{12.2}$$

If we could calculate the integral on the left, we could find the velocity as a function of position without finding the complete motion.

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_12

Conservation of momentuum: This techinque is sometimes called an integration method. To get energy conservation we integrated Newton's second law over space. But we can also integrate Newton's second law over time:

$$\int_{t_0}^{t_1} \mathbf{F}^{\text{net}} dt = \int_{t_0}^{t_1} m\mathbf{a} dt = m\mathbf{v}_1 - m\mathbf{v}_0 \ . \tag{12.3}$$

If the integral on the left is zero, that is, if the net force is zero, then we find that $m\mathbf{v}$ does not change. This gives us another conservation law: Conservation of momentum, $m\mathbf{v}$. But how can that be useful? Didn't we already know from Newton's first law that if the net external force is zero, the velocity does not change? It turns out that conservation of momentum is not that useful for a single object, but it is very useful for systems consisting of *several objects*. For systems of several objects, we will demonstrate that the total momentum is conserved if there are no external forces acting on the system. It does not matter what internal forces are acting, the total momentum is conserved at all times as long as there are no external forces.

Collisions: Conservation of momentum is particularly useful for collisions. During a collision between two objects, the interactions between the objects can be very complicated, and may consist of both conservative and non-conservative forces, but as long as the objects are not affected by any external forces, their total momentum is conserved. We use this to find the velocities of each object after a collision from the velocities before a collision, without finding the motion of each object. Conservation of momentum is more general than the conservation of energy, since it is valid for any internal force, and not only for conservative forces.

Overview: In order to introduce these concepts, we will start by introducing tranlational momentum, $\mathbf{p} = m\mathbf{v}$. We reformulate Newton's second law using momentum, and find that the integral of the net forces acting on an object corresponds to the change of momentum. We will then use these concepts to address systems with several objects, with a particular focus on collisions.

12.1 Motivating Example—Meteor Impact

You are now an expert solver of mechanics problems: Given a set of force models, you can find the motion of an object from Newton's second law of motion using analytical or numerical tools. However, in some cases we may not know (or care to model) the detailed forces acting between two objects, but you still would like to know what happens to the objects. For example, you may observe a large meteor head directly towards a small planet. You know the masses and velocities of both

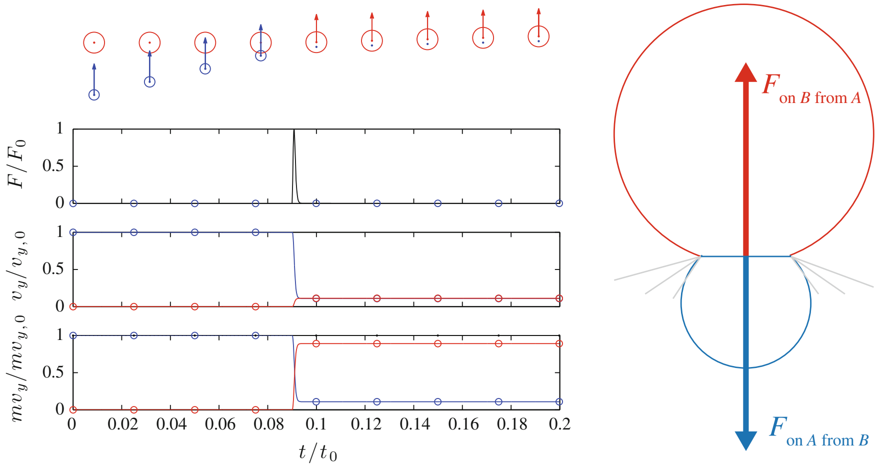

Fig. 12.1 Illustration a collision between a meteor and a planet

objects at some time before the collision, and want to know the velocity of the planet after the hit. Can you do that without having a detailed model for the collision?[1]

Identify: The meteor impact is illustrated in Fig. 12.1. We describe the meteor (object A) with its position $\mathbf{r}_A(t)$ and its mass m_A, and the motion of the planet (object B) with its position $\mathbf{r}_B(t)$ and its mass m_B. At some time t_0 before the impact you know the velocities of both objects, $\mathbf{v}_A(t_0) = \mathbf{v}_{A,0}$ and $\mathbf{v}_B(t_0) = \mathbf{v}_{B,0}$.

Model: You know you can find the motion of both objects by applying Newton's second law to get an equation of motion. If we assume that the only interactions are between the planet and the meteor, Newton's second law for objects A and B are:

$$\begin{aligned} \mathbf{F}_A^{\text{net}} &= \mathbf{F}_{\text{from } B \text{ on } A} = m_A \mathbf{a}_A \\ \mathbf{F}_B^{\text{net}} &= \mathbf{F}_{\text{from } A \text{ on } B} = m_B \mathbf{a}_B \end{aligned} \; . \tag{12.4}$$

Unfortunately, we do not have a simple force model for the force from the meteor on the planet during the collision. How can we then solve the problem?

Using Newton's third law: Fortunately, we can use a common trick: We know that the force from the meteor on the planet is the reaction force to the force from the planet on the meteor: Newton's third law tells us that:

$$\mathbf{F}_{\text{from } A \text{ on } B} = -\mathbf{F}_{\text{from } B \text{ on } A} = \mathbf{F} \; . \tag{12.5}$$

---

[1]A model for such a collision would be very complicated, as there are many different processes involved.

We can therefore rewrite Newton's second law in (12.4) to be:

$$\begin{aligned} m_A \mathbf{a}_A &= -\mathbf{F} \\ m_B \mathbf{a}_B &= \mathbf{F} \end{aligned} \tag{12.6}$$

which is valid as long as there are no other forces acting on the meteor or the planet. Now, we see that we can get rid of the unknown force, **F**, altogether by adding the two equations, getting:

$$m_A \mathbf{a}_A + m_B \mathbf{a}_B = -\mathbf{F} + \mathbf{F} = 0 \ . \tag{12.7}$$

Integration method: Now, we can integrate this equation, from the initial time t_0, where we know the velocities, to the

$$\int_{t_0}^{t_1} m_A \mathbf{a}_A + m_B \mathbf{a}_B dt = 0 \ , \tag{12.8}$$

$$m_A \int_{t_0}^{t_1} \mathbf{a}_A dt + m_B \int_{t_0}^{t_1} \mathbf{a}_B dt = 0 \ , \tag{12.9}$$

$$m_A \left(\mathbf{v}_A(t_1) - \mathbf{v}_A(t_0) \right) + m_B \left(\mathbf{v}_B(t_1) - \mathbf{v}_B(t_0) \right) = 0 \ . \tag{12.10}$$

Let us now group the quantities relating to t_1 on the left side, and the quantities relating to t_0 on the right side:

$$m_A \mathbf{v}_A(t_1) + m_B \mathbf{v}_B(t_1) = m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \ . \tag{12.11}$$

This looks like what we have previously found for energy: It is a conservation law. But for what? For the quantity:

$$\mathbf{P} = m_A \mathbf{v}_A + m_B \mathbf{v}_B \ , \tag{12.12}$$

which we call the *total momentum* of the system consisting of the planet and the meteor.

Solve: How can we use (12.11) to find the velocity of the planet and the meteor after the collision? First, we notice that (12.11) actually is valid for all times, also at any time during the collision. But it is not sufficient to find the velocities of each object after the collision. We only know what the awkward sum, $m_A \mathbf{v}_A + m_B \mathbf{v}_B$ is after the collision. We do not know how this sum is distributed between the two objects. But since this is meteor impact we know something else, we know that the two objects move as one object after the collision: The meteor and the planet have the same velocity afterwards:

$$\mathbf{v}_A(t_1) = \mathbf{v}_B(t_1) = \mathbf{v}_1 \ . \tag{12.13}$$

Now, we can determine the velocity after the collision. Starting from (12.11) we have:

$$
\begin{aligned}
m_A \mathbf{v}_A(t_1) + m_B \mathbf{v}_B(t_1) &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
m_A \mathbf{v}_1 + m_B \mathbf{v}_1 &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
(m_A + m_B)\,\mathbf{v}_1 &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
\mathbf{v}_1 &= \frac{m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0)}{m_A + m_B}
\end{aligned}
\tag{12.14}
$$

We have found the velocity of the planet and the meteor after the collision, without solving the equations of motion!

Discussion: The method we applied here is very similar to the energy conservation method, but it is based on a different conservation law. But notice that we could not have found the velocity after the collision, if we did not know that the objects were moving with the same velocity after the collision. Since the equation we have used here, conservation of momentum, only provides one equation, we cannot use it to find two velocities. We need more equations. We therefore need to know something more about the collision. For example, we may use that the energy is conserved during the collision, or that we know how much energy is lost during the collision.

In the remaining of this chapter we will more thoroughly introduce the concepts briefly mentioned here, and apply the concepts systematically to study two-particle and multi-particle collisions.

12.2 Translational Momentum

The **translational momentum** of an object is defined as:

$$
\mathbf{p} = m\mathbf{v} \ , \tag{12.15}
$$

The translational momentum is a property of the object that depends on both the objects inertial mass, m, and the objects velocity, $\mathbf{v}$. The *translational momentum* is often also called the *linear momentum*. We prefer the term translational momentum to discern it from rotational momentum, which we encounter when discussing rotational motion. In the following we use the short term momentum instead of translational momentum.

The translational momentum of an object is a *vector*, which is in contrast to energy, which is a *scalar*. Translational momentum is a general property that may be extended also to particles without mass. For example, photons have a translational momentum even though they have no mass.

Newton's Second Law

We have previously introduced Newton's second law of motion as:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a} \ . \tag{12.16}$$

However, the most fundamental, and the original, form of Newton's second law is:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{p} \ , \tag{12.17}$$

The net force acting on an object causes a change in the momentum of the object.[2]

For an object with constant mass, this formulation reduces to the original formulation:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(m\mathbf{v}) = \underbrace{\frac{dm}{dt}}_{=0}\mathbf{v} + m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \ . \tag{12.18}$$

This law is a fundamental principle in physics, on the same level as the energy-principle. It is the general formulation of Newton's second law, and we use the term Newton's second law for this law as well as the special case when the mass is constant.

12.3 Impulse and Change in Momentum

What is causing a change in the momentum of an object? Let us study an object that is affected by a force during a short time interval, such as a tennis ball during a serve. While the ball is in contact with the racket, the contact force $\mathbf{F}(t)$ on the ball varies as illustrated in Fig. 12.2. When the racket makes contact with the ball, the contact force is small, but it grows rapidly as the racket deforms against the ball. As the ball speeds up, the force decreases while the racket returns to its original shape, until the force reaches zero as the ball leaves the racket.

During the collision, other forces such as gravitational forces are negligible compared with the contact force from the racket. We can therefore assume that the contact force $\mathbf{F}(t)$ is approximately equal to the net force on the ball.

The change of momentum of the ball from the time t_0 before contact with the racket to the time t_1 after the ball has left the racket is

$$\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 = \int_{t_0}^{t_1} \frac{d\mathbf{p}}{dt}\,dt = \int_{t_0}^{t_1} \mathbf{F}(t)\,dt \ . \tag{12.19}$$

[2] This formulation of Newton's second law can also be exteded to relativistic mechanics.

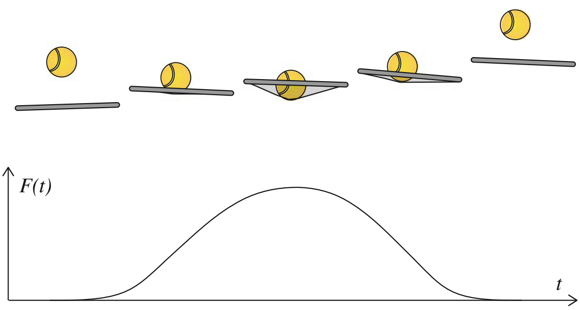

Fig. 12.2 Illustration a ball being hit by a tennis racket, showing an illustration of the collision as a function of time, and a plot of the force $F(t)$ from the racket on the ball as a function of time

The left side is the change in momentum of the ball. The right side includes both the strength of the interaction—the force **F**—and the duration of the interaction. We call this quantity the **impulse**, **J**, experienced by the object during the collision:

Impulse:

$$\mathbf{J} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(t)\, dt \ . \tag{12.20}$$

If the direction of the net force $\mathbf{F}(t)$ does not change during the collision, the impulse is directed in the same direction as $\mathbf{F}(t)$. In this case, the impulse, J, is the area under the curve, $F(t)$, in Fig. 12.2.

Time-Averaged Force

Unfortunately, we generally do not know the time dependency of the net force, since this requires a detailed force model for the collision, or a detailed measurement of the forces acting. Instead, we can use our knowledge of the change in momentum to determine the *average force* acting on the ball during the collision. The time-average force is defined as:

$$\mathbf{F}^{\text{net}}_{\text{avg}} = \langle \mathbf{F}^{\text{ext}}(t) \rangle = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{F}(t)\, dt \ . \tag{12.21}$$

where $\Delta t = t_1 - t_0$ is the duration of the collision. We define the start of the collision as the time t_0 when the ball comes in contact with the racket (when the contact force becomes non-zero), and the end of the collision as the time t_1 when the ball loses contact with the racket (when the contact force becomes zero). We recognize the

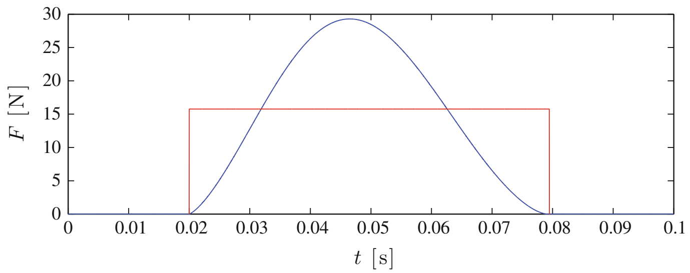

Fig. 12.3 A plot of the force, $F(t)$, as a function of time during a collision, and the average force F_{avg}. In most cases, the average force is a good estimate for the typical force and the maximum force during the collisions

integral on the right-hand side as the impulse of the net force, which is equal to the change in momentum:

$$\mathbf{F}_{\text{avg}}^{\text{net}} = \frac{1}{\Delta t} \int_{t_0}^{t_1} \mathbf{F}(t)\, dt = \frac{1}{\Delta t} \mathbf{J} = \frac{1}{\Delta t} \Delta \mathbf{p} \; . \tag{12.22}$$

Momentum Change During a Collision

The momentum change during a collision gives useful insight into the collision: While the net force may vary throughout the collision, and the maximum force may be much larger than the average force, the average force is still a reasonable estimate for the force acting on the object. For example, if we want to estimate the damage done to an object during a collision, the average force is a good estimate also of the maximum force, because for most physical interactions (for most force models), the force does not display a very narrow peak, but instead varies gradually over a wider time interval, and hence the maximum force is often just a few times the maximum force. (See Fig. 12.3 for an illustration).

Test your understanding: You jump down from a window 5 meters above the ground. How should you land in order to minimize the force on you from the ground? What determines the change in momentum during the impact? Demonstrate that you can answer this question using both energy and momentum considerations.

12.3.1 Example: Ball Colliding with Wall

Problem: A ball falls vertically and collides with a horizontal floor. The force from the floor on the ball during the contact is:

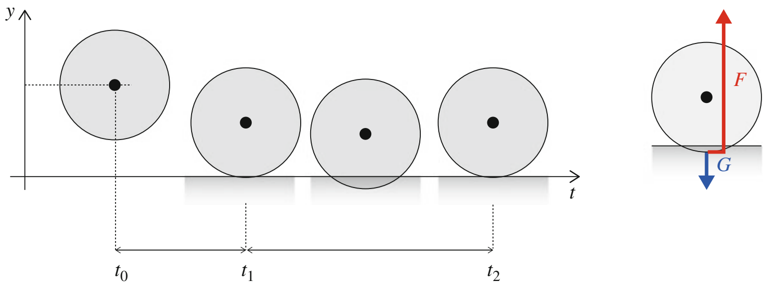

Fig. 12.4 Free-body diagram for a ball colliding with a floor

$$N = \begin{cases} k\,(R - y)^{3/2} & y < R \\ 0 & y \geq R \end{cases} \tag{12.23}$$

You can neglect air drag. Find the motion of the ball and visualize the change in momentum during the collision.

Model: We describe the motion of the ball by its vertical position, $y(t)$. The ball is affected by two forces, the contact force N and gravity, G, as illustrated in Fig. 12.4, where we have neglected air drag.

We find the acceleration of the ball from Newton's second law. The forces acting on the ball are the contact force from the surface and gravity:

$$ma = F^{\text{net}} = N - mg \;\Rightarrow\; a = \frac{1}{m}N - g \,. \tag{12.24}$$

Solve: The ball starts at $y(t_0) y_0$ with the velocity $v(t_0) = v_0$. While we may be able to find the motion $y(t)$ analytically, a numerical approach based on Euler-Cromer's method is sufficient. This is implemented in the following program:

```
from pylab import *
g = 9.8 # m/s^2
R= 0.02 # m
m = 0.1 # kg
y0 = 0.021 # m
v0 = -2.8 # m/s
k = 1000000.0 #
time = 0.005
dt = 0.00001
n = int(round(time/dt));
t = zeros(,n,1);
y = zeros(n,1);
v = zeros(n,1);
Fnet = zeros(n,1);
y[0] = y0
v[0] = v0
for i in range(n):
    dy = R-y[i]
    if (dy<=0.0):
        N = 0.0
    else:
        N = k*dy**1.5
```

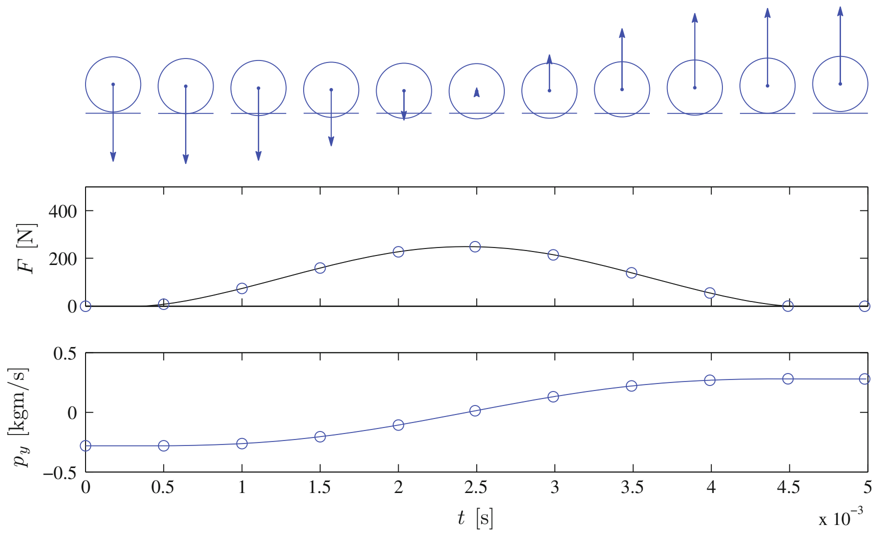

Fig. 12.5 Illustration of a simulation of a ball bouncing on the floor

```
    Fnet[i] = N - m*g
    a = Fnet[i]/m
    v[i+1] = v[i] + a*dt
    y[i+1] = y[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
subplot(2,1,1), plot(t,Fnet)
xlabel('t [s]'),ylabel('F [N]')
p = m*v
subplot(2,1,2), plot(t,p)
xlabel('t [s]'), ylabel('P [kgm/s]')
```

Figure 12.5 illustrates a simulation with this model. Here, you can see the time development of the net force and the momentum of the ball throughout the collision.

Change in momentum: What is the change in momentum of the ball? Since the force only depends on the position of the ball, the force is conservative and the mechanical energy of the ball is conserved throughout the collision. Hence, the kinetic energy of the ball is the same when the ball comes in contact with the surface and when it loses contact with the surface, since the vertical position is the same:

$$E_0 = U(y_0) + \frac{1}{2}mv_0^2 = E_1 = U(y_1) + \frac{1}{2}mv_1^2 \, , \tag{12.25}$$

where $y_0 = y_1$ and therefore

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 \, , \tag{12.26}$$

which gives $v_1 = -v_0$, since we know that the velocity after the collision is in the opposite direction of the velocity before the collision.

The change in momentum is therefore

$$\Delta p = mv_1 - mv_0 = m(-v_0) - mv_0 = -2mv_0 = -2p_0 \ . \tag{12.27}$$

If we know the duration, Δt, of the collision, we can find the average net force on the ball during the collision from

$$F_{\text{avg}}^{\text{net}} = \frac{-2mv_0}{\Delta t} \ . \tag{12.28}$$

12.3.2 Example: Hitting a Tennis Ball

Problem: A tennis ball of mass 57 g is approaching you with a horizontal velocity $v_0 = 20$ m/s. You hit the ball, returning it with a horizontal velocity $v_1 = 20$ m/s, now in the opposite direction. (a) What is the impulse $\mathbf{J}$ on the ball while it is in contact with the racket during the collision? (b) The ball and racket are in contact for 2.0 ms. What is the average net force on the racket during the collision? (c) You want to return the ball as a high lob and give the ball a velocity $v_1 = 15$ m/s at angle of 45° upward. What is now the impulse on the ball and the net force from the racket on the ball?

Approach: We may solve this problem by determining the motion of the ball from Newton's laws of motion, but this would require a detailed force model for the force from the tennis racket on the ball. In this case, we do not have such a model. Instead, we want to use the measured change in velocity to determine the average force on the ball.

Identify: In this problem we address the motion of the tennis ball, described by the position $\mathbf{r}(t)$. The ball starts with the velocity $\mathbf{v}_0 = -v_0\,\mathbf{i}$ at the time t_0 (before the collision), and gets the velocity $\mathbf{v}_1$ after the collision.

Model: The ball is affected by a force, $\mathbf{F}(t)$, from the racket on the ball, and by gravity. However, we assume that gravity is small compared with the typical force from the racket, and ignore the effects of gravity.

Solve: The impulse is defined as the integral of the net force on the ball, and it is equal to the change in momentum of the ball:

$$\mathbf{J} = \Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0 = m\mathbf{v}_1 - m\mathbf{v}_0 \ , \tag{12.29}$$

Solution part a: In part (a) of the problem, the final velocity is $\mathbf{v}_1 = v_1\,\mathbf{i}$. The key idea is that momentum is a vector quantity. The ball therefore experiences a change in momentum, even though the magnitude of the momentum does not change, because the direction of the momentum changes.

The impulse on the ball in the collision is:

$$\mathbf{J} = m\,(v_1\,\mathbf{i} - (-v_0\,\mathbf{i})) = m(v_1 + v_0)\,\mathbf{i} \tag{12.30}$$
$$= 0.057\,\text{kg}\ (20.0\,\text{m/s} + 20.0\,\text{m/s})\ \mathbf{i} = 2.28\,\text{kg m/s}\,\mathbf{i}\ . \tag{12.31}$$

The impulse is positive, since the force acting on the ball is in the positive x-direction—this is also the direction of the acceleration of the ball.

Solution of part b: In part (b) of the problem, we find the average force from the change in momentum during the collision:

$$\mathbf{F}_{\text{avg}} = \frac{1}{\Delta t}\int_{t_0}^{t_1} \mathbf{F}dt = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{2.28\,\text{kg m/s}}{2\,10^{-3}\,\text{s}} = 1140\text{N}\ , \tag{12.32}$$

This is the average net force on the ball. We recall from Fig. 12.3 that the net force is smaller than the maximum force, although they are typically of comparable magnitude.

Dicussion: What about gravity? We neglected gravity because we assumed it to be small compared with the force from the racket. We could check this assumption in two ways. First, we could check that the impulse of gravity is much smaller than the total impulse on the ball—we can do this without calculating the average force. The magnitude of gravity is:

$$W = mg = 0.057\,\text{kg}\,9.8\,\text{m/s}^2 = 0.56\,\text{N}\ . \tag{12.33}$$

The impulse of gravity is therefore:

$$J_g = mg\Delta t = 0.56\,\text{M}\,2\,10^{-3}\,\text{s} = 1.1\,10^{-2}\,\text{kg m/s}\ , \tag{12.34}$$

which is much smaller than the impulse of the net force.

From this calculation we also found the force from gravity, which is much smaller than the average net force on the ball. We were therefore right in neglecting the effect of gravity.

Solution to part c: Finally, we address question (c), where the collision is not head on, and the ball leaves the racket at an angle α, as illustrated in Fig. 12.6. In this case, we need to treat the collision as two-dimensional. First, we introduce the velocity of the ball after the collision as:

$$\mathbf{v}_1 = v_1\,(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})\ , \tag{12.35}$$

where $v_1 = 15\,\text{m/s}$ and $\alpha = 45° = \pi/4$. The impulse on the ball is still given as the change in momentum:

$$\mathbf{J} = \mathbf{p}_1 - \mathbf{p}_0 = m\,(\mathbf{v}_1 - \mathbf{v}_0)\ . \tag{12.36}$$

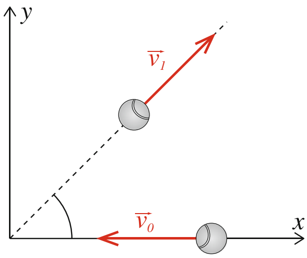

Fig. 12.6 Illustration of the velocity of a tennis ball before and after it was hit by a racket

We find the x- and y-components of the impulse:

$$J_x = \mathbf{J} \cdot \mathbf{i} = m \left(v_{x,1} - v_{x,0}\right) = 57\,\text{g}\ (15 \cos \alpha + 20)\ \text{m/s} = 1.74\,\text{kg m/s} \quad (12.37)$$

$$J_y = \mathbf{J} \cdot \mathbf{j} = m \left(v_{x,1} - v_{x,0}\right) = 57\,\text{g}\ (15 \sin \alpha - 0)\ \text{m/s} = 0.60\,\text{kg m/s}\ . \quad (12.38)$$

The impulse is therefore:

$$\mathbf{J} = 1.74\,\text{kg m/s}\,\mathbf{i} + 0.60\,\text{kg m/s}\,\mathbf{j}\ , \quad (12.39)$$

The force is given as the impulse divided by the time interval. We assume the time interval to be the same for this process, $\Delta t = 2\,\text{ms}$. The average net force is therefore:

$$\mathbf{F}_{\text{avg}} = \frac{\mathbf{J}}{\Delta t} = \frac{1.74\,\text{kg m/s}\,\mathbf{i} + 0.60\,\text{kg m/s}\,\mathbf{j}}{2\,10^{-3}\,\text{s}} = 870\,\text{N}\,\mathbf{i} + 300\,\text{N}\,\mathbf{j}\ , \quad (12.40)$$

Interestingly, this means that the direction of the net force is in the direction β:

$$\beta = \tan^{-1} \frac{F_y}{F_x} = 19^\circ\ . \quad (12.41)$$

12.4 Isolated Systems and Conservation of Momentum

During a collision between two objects the forces acting between the objects generally have a complicated time dependence—the curve of $F(t)$ is non-trivial. It is therefore not a simple task to calculate the impulse integral and use this to determine the change in momentum of the objects. Fortunately, it turns out that the problem can be significantly simplified for an isolated system where the net external force is zero. In this case the total momentum of the system is conserved throughout the collision. The total momentum is therefore the same before and after the collision. This is a powerful principle we use to analyze complex interactions without determining the detailed motion and forces in the system.

Momentum and Motion of Two Objects

We will now demonstrate that the total momentum is conserved when there are no external forces by discussing the collision between two objects A and B, illustrated in Fig. 12.7. We know that we can determine the motion of each object from Newton's second law of motion applied to each of the objects. For each object we separate the forces into *external forces*, forces having an origin outside the system, and *internal forces*, forces that are acting between the two objects:

> **Internal forces** act between the objects in the system.
> **External forces** act between objects in the system and the environment.

For the collision in Fig. 12.7 the only internal forces are the forces between the objects: The force from A on B and the reaction force from B on A. With this notation, Newton's second law for object A can be written as:

$$\sum \mathbf{F}_A = \sum \mathbf{F}_A^{\text{ext}} + \mathbf{F}_{B \text{ on } A} = \frac{d\mathbf{p}_A}{dt} , \tag{12.42}$$

where the sum is over all the external forces acting on object A. Similarly, Newton's second law for object B is:

$$\sum \mathbf{F}_B = \sum \mathbf{F}_B^{\text{ext}} + \mathbf{F}_{A \text{ on } B} = \frac{d\mathbf{p}_B}{dt} , \tag{12.43}$$

where we again have summed over all the external forces acting on object B. Now, we do not want to address the internal forces acting between A and B. How can we

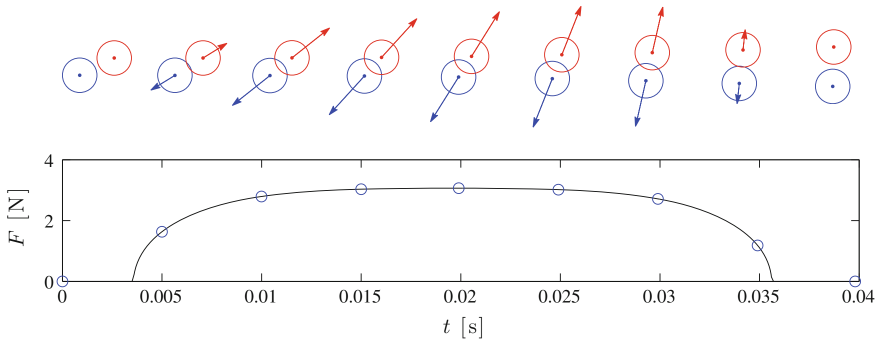

Fig. 12.7 Illustration of a collision between two objects A (*red*) and B (*blue*). The *top* figure shows the forces $\mathbf{F}_{B \text{ on } A}$ (*red*) and $\mathbf{F}_{A \text{ on } B}$ (*blue*) acting between the objects at various times t_i throughout the collision. The *bottom* figure shows the magnitude of the force, $F(t)$, as a function of time. The time t_i that are shown in the *top* figure is illustrated by circles

get rid of them in these equations? There is a commonly used trick: we recall from Newton's thirds law that the reaction force $\mathbf{F}_{A \text{ on } B}$ is equal, but oppositely directed to $\mathbf{F}_{B \text{ on } A}$:

$$\mathbf{F}_{A \text{ on } B} = -\mathbf{F}_{B \text{ on } A} \ , \tag{12.44}$$

If we insert this into (12.43), we get two equations for the motion of object A and B:

$$\begin{aligned} \sum \mathbf{F}_A^{\text{ext}} + \mathbf{F}_{B \text{ on } A} &= \frac{d\mathbf{p}_A}{dt} \\ \sum \mathbf{F}_B^{\text{ext}} - \mathbf{F}_{B \text{ on } A} &= \frac{d\mathbf{p}_B}{dt} \end{aligned} \tag{12.45}$$

If we add the equations, we get rid of the internal forces, $\mathbf{F}_{B \text{ on } A}$:

$$\sum \mathbf{F}_A^{\text{ext}} + \sum \mathbf{F}_B^{\text{ext}} = \frac{d\mathbf{p}_A}{dt} + \frac{d\mathbf{p}_B}{dt} \ , \tag{12.46}$$

We introduce the sum over all the external forces on all the objects in the system: Over all the forces acting on object A *and* all the forces acting on object B:

$$\sum \mathbf{F}^{\text{ext}} = \sum \mathbf{F}_A^{\text{ext}} + \sum \mathbf{F}_B^{\text{ext}} \ . \tag{12.47}$$

We use this to simplify (12.46):

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt} \left(\mathbf{p}_A + \mathbf{p}_B \right) \ . \tag{12.48}$$

We call the sum of the momenta for each of the objects the *total momentum* of the system:

Total momentum:

$$\mathbf{P} = \sum \mathbf{p} = \mathbf{p}_A + \mathbf{p}_B \ , \tag{12.49}$$

This provides a generalization of Newton's second law for a two-particle-system:

Generalization of Newton's second law:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt} \sum \mathbf{p} = \frac{d}{dt} \left(\mathbf{p}_A + \mathbf{p}_B \right) \ , \tag{12.50}$$

This law is completely general. We have not made any assumptions about the interactions between the two objects. The internal and external forces may be of any kind, conservative or non-conservative. The law is valid in all cases.

Conservation of Momentum in Isolated Systems

As a special case of this law, we observe that if the net external force on the system is zero (or negligible), the total momentum of the system is conserved:

$$\sum \mathbf{F}^{\text{ext}} = 0 \;\Rightarrow\; \frac{d}{dt}\left(\mathbf{p}_A + \mathbf{p}_B\right) = 0 \,. \tag{12.51}$$

We call a system **isolated** if the net external force is zero (or negligible):

An **isolated system** is a collection of objects that may interact internally, but where the net external force on all the objects is zero (or negligible).

For an isolated system, the total momentum is conserved:

$$\mathbf{p}_A + \mathbf{p}_B = \text{constant (for an isolated system)} \,. \tag{12.52}$$

- This is a completely general law for the conservation of momentum of a system. It only requires the *net external* force on the system to be zero. It is valid not only at the beginning and at the end of the collision, but at all times during the collision as well.
- Notice: A common mistake is to forget the absolutely necessary condition that the net external force on the system must be zero (or negligible). Whenever you employ this law, you should make a habit of always asking yourself if the net external force is zero, or if it is reasonable to neglect it compared with other forces.
- Notice that the conservation law is a vector equation, and that it can be valid in one direction independently of an orthogonal direction. If the net external force in the x-direction is zero, the total momentum in this direction is conserved even though there is a net external force in the y-direction.
- Notice that it is not only valid for contact forces, as illustrated in Fig. 12.7, but for any type of force, including long-reaching forces such as gravity. The gravitational forces between object in the system are internal forces, while gravitational forces between objects in the system and objects outside the system are external forces.

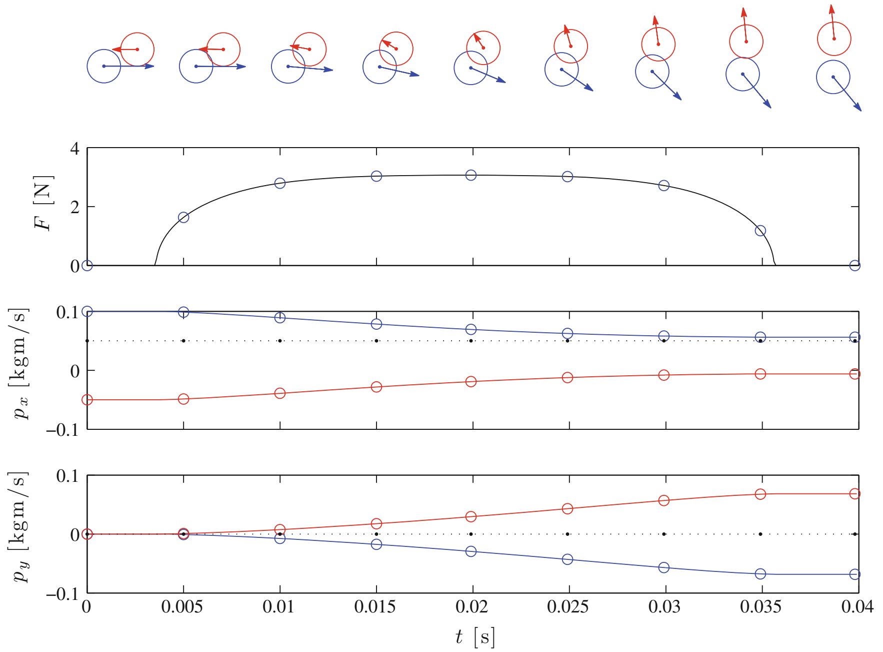

Fig. 12.8 Illustration of the same process as in Fig. 12.7, but now the *arrows* in the *top* figure illustrates the momentum of the object, and the two *bottom* figures show the momentum $p_x(t)$ and $p_y(t)$ as a function of time. The total momentum is shown with a *dotted line*

Conservation of Total Momentum During a Collision

The conservation of total momentum during the collision between objects A and B is illustrated in Fig. 12.8. The arrows indicate the momentum of each object and the plots show the momentum in the x- and y-direction for each object and the total momentum. Since there are only internal forces acting in this system, there are no external forces affecting either object, and the total momentum is conserved in both the x- and the y-direction.

Conservation of Total Momentum During a Collision with an External Force

What happens if we addressed the same collision, but also include a gravitational force in the y-direction for both objects? As illustrated in Fig. 12.9, the behavior looks similar: The force as a function of time, $F(t)$, is similar, and the total momentum in the x-direction is conserved. However, the momentum in the y-direction is not conserved. It is decreasing throughout the motion due to the external force, gravity, acting on both object A and object B.

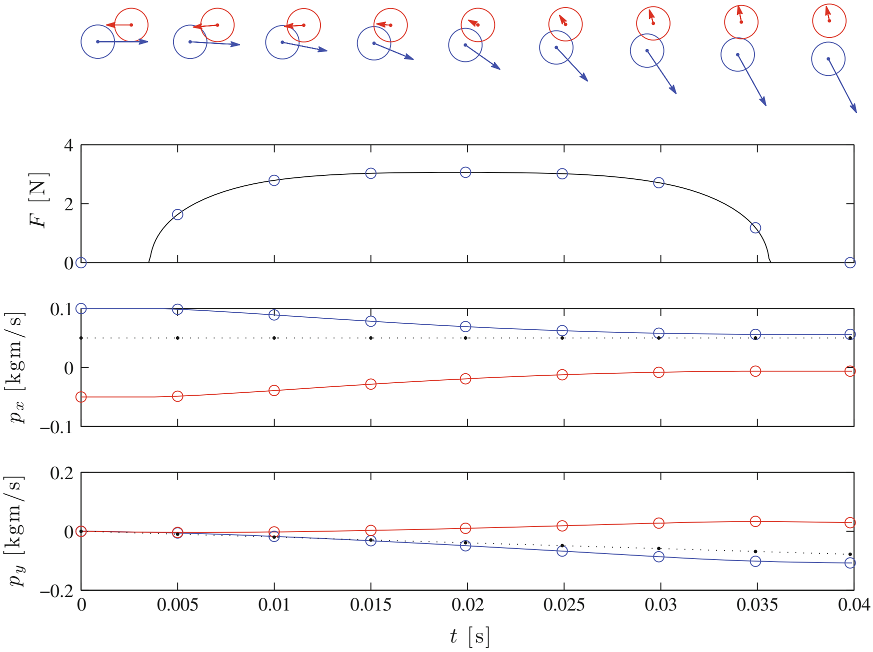

Fig. 12.9 Illustration of the same objects as in Figs. 12.7 and 12.8., but now with gravity acting in the y-direction. The *arrows* in the *top* figure illustrates the momentum of the objects. The *bottom* figures show the momentum $p_x(t)$ and $p_y(t)$ as a function of time. The total momentum (*dotted line*) is constant in the x-direction, but is changing in the y-direction due to the net force from gravity affecting the objects

How large is the effect of an external force, like gravity in this case? The change in momentum is given by the impulse of the external forces:

$$\Delta \mathbf{P} = \mathbf{J}^{\text{ext}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{ext}} dt \ , \tag{12.53}$$

To determine the effect of the external forces, we therefore need to compare the change of momentum due to the external force, $\Delta \mathbf{P}$, to the total momentum of the system, $\mathbf{P}$. For the gravitational force $\mathbf{G} = -mg æ$, the impulse of gravity is:

$$J_{G,y} = \int_{t_0}^{t_1} (-m_A g - m_B g)\, dt = -\left(m_A + m_B\right) \Delta t \ , \tag{12.54}$$

and the total momentum in the y-direction of the whole system,

$$P_y = p_{A,y} + p_{B,y} = m_A v_{A,0,y} + m_B v_{B,0,y} \ . \tag{12.55}$$

If $J_G \ll P_y$, the impulse of the external forces are negligible during the collision, and we can assume that momentum is approximately conserved. Notice that the impulse depends on the duration of the collision, Δt. If the collision takes a short time, the momentum change due to the external forces will be small during the collision, and the momentum will be approximately conserved from the beginning to the end of the collision. This is why we often assume that collisions are instantaneous. Then we may assume that the impulse due to external forces is small compared with the total momentum, and therefore that the total momentum is approximately conserved in the collision.

Conservation of Momentum for Multi-particle Systems

The generalization of Newton's law can be extended to any number of objects. For a system with N objects, the total momentum is:

$$\mathbf{P} = \sum_{j=1}^{N} \mathbf{p}_j \ , \tag{12.56}$$

and the generalization of Newton's second law is:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt}\mathbf{P} = \frac{d}{dt}\sum_{j=1}^{N} \mathbf{p}_j \ . \tag{12.57}$$

The derivation follows the same principles used for two particle system: Also for the N-particle system Newton's third law ensures that all internal forces come in pairs that cancel each other, so that the sum of all the internal forces is zero.

12.5 Collisions

The conservation of momentum allows us to determine the velocities of objects after a collision from the velocities before a collision without knowing the details of the interactions during the collision: We can get by without solving the complete equations of motion, even without knowing the details of the interactions, for all the objects. Let us therefore apply the law we have introduced, the conservation of momentum for isolated systems, to address collisions, first in one dimension and then in two and three dimensions.

What is a collision? For our purposes, a collision between two objects is a process where large forces are acting between the two objects over a short time interval:

A **collision** between two or more objects is a process

- where the internal forces between the objects are much larger than the external forces from the environment
- that occur over a short time interval compared to the time scale of the motion.

Collisions Along a Line

We start from the simplest case: a one-dimensional collision between two blocks moving along a line. We want to find the velocities of the objects after the collision. Blocks A and B with masses m_A and m_B slide along a horizontal, frictionless straight track. Before the collision, the x-component of the velocities of the blocks are $v_{A,0}$ and $v_{B,0}$. The blocks collide during a short time interval Δt. What are their velocities after the collision? The process is illustrated in Fig. 12.10.

We assume that the net external force in the x-direction is zero. Hence, the momentum in the x-direction is conserved, and the total momentum P is the same before and after the collision:

$$
\begin{aligned}
P_0 &= P_1 \\
p_{A,0} + p_{B,0} &= p_{A,1} + p_{B,1} \\
m_A v_{A,0} + m_B v_{B,0} &= m_A v_{A,1} + m_B v_{B,1} \ .
\end{aligned}
\tag{12.58}
$$

This equation holds at any time during the collision. Unfortunately, we cannot find the velocities after the collision from this equations alone, because there are two unknown velocities, $v_{A,1}$, and $v_{B,1}$, but only one equation. We need more information about the process—we need an additional equation!

We need to know more about the collision process to find another relation between the initial and the final velocities. For example, if we know that all the forces acting between the objects are conservative, we would get an additional equation from the conservation of mechanical energies. If the objects are not interacting, which is the case before and after, but not during the collision, the total energy is equal to the kinetic energies plus a constant potential energy (because the potential energies are constant when the objects are not interacting):

$$
\frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B v_{B,0}^2 + U_0 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 + U_1 \ , \tag{12.59}
$$

where $U_0 = U_1$. In this case, we would have two equations and two unknowns, and we can find one (or a few) unique solutions, which gives us the values of the velocities after the collision.

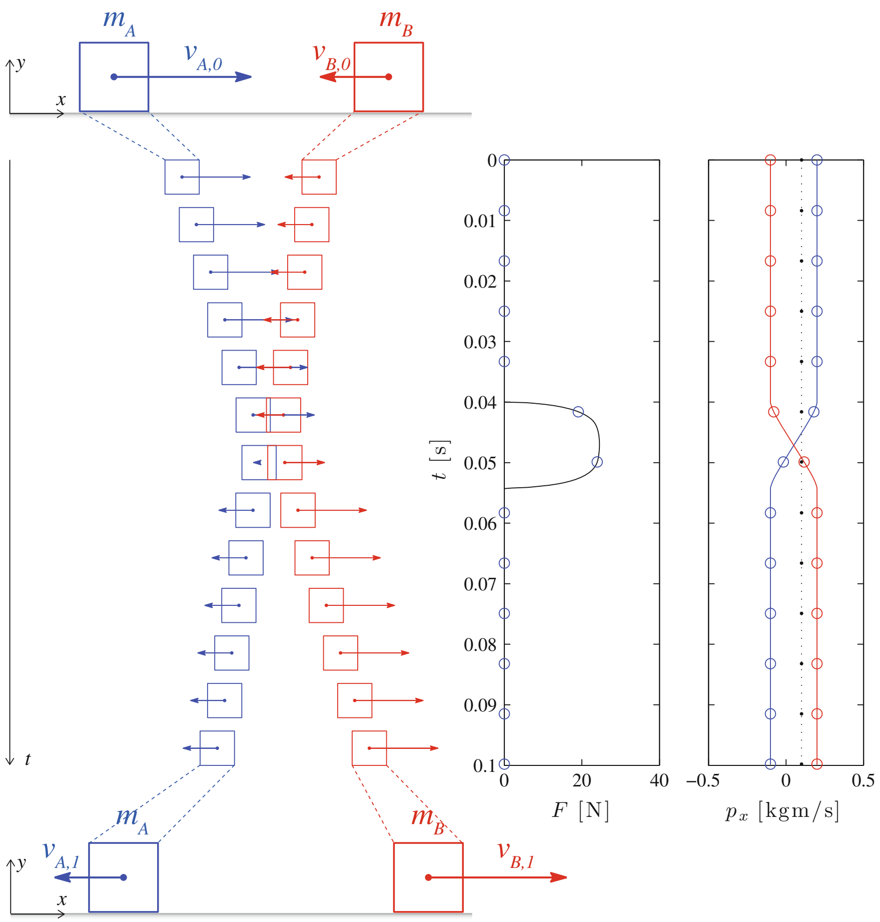

Fig. 12.10 Illustration of a collision between two blocks A and B showing the situations before and after the collision. Snapshots of the time development are shown in the cartoons. The force, F, acting between the blocks and the momentum of block A, block B, and the total momentum (*dotted line*) are all plotted as functions of time

Conservation of mechanical energy is one possibility. Many things can happen between the objects during a collision, which may give rise to other relations and other equations relating the initial and the final velocities. It is customary to describe collisions by their degree of energy conservation, ranging from *elastic* collisions, where mechanical energy is conserved, through various *inelastic* collisions where mechanical energy is not conserved, to a *perfectly inelastic* collision, which corresponds to the maximum loss of mechanical energy while still conserving momentum. We discuss these situations in the following.

Coefficient of Restitution

The energy loss in a collision between two objects is often described by the coefficient of restitution, r. For example, we can characterize the collision between a ball and a massive wall by the velocity, v_1, of the ball after the collision:

$$v_1 = -r v_0 \ , \tag{12.60}$$

as a function of the velocity v_0 before the collision. The change in kinetic energy is:

$$E_1 - E_0 = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 = (r^2 - 1) E_0 \ , \tag{12.61}$$

This gives the following cases:

- A collision is called **elastic** if the mechanical energy is conserved. This corresponds to $r = 1$.
- A collision is called **inelastic** if the mechanical energy is not conserved. This corresponds to $0 \le r < 1$.
- A collision is called **perfectly inelastic** the two objects have the same velocity after the collision. This corresponds to $r = 0$.

If you bounce a ball on the floor, we can relate the coefficient of restitution to how high the ball bounces back up. If we drop the ball from a height h_0, the velocity of the ball is given by $v_0^2 = 2gh_0$ before the collision, and $v_1^2 = r^2 v_0^2 = 2gh_0$ after the collision. The maximum height the ball reaches is found from $2gh_1 = v_1^2 = r^2 v_0^2 = r^2 2gh_0$, which gives $h_1 = r^2 h_0$ as illustrated in Fig. 12.11.

The coefficient of restitution for a collision between two objects is defined as:

$$r = -\frac{v_{A,1} - v_{B,1}}{v_{A,0} - v_{B,0}} \ . \tag{12.62}$$

(Notice that the definition does not change if we exchange A with B, which is necessary for such a definition to make sense.)

It is possible to show that $r = 0$ corresponds to the maximum loss of energy while conserving momentum.

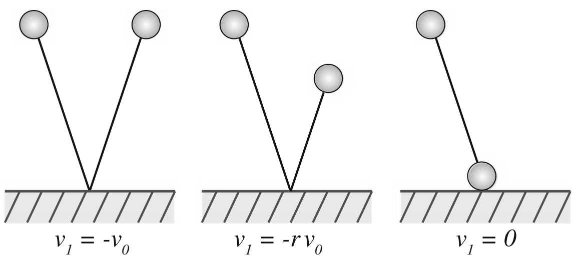

Fig. 12.11 Illustration of a ball bouncing on the floor

What does r depend on? Since any conservative force would lead to energy conservation and therefore $r = 1$, the coefficient of restitution characterizes the non-conservative parts of the contact forces. In reality, there are many processes during a collision that result in changes in kinetic energy, such as internal oscillations in the objects, permanent deformations, and viscous- and frictional forces. For a given force model, such as for a spring model with a viscous damping term, we can calculate the velocity after a collision and therefore determine the coefficient of restitution.

Elastic Collisions

For an elastic collision between the two blocks A and B in Fig. 12.10, both the total momentum and the mechanical energy are conserved:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.63}$$

$$\frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B v_{B,0}^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ . \tag{12.64}$$

Since we are free to choose the coordinate system, we simplify the problem by choosing a system where block B is initially at rest: $v_{B,0} = 0$, giving:

$$m_A v_{A,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.65}$$

$$m_A v_{A,0}^2 = m_A v_{A,1}^2 + m_B v_{B,1}^2 \ . \tag{12.66}$$

These two equations have a unique solution (see Extra material for a derivation).

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} \ , \ v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} \ . \tag{12.67}$$

These formulas are general. Let us get to know them by studying a few special cases:

Equal Masses

If the two blocks have the same masses: $m_A = m_B$, we find that:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} = 0 \ , \ v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} = v_{A,0} \ . \tag{12.68}$$

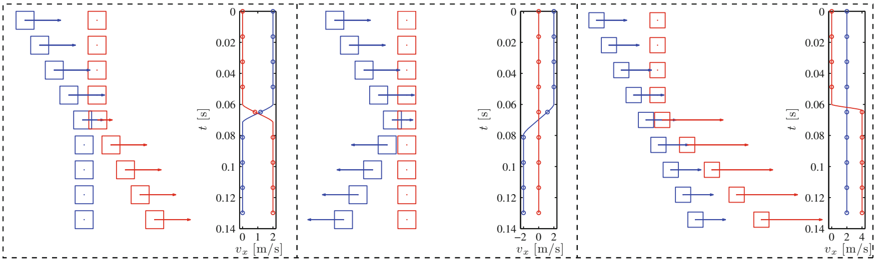

Fig. 12.12 Illustration of collision with (*left*) equal masses, (*middle*) a large mass, (*right*) a small mass

The two blocks exchange velocities! Before the collision block B is at rest and block A is moving with velocity $v_{A,0}$, and after the collision block A is at rest, and block B is moving with velocity $v_{A,0}$ (see Fig. 12.12).

You are probably familiar with this effect. You notice it when two equally sized balls collide head on, such as two billiard balls.

Collision with a Large Mass

What happens if block A collides with a large, stationary mass? That is, if $m_B \gg m_A$. We expect this to be like a collision with a stationary wall. We find:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} = \frac{(m_A/m_B) - 1}{(m_A/m_B) + 1} v_{A,0} \simeq -v_{A,0} \; , \tag{12.69}$$

where we have used the $m_A/m_B \ll 1$, that is $m_A/m_B \simeq 0$ when $m_B \gg m_A$. Similarly, for $v_{B,1}$:

$$v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} \simeq 0 \; , \tag{12.70}$$

For an elastic collision with a wall (or a very large object), the velocity is simply reversed (see Fig. 12.12).

Collision with Small Mass

What happens if block A collides with a tiny block B? That is, if $m_A \gg m_B$? We find that:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} = \frac{1 - (m_B/m_A)}{1 + (m_B/m_A)} v_{A,0} \simeq v_{A,0} \; , \tag{12.71}$$

and

$$v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} = \frac{2}{1 + (m_B/m_A)} v_{A,0} \simeq 2v_{A,0} \; . \tag{12.72}$$

If a large object collides head on into a much smaller, stationary object, the large object continues with approximately the same velocity, but the small object is shot forward with twice the velocity of the large object (see Fig. 12.12).

Collisions and Relative Motion

So far we have only addressed the case when block B starts at rest, but our solutions are more general than you may believe: We can use a coordinate system change to map any collision onto the solutions we have found. Let us demonstrate how to do this by addressing an elastic collision between a ball (object A) with mass m_A and a moving wall, the much more massive object B: $m_B \gg m_A$.

The situation is illustrated in Fig. 12.13. Before the collision the velocity of the ball is $v_{A,0}$ and the velocity of the wall is $v_{B,0}$ measured relative to a coordinate system S, which we call the laboratory system. However, if we follow the motion of the wall, in the system S', the wall has velocity zero before the collision: $v'_{B,0} = 0$.

How are the two coordinate systems related? In Sect. 6.4 who found that the position $\mathbf{r}$ measured in system S is related to the position $\mathbf{r}'$ measured in system S' through:

$$\mathbf{r} = \mathbf{R} + \mathbf{r}' \; , \tag{12.73}$$

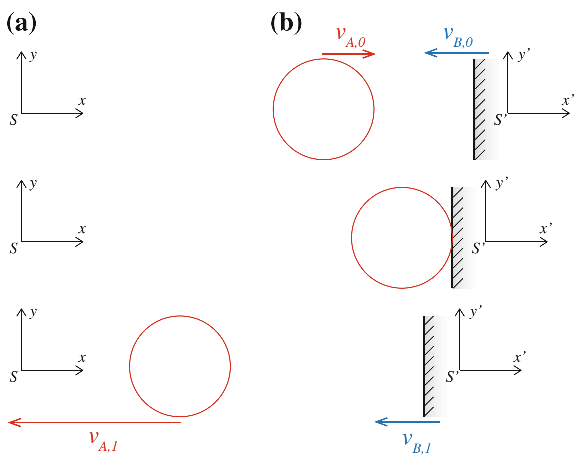

Fig. 12.13 Illustration of a collision between a ball (**a**) and a wall (**b**). The collision is addressed in both the system S, where the wall has a velocity toward A, and in system S', which follows the motion of the wall

where $\mathbf{R}$ is the position of system S' measured in system S. We take the time derivative on both sides to find a relation between the velocities:

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' \ , \tag{12.74}$$

where $\mathbf{u}$ is the velocity of system S' measured in system S. We apply this to the current situation, where system S' is moving with the (initial) velocity of the wall $u = v_{B,0}$, we get:

$$v'_{A,0} = v_{A,0} - u = v_{A,0} - v_{B,0} \ , \tag{12.75}$$

and

$$v'_{B,0} = v_{B,0} - u = v_{B,0} - v_{B,0} = 0 \ , \tag{12.76}$$

which was the whole point—since $v'_{B,0} = 0$ we can use the solution from (12.69) to determine the velocity after the collision in the S' system:

$$v'_{A,1} = -v'_{A,0} = -\left(v_{A,0} - v_{B,0}\right) \ . \tag{12.77}$$

We find the velocity in the original system by the reverse transformation:

$$\begin{aligned} v_{A,1} &= v'_{A,1} + u = v'_{A,1} + v_{B,0} = -v'_{A,0} + v_{B,0} \\ &= -\left(v_{A,0} - v_{B,0}\right) + v_{B,0} = 2v_{B,0} - v_{A,0} \ , \end{aligned} \tag{12.78}$$

This is what happens when a ball hits a wall that moves toward the ball, such as a collision between a golf club (massive) and a golf ball (small mass). Similar techniques can be used to address other solutions.

General Solutions to Elastic Collisions

We can find the general solution for an elastic collision between blocks A and B as illustrated in Fig. 12.10:

$$v_{A,1} = \frac{(m_A - m_B)\, v_{A,0} + 2m_B v_{B,0}}{m_A + m_B} \tag{12.79}$$

$$v_{B,1} = \frac{(m_B - m_A)\, v_{B,0} + 2m_A v_{A,0}}{m_A + m_B} \ . \tag{12.80}$$

(A proof of this result is given in Sect. A.2).

Perfectly Inelastic Collisions

For a perfectly inelastic collision the two blocks A and B become attached after the collision, and they continue with the same velocity, v_1:

$$v_{A,1} = v_{B,1} = v_1 \ , \tag{12.81}$$

In this case, only the total momentum is conserved, and not the kinetic energy. Conservation of momentum gives:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.82}$$

Velocity After Perfectly Inelastic Collision

In the case when $v_{B,0} = 0$, we find:

$$m_A v_{A,0} = m_A v_1 + m_B v_1 = (m_A + m_B)\, v_1 \ , \tag{12.83}$$

which gives:

$$v_1 = \frac{m_A}{m_A + m_B} v_{A,0} = v_{A,1} = v_{B,1} \ . \tag{12.84}$$

Loss of Energy After Perfectly Inelastic Collision

The mechanical energy is not conserved for a perfectly inelastic collision. This means that the initial kinetic energy of the system is transformed into thermal energy and has been used to deform the objects permanently. The perfectly inelastic collision gives the maximum loss of energy. Let us find the change in kinetic energy in the collision.

Before the collision (when B is at rest), the kinetic energy is:

$$K_0 = \frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B \underbrace{v_{B,0}^2}_{=0} = \frac{1}{2} m_A v_{A,0}^2 \ . \tag{12.85}$$

After the collision the kinetic energy is:

$$\begin{aligned} K_1 &= \frac{1}{2}(m_A + m_B)\, v_1^2 = \frac{1}{2}(m_A + m_B)\, \frac{m_A^2}{(m_A + m_B)^2} v_{A,0}^2 \\ &= \frac{1}{2} \frac{m_A^2}{m_A + m_B} v_{A,0}^2 = \frac{m_A}{m_A + m_B} \cdot \frac{1}{2} m_A v_{A,0}^2 = \frac{m_A}{m_A + m_B} K_0 \ , \end{aligned} \tag{12.86}$$

How large fraction of the original kinetic energy remains after the collision?

$$\frac{K_1}{K_0} = \frac{m_A}{m_A + m_B} < 1 \ . \tag{12.87}$$

The loss in kinetic energy is:

$$\Delta K = K_1 - K_0 = \left(\frac{m_A}{m_A + m_B} - 1 \right) K_0 = -\frac{m_B}{m_A + m_B} K_0 \ . \tag{12.88}$$

12.5.1 Example: Ballistic Pendulum

Problem: A 10 g bullet is fired into a 1 kg wooden block hanging in a 1 m long rope. The block reaches a height 30 cm above its initial level. (a) What was the velocity of the bullet? (b) What was the loss of energy in the system?

Identify: In this problem we address the motion of two objects: The bullet (object A) and the block (object B). The block is hanging from a massless rope. This means that the block behaves as a pendulum after the collision between the bullet and the pendulum. After the collision, the block swings to a height h above its initial position. The process is illustrated in Fig. 12.14.

Model: In this problem, we do not know the detailed interactions between the bullet and the block. Therefore, we use conservation laws to address the collision between the bullet and the block. The bullet becomes stuck in the block—this means that the bullet and the block has the same velocity after the collision—the collision is perfectly inelastic. Before the collision the bullet has a horizontal velocity, $\mathbf{v}_{A,0} = v_0 \, \mathbf{i}$, and the block is at rest, $\mathbf{v}_{B,0} = 0$. After the collision, both objects have the velocity $\mathbf{v}_{A,1} = \mathbf{v}_{B,1} = v_1 \, \mathbf{i}$.

Solution to part a: There are no external forces acting on the system in the x-direction during the collision. (There may be forces acting on the system in the y-direction—depending on the motion of the system and the effect of the rope. We

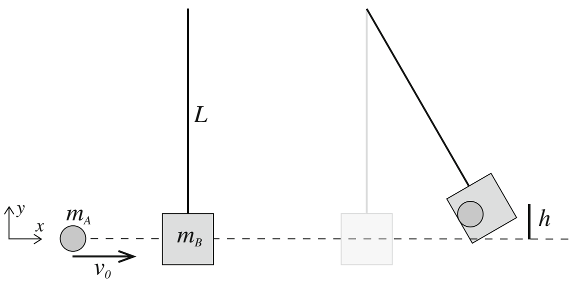

Fig. 12.14 Illustration of the motion of a ballistic pendulum. First, there is a collision between the bullet and the block, and afterward the bullet and the block swings as a pendulum to a height h above their initial position

assume that during the collision, the rope is vertical, so that the rope does not exert a horizontal force on the system.) From Newton's second law in the x-direction we get:

$$\sum F_x = 0 = \frac{d}{dt}\left(p_{A,x} + p_{B,x}\right) . \tag{12.89}$$

The total momentum in the x-direction is therefore conserved during the collision:

$$m_A v_{A,0} + m_B \underbrace{v_{B,0}}_{=0} = m_A v_{A,1} + m_B v_{B,1} \Rightarrow v_1 = \frac{m_A}{m_A + m_B} v_{A,0} . \tag{12.90}$$

In the second part of the process, the block and bullet swings up as a pendulum. In this process, the rope does not do any work on the system, and we assume that the air resistance is negligible. The only force doing work on the system is therefore gravity. In this second part of the process, the mechanical energy is conserved!

Notice that in this case the process consists of two separate subprocesses. In the first subprocess, the collision, the mechanical energy is not conserved, but in the second subprocess, the swinging pendulum, the mechanical energy is conserved.

We use energy considerations to determine how high the pendulum swings. The mechanical energy at the beginning of this motion, immediately after the collision, is the same as the mechanical energy when the pendulum has reached its maximum height. At its maximum height the kinetic energy of the pendulum is zero. Conservation of mechanical energy therefore gives:

$$\frac{1}{2}\left(m_A + m_B\right) v_1^2 = \left(m_A + m_B\right) g h \Rightarrow h = \frac{v_1^2}{2g} = \left(\frac{m_A}{m_A + m_B}\right)^2 \frac{v_0^2}{2g} . \tag{12.91}$$

Now, the problem posed was to find the initial velocity, v_0, as a function of h. We find v_0 from (12.91):

$$v_0 = \sqrt{2gh}\left(\frac{m_A + m_B}{m_A}\right) . \tag{12.92}$$

Let us now insert the numbers given in the problem:

$$v_0 = \sqrt{2\left(9.8\,\mathrm{m/s^2}\right)(0.3\,\mathrm{m})}\left(\frac{1.01\,\mathrm{kg}}{0.01\,\mathrm{kg}}\right) \simeq 245\,\mathrm{m/s} . \tag{12.93}$$

Solution to part b: In the second part of the problem, we are asked to find the loss of energy. The kinetic energy before the collision is

$$K_0 = \frac{1}{2} m_A v_0^2 , \tag{12.94}$$

and after the collision the kinetic energy is:

$$K_1 = \frac{1}{2}(m_A + m_B)v_1^2 = \frac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B}v_0\right)^2 \tag{12.95}$$

$$= \frac{1}{2}m_A v_0^2 \frac{m_A}{m_A + m_B} = K_0 \frac{m_A}{m_A + m_B} \, . \tag{12.96}$$

The relative loss of kinetic energy is therefore:

$$\frac{\Delta K}{K_0} = \frac{m_A}{m_A + m_B} - 1 = -\frac{m_B}{m_A + m_B} \, , \tag{12.97}$$

which means that practically all the energy was lost in the collision!

12.5.2 Example: Super-Ball

Problem: In super-ball we take two balls, one small and one large, and release them together from a height h_0 above the ground, as illustrated in Fig. 12.15. What is the maximum height h_1 reached by the top ball after the collision? Assume that all collisions are conservative.

Identify: In this problem we address the motion of two objects: The bottom ball, A, and the top ball, B. The whole process may be subdivided into three separate parts. In the first part both objects are falling until they hit the ground. In the second part they are colliding with the ground and each other, and in the third part, they are both moving upward. The various subprocesses are illustrated in Fig. 12.15.

Model: We do not know the interactions between the balls, or between the balls and the ground, but we know that all collisions are elastic. We will also consider the problem to consist of a sequence of collisions between two objects: First the bottom

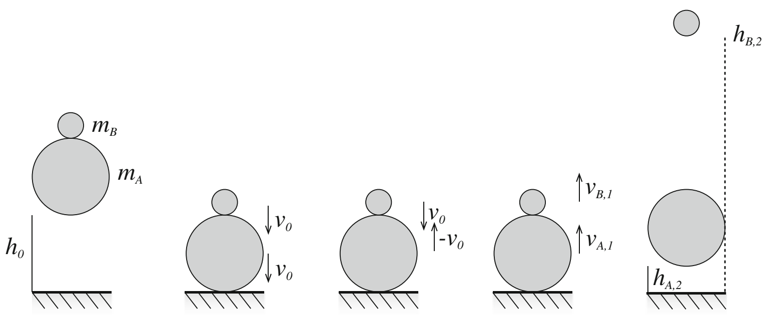

Fig. 12.15 Illustration of the motion of two balls A and B bouncing off the ground and each other

ball collides with the ground, and then the bottom ball collides with the top ball. Let us look at each of these subprocesses individually:

In the *first part*, the balls fall from a height h_0. We use energy conservation to find the velocity:

$$K_0 + mgh_0 = K_1 + mg0 \ , \tag{12.98}$$

$$mgh_0 = \frac{1}{2}mv_0^2 \ , \tag{12.99}$$

and

$$v_0 = \sqrt{2gh_0} \ , \tag{12.100}$$

is the velocity of both the balls.

In the *second part*, the bottom ball hits the floor. This is an elastic collision, where we know tthat the velocity is reversed. After the collision, the bottom ball therefore has an upward velocity v_0.

In the *third part*, the bottom ball collides with the top ball. Since the only external force is gravity, which has a small impulse compared to the forces between the balls, the net vertical force is approximately zero, and conservation of momentum in the y-direction gives:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.101}$$

$$m_A v_0 + m_B \left(-v_0\right) = m_A v_{A,1} + m_B v_{B,1} \ . \tag{12.102}$$

Conservation of kinetic energy gives:

$$\frac{1}{2}m_A v_{A,0}^2 + \frac{1}{2}m_B v_{B,0}^2 = \frac{1}{2}m_A v_{A,1}^2 + \frac{1}{2}m_B v_{B,1}^2 \ , \tag{12.103}$$

where we insert $v_{A,0} = v_0$ and $v_{B,0} = -v_0$:

$$\frac{1}{2}m_A v_0^2 + \frac{1}{2}m_B v_0^2 = \frac{1}{2}m_A v_{A,1}^2 + \frac{1}{2}m_B v_{B,1}^2 \ . \tag{12.104}$$

We can rewrite the two equations to be:

$$m_A \left(v_0 - v_{A,1}\right) = m_B \left(v_0 + v_{B,1}\right) \ , \tag{12.105}$$

and

$$m_A \left(v_0^2 - v_{A,1}^2\right) = m_B \left(v_{B,1}^2 - v_0^2\right) \ , \tag{12.106}$$

This last equation can also be written as:

$$m_A \left(v_0 - v_{A,1}\right) \left(v_0 + v_{A,1}\right) = m_B \left(v_{B,1} - v_0\right) \left(v_{B,1} + v_0\right) \ . \tag{12.107}$$

Dividing the two equations, we get:

$$v_0 + v_{A,1} = v_{B,1} - v_0 \ \Rightarrow \ 2v_0 = v_{B,1} - v_{A,1} \ . \tag{12.108}$$

We eliminate $v_{A,1}$, finding:

$$v_{B,1} = \frac{3m_A - m_B}{m_A + m_B} v_0 \ . \tag{12.109}$$

Discussion: Let us address three special cases:

- For $m_A = \frac{m_B}{3}$ we find $v_{B,1} = 0$
- For $m_A = m_B$ we find $v_{B,1} = v_0$
- For $m_A = 3m_B$ we find $v_{B,1} = 2v_0$
- For $m_A \gg m_B$ we find $v_{B,1} = 3v_0$

We find maximum height from the case when $m_A \gg m_B$, using conservation of energy for the motion of ball B:

$$m_B g h_{B,2} = \frac{1}{2} m_B v_{B,1}^2 = \frac{1}{2} m_B 9 v_0^2 \ , \tag{12.110}$$

and therefore we find that:

$$h_{B,2} = 9h_0 \ , \tag{12.111}$$

which is the maximum height, in the case when the large ball has much larger mass than the small ball.

Non-central Elastic Collisions

So far we have only studied one dimensional collisions—collisions where all the objects move along a line so that all velocities also are directed along the line. Such collisions are called central collisions:

> In a **central collision** the momentum of both objects is directed along the line between the two objects before and after the collision.

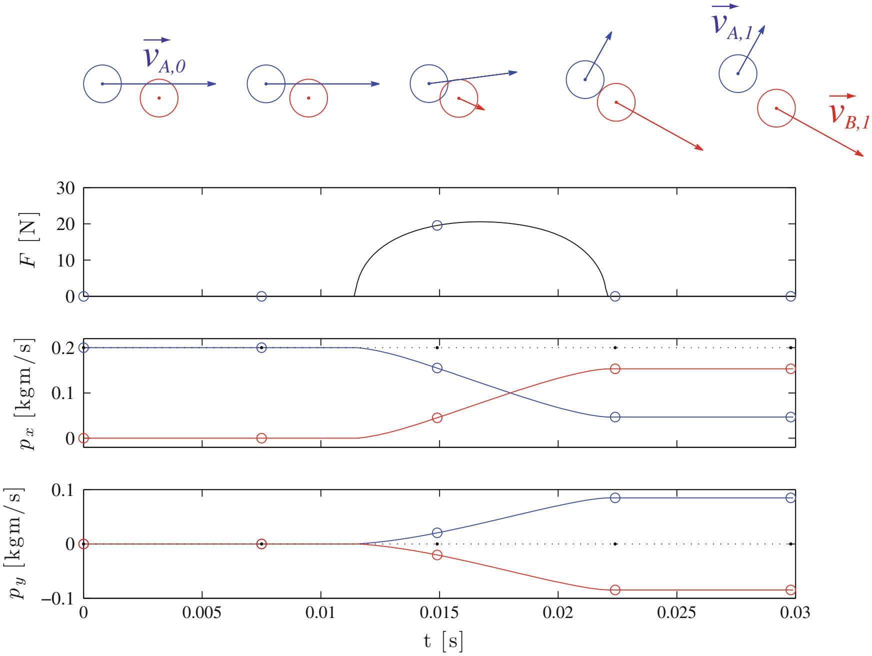

Fig. 12.16 Illustration of a non-central collision between two objects

Let us now address the more general case where two objects are colliding, but where the momentum of at least one of the objects is not directed along the line connecting the centers of the objects (before the collision). An example of such a collision is illustrated in Fig. 12.16.

We study this collision in a reference system where object B is initially at rest, $\mathbf{v}_{B,0} = 0$. Since there are no external forces acting on the system during the collision, the total momentum is conserved:

$$m_A \mathbf{v}_{A,0} + m_B \underbrace{\mathbf{v}_{B,0}}_{=0} = m_A \mathbf{v}_{A,1} + m_B \mathbf{v}_{B,1} \ . \tag{12.112}$$

If the collision is also elastic, the kinetic energy is conserved:

$$\frac{1}{2} m_A v_{A,0}^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ . \tag{12.113}$$

For this very general case we cannot make any further general statements. But for a collision between two objects with equal masses, $m_A = m_B$, we find that the conservation of momentum is:

$$\mathbf{v}_{A,0} = \mathbf{v}_{A,1} + \mathbf{v}_{B,1} \ , \tag{12.114}$$

and conservation of kinetic energy is now:

$$v_{A,0}^2 = v_{A,1}^2 + v_{B,1}^2 \; . \tag{12.115}$$

We insert $\mathbf{v}_{A,0}$ into (12.115):

$$\begin{aligned} v_{A,0}^2 &= v_{A,1}^2 + v_{B,1}^2 \\ \left(\mathbf{v}_{A,1} + \mathbf{v}_{B,1}\right)^2 &= v_{A,1}^2 + v_{B,1}^2 \\ v_{A,1}^2 + 2\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} + v_{B,1}^2 &= v_{A,1}^2 + v_{B,1}^2 \\ 2\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} &= 0 \; , \end{aligned} \tag{12.116}$$

We have found that $\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} = 0$, which means that the two velocities are orthogonal! Notice that we still do not have enough equations to determine the vectors: We have 3 equations, but 4 unknown components in the velocity vectors after the collision. In order to determine the velocities after the collision we need more information about the collision. We need to know something about the force acting between the particles throughout the collision.

12.6 Modeling and Visualization of Collisions

We can gain better insights into the concepts introduced in this chapter by studying collisions in detail. If we know the details of the interactions between two objecst, that is, if the have models for the interaction forces, we can find their motion from Newton's second law.Let us use this to get a better understanding of elastic, inelastic and perfectly inelastic collisions.

We model two objects, A and B, with masses m_A and m_B. The force from B on A is:

$$\mathbf{F}_{\text{B on A}} = \mathbf{F}(\mathbf{r}_A, \mathbf{r}_B, \mathbf{v}_A, \mathbf{v}_B) \; , \tag{12.117}$$

and from Newton's third law, we know that

$$\mathbf{F}_{\text{A on B}} = -\mathbf{F} \; . \tag{12.118}$$

For example, for two solid spheres of radius R, a reasonable force model is:

$$\mathbf{F} = \begin{cases} k\,|\Delta r - 2R|\,\frac{\Delta \mathbf{r}}{\Delta r} - \eta\,(\Delta \mathbf{v}) & , \; \Delta r < 2R \\ \mathbf{0} & , \; \Delta r \geq 2R \end{cases} \; , \tag{12.119}$$

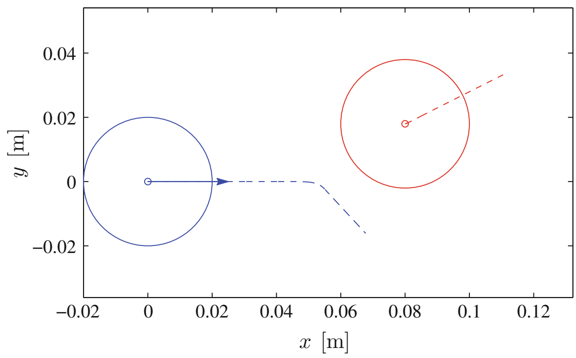

Fig. 12.17 Illustration of object trajectories and initial conditions

where $\Delta\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, $\Delta r = |\Delta\mathbf{r}|$ and $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$. We find the equation of motion from Newton's second law:

$$m_A\mathbf{a}_A = \mathbf{F} \;\Rightarrow\; \mathbf{a}_A = \frac{\mathbf{F}}{m_A} \,. \tag{12.120}$$

We find the motion of both objects from Euler-Cromer's method, as implemented in the following program. First, we define the masses, the radius, and the initial positions and velocities of the objects:

```
from pylab import *
R = 0.02 # m
mA = 0.1 # kg
mB = 0.1 # kg
rA0 = array([0.0,0.0]) # m
vA0 = array([1.0,0.0]) # m/s
rB0 = array([0.08,0.018]) # m
vB0 = array([0.0,0.0]) # m/s
time = 0.10 # s
```

This set of initial conditions are illustrated in Fig. 12.17. These conditions will result in an non-central collision. Then, we define the parameters used by the force model, such as the force constant k and the viscous term η. We choose an unrealistically small value for k in order to make the collision extend over some time so that we can observe the interactions during the collision. The time step is chosen small enough, so that we are sure to have good resolution for the motion during the collision:

```
# Force model
eta = 1.0
k = 20000.0 # Nm
dt = 0.0001 # s
```

We initialize by generating arrays for all the variables:

```
# Initialization
n = int(round(time/dt))
t = zeros(n,float)
rA = zeros((n,2),float)
vA = zeros((n,2),float)
rB = zeros((n,2),float)
vB = zeros((n,2),float)
```

```
F = zeros((n,2),float)
rA[0] = rA0
vA[0] = vA0
rB[0] = rB0
vB[0] = vB0
D = 2*R # Diameter
```

Where we have introduce the diameter, $D = 2R$, to simplify the expressions. The integration loop follows the mathematical formulation of the force law in (12.119) as closely as possible:

```
# Integration loop
for i in range(n-1):
    Deltar = rB[i]-rA[i]
    Deltarnorm = sqrt(dot(Deltar,Deltar))
    Deltav = vB[i]-vA[i]
    if (Deltarnorm>=D):
        Fnet = array([0,0])
    else:
        Fnet = -k*abs(Deltarnorm-D)**1.5*Deltar/Deltarnorm + eta*Deltav;
    F[i] = Fnet
    aA = Fnet/mA
    aB = -Fnet/mB
    vA[i+1] = vA[i] + aA*dt
    rA[i+1] = rA[i] + vA[i+1]*dt
    vB[i+1] = vB[i] + aB*dt
    rB[i+1] = rB[i] + vB[i+1]*dt
    t[i+1] = t[i] + dt
```

Finally, we plot the resulting trajectories and the momentum in the x and y direction as functions of time:

```
# Plot trajectories and momentum
figure(1)
plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r')
xlabel('x [m]')
ylabel('y [m]')
axis('equal')
figure(2)
pA = vA.copy()*mA
pB = vB.copy()*mB
subplot(2,1,1)
plot(t,pA[:,0],'-b',t,pB[:,0],'-r')
xlabel('t [s]')
ylabel('p_x [kgm/s]')
subplot(2,1,2)
plot(t,pA[:,1],'-b',t,pB[:,1],'-r');
xlabel('t [s]')
ylabel('p_y [kgm/s]');
```

While these plots provide useful information about the collision, and we can use them to gain intuition about collisions, we may also learn from seeing the dynamics of the collision—how the objects move. This can be done by generating a simple animation using the `plot` command:

```
# Animate using plot
figure(3)
for i in range(0,n,50):
    plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r',
          [rA[i,0]],[rA[i,1]],'ob',[rB[i,0]],[rB[i,1]],'or')
    xlabel('x [m]')
    ylabel('y [m]')
    axis('equal')
```

However, you get a better impression from an animation that shows the extent of the colliding spheres. In addition, we need to scale the coordinate system so that we have room for not only the centers of the spheres, but also their whole extend, so that the plotting region does not move around during the animation, since this will make the animation difficult to interpret. We therefore find the maximum range of the objects positions and scale the axes accordingly:

```
# Animate by drawing
figure(4)
xmin = min(min(rA[:,0]-R),min(rB[:,0]-R))
xmax = max(max(rA[:,0]+R),max(rB[:,0]+R))
ymin = min(min(rA[:,1]-R),min(rB[:,1]-R))
ymax = max(max(rA[:,1]+R),max(rB[:,1]+R))
theta = linspace(0,2*pi,100)
xcirc = R*cos(theta)
ycirc = R*sin(theta)
for i in range(0,n,50):
    plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r',
         [rA[i,0]],[rA[i,1]],'ob',[rB[i,0]],[rB[i,1]],'or')
    hold('on')
    x = rA[i,0] + xcirc
    y = rA[i,1] + ycirc
    plot(x,y,'-b')
    x = rB[i,0] + xcirc
    y = rB[i,1] + ycirc
    plot(x,y,'-r')
    hold('off')
    axis('equal')
    xlabel('x [m]')
    ylabel('y [m]')
```

This program serves as the basis for the illustrations shown in this chapter and Fig. 12.18. You can now use this code to study various collision—central elastic collisions, non-central elastic collision, and inelastic collisions by introducing a finite value for η. Reasonable values for η are in the range $1\,\mathrm{kg/s} < \eta < 10\,\mathrm{kg/s}$.

Test your understanding: Based on this modeling framework, you are now ready to model other interactions. For example, you may introduce an ionic inteaction for the force between two ions: $\mathbf{F} = -C\,\Delta r^{-2}\,(\Delta\mathbf{r}/\Delta r)$. Try to implement and test this model using a similar approach as introduced here.

12.7 Rocket Equation

A rocket accelerates forward by ejecting exhaust backward at high velocity. The rocket is essentially throwing out mass backward in order to move forward. How can we address the motion of a rocket using Newton's second law?

We start from a specific example: A rain drop is falling down through the atmosphere and on the way it adsorbs water vapor that condensates on the drop. Let us address the motion of the drop over a small time interval from t to $t + \Delta t$. During this time interval the drop adsorbs a small drop of mass Δm and initial velocity $\mathbf{u}$. At the time t the mass of the drop is m and its velocity $\mathbf{v}$. At the time $t + \Delta t$, the mass is $m + \Delta m$ and the velocity has changed to $\mathbf{v} + \Delta\mathbf{v}$.

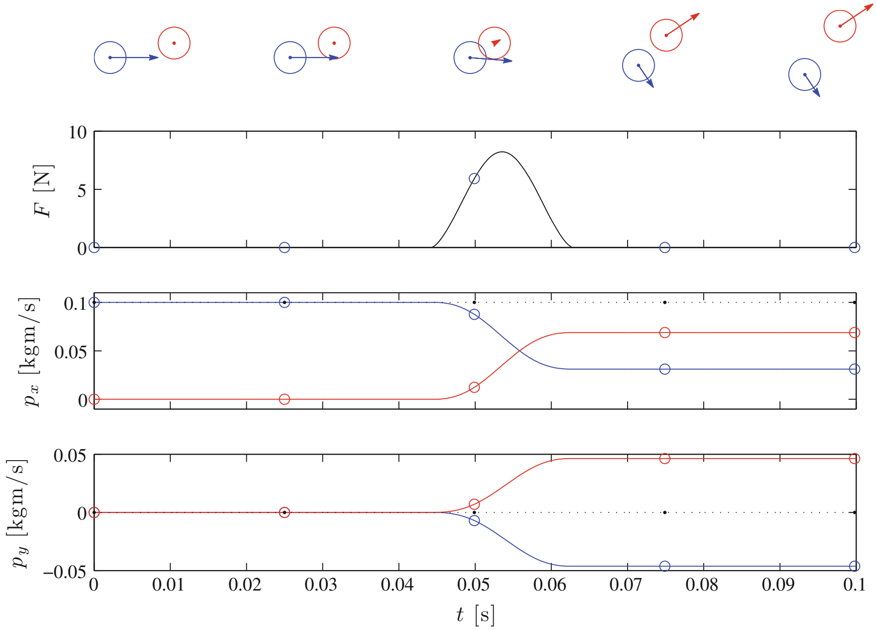

Fig. 12.18 Illustration of object trajectories and $p_x(t)$ and $p_y(t)$ throughout the collision

The change in momentum of the drop is related to the external forces acting on the drop. What is the change in momentum? At the time t, the total momentum (of the drop and the small drop being adsorbed) is:

$$\mathbf{p}(t) = m\mathbf{v} + \Delta m\mathbf{u} \tag{12.121}$$

after the time interval Δt, the small drop is adsorbed, and the new momentum is:

$$\mathbf{p}(t + \Delta t) = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) = m\mathbf{v} + m\Delta\mathbf{v} + \Delta m\mathbf{v} + \Delta m\Delta\mathbf{v} \ . \tag{12.122}$$

The change in momentum is therefore:

$$\Delta\mathbf{p} = \mathbf{p}(t + \Delta t) - \mathbf{p}(t) = m\Delta\mathbf{v} + (\mathbf{v} - \mathbf{u})\,\Delta m + \Delta m\Delta\mathbf{v} \ . \tag{12.123}$$

Newton's second law for the system related the change in momentum to the net forces acting on the system:

$$\sum \mathbf{F}^{\text{ext}} = \frac{\Delta\mathbf{p}}{\Delta t} = m\frac{\Delta\mathbf{v}}{\Delta t} + (\mathbf{v} - \mathbf{u})\,\frac{\Delta m}{\Delta t} + \Delta m\frac{\Delta\mathbf{v}}{\Delta t} \ . \tag{12.124}$$

This is only valid in the limit when $\Delta t \to 0$. If we assume the adsorption process also to be continuous, the change in mass and the change in velocity also goes to zero when Δt goes to zero. Hence, the term:

$$\Delta m \frac{\Delta \mathbf{v}}{\Delta t} \to 0 \text{ when } \Delta t \to 0 \; . \tag{12.125}$$

In the limit of small Δt we therefore find the **Rocket equation**:

Rocket equation:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\frac{dm}{dt} \; . \tag{12.126}$$

The rocket equation is used to describe the motion of an object that is expelling or adsorbing mass with a velocity $\mathbf{u}$. A special case is when the expelled or adsorbed mass has zero velocity, $\mathbf{u} = 0$:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{12.127}$$

which is what we get when taking the derivative of $\mathbf{p} = m\mathbf{v}$ with a time-dependent mass.

For a rocket, we may not generally know at what speed relative to the ground the exhaust is ejected, but rather at what speed relative to the rocket the mass is expunged. We introduce the velocity of Δm relative to m as:

$$\mathbf{v}_{\text{rel}} = \mathbf{u} - \mathbf{v} \; , \tag{12.128}$$

using this notation we can write the rocket equation (12.126) as:

Rocket equation:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}}\frac{dm}{dt} = m\frac{d\mathbf{v}}{dt} \; . \tag{12.129}$$

Written in this form, the equation looks like the usual formulation of Newton's second law, and we interpret all the terms on the left side as forces, and the term on the right side is the mass multiplied by the acceleration. Expelling mass with a velocity $\mathbf{v}_{\text{rel}}$ relative to the object at a rate dm/dt, has the same effect as pushing on the object with a force:

$$\mathbf{F} = \mathbf{v}_{\text{rel}}\frac{dm}{dt} \; . \tag{12.130}$$

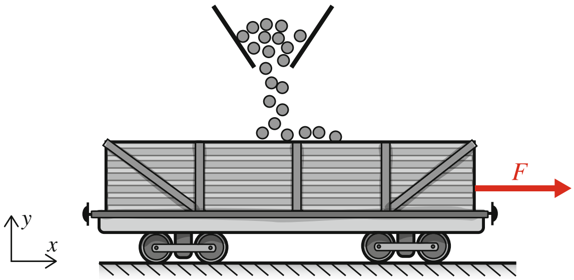

Fig. 12.19 A railway car is pulled with a force F so that it retains a constant velocity while sand is added from above at a constant rate

12.7.1 Example: Adding Mass to a Railway Car

Problem: A railway car is moving with constant velocity along a straight railway track. While it is moving, sand is dropped onto the car from above at a constant rate, $R = dm/dt$, as illustrated in Fig. 12.19. With how large force, F, must the car be pulled in order for the car to move with constant velocity?

Solution: We use the rocket equation to relate the forces acting on the car to its acceleration. The only horizontal force acting on the car is the external force F, friction and air resistance. We assume that friction and air resistance forces are negligible. The sand is falling onto the car from above. This means that the sand has a vertical velocity but no horizontal velocity as it hits the car. We can therefore use the rocket equation, with $\mathbf{u} = 0$:

$$\sum \mathbf{F}^{\text{ext}} = (\mathbf{v} - \mathbf{u}) \frac{dm}{dt} + m \frac{d\mathbf{v}}{dt} \ , \tag{12.131}$$

The net horizontal force on the car is $\mathbf{F} = F\,\mathbf{i}$, and the velocity is constant, $dv_x/dt = 0$, and the velocity $u_x = 0$ therefore:

$$F = (v_x - u_x) \frac{dm}{dt} + m \frac{dv}{dt} = (v_x - 0) \frac{dm}{dt} + m \times 0 = v \frac{dm}{dt} \ . \tag{12.132}$$

The force is therefore given by the velocity and the rate at which mass is added.

12.7.2 Example: Rocket with Diminishing Mass

Problem: A rocket is at rest in outer space, where the net force acting on the rocket is zero. In order to accelerate, it turns on its thrusters, which propels exhausting gases backward with a velocity v_{rel} relative to the spaceship. Find how the velocity increases as the fuel is used.

Solution: Let us describe the motion with the x-axis directed forward—in the direction of motion of the rocket. The rocket starts with the velocity $v_0 = 0$. We use the rocket equation to find an expression for the velocity of the rocket:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}} \frac{dm}{dt} = m \frac{d\mathbf{v}}{dt} \,, \tag{12.133}$$

Here, the net external force is zero, and all motion is in the x-direction:

$$-v_{\text{rel}} \frac{dm}{dt} = m \frac{dv}{dt} \;\Rightarrow\; \frac{dv}{dt} = -v_{\text{rel}} \frac{1}{m} \frac{dm}{dt} \,. \tag{12.134}$$

We find the velocity by integrating from t_0 to t_1:

$$\int_{t_0}^{t_1} \frac{dv}{dt} dt = -v_{\text{rel}} \int_{t_0}^{t_1} \frac{1}{m} \frac{dm}{dt} dt = -v_{\text{rel}} \int_{m(t_0)}^{m(t_1)} \frac{dm}{m} \,, \tag{12.135}$$

which gives

$$v(t_1) - v(t_0) = -v_{\text{rel}} \left(\ln m(t_1) - \ln m(t_0) \right) = -v_{\text{rel}} \ln \frac{m(t_1)}{m(t_0)} \,. \tag{12.136}$$

This is the increase in velocity of the rocket as the mass changes from $m(t_1)$ to $m(t_0)$. We notice that the velocity is increasing, since the final mass, $m(t_1)$ is always smaller than the initial mass $m(t_0)$.

Summary

Translational momentum: is defined as $\mathbf{p} = m\mathbf{v}$
Newton's second law: on general form is: $\sum \mathbf{F}^{\text{ext}} = d\mathbf{p}/dt$
Impulse: Change in momentum is related to the *impulse*, $\mathbf{J}$, of the net force:

$$\mathbf{J} = \int_{t_0}^{t_1} \mathbf{F}^{\text{ext}} dt = \mathbf{p}(t_1) - \mathbf{p}(t_0) \,.$$

The average net force: during a collision is:

$$\mathbf{F}^{\text{avg}} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{F} dt = \frac{\Delta \mathbf{p}}{t_1 - t_0} \,.$$

System of particles: For a system of particles we distinguish between:

- **internal forces** acting between particles in the system, and
- **external forces** acting between particles in the system and objects in the environment.

Newton's second law for a system of particles: is

$$\frac{d}{dt}\sum_j \mathbf{p}_j = \sum \mathbf{F}^{\text{ext}}$$

Isolated system: A system is isolated if the *net external force* is zero.
Conservaton of momentum: For an isolated system, the total momentum is conserved, $\sum_j \mathbf{p}_j =$ constant. Conservation of momentum is a *vector equation*, which can be applied for each direction independently of other directions.
Collisions: For a collision between two objects A and B, the total momentum is conserved if there are no external forces: $\mathbf{p}_{A,0} + \mathbf{p}_{B,0} = \mathbf{p}_{A,1} + \mathbf{p}_{B,1}$.
Elastic collisions: In an *elastic* collision, the kinetic energy is conserved.
Elastic collision in one dimension: The velocities of the objects after the collision are:

$$v_{A,1} = \frac{(m_A - m_B)\, v_{A,0} + 2m_B v_{B,0}}{m_A + m_B} \;,\; v_{B,1} = \frac{(m_B - m_A)\, v_{B,0} + 2m_A v_{A,0}}{m_A + m_B}$$

Perfectly inelastic collision: is a collision where the two objects have the same velocity after the collision: $\mathbf{v}_{A,1} = \mathbf{v}_{B,1}$
Perfectly inelastic collision in one dimension: The velocity after the collision is:

$$v_1 = v_{A,1} = v_{B,1} = \frac{m_A v_A + m_B v_B}{m_A + m_B} \;,$$

Inelastic collision: is a collision with energy loss, characterized by the *coefficient of restitution*, $r = -\left(v_{B,1} - v_{A,1}\right) / \left(v_{B,0} - v_{A,0}\right)$
Rocket equation: The motion of object that in a small time Δt is absorbing a mass Δm with a velocity $\mathbf{u}$, is given by the *rocket equation*:

$$\sum \mathbf{F}^{\text{ext}} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\,\frac{dm}{dt}$$

If the velocity of the absorbed/ejected material is $\mathbf{v}_{\text{rel}}$ relative to the object, the rocket equation can be written:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}}\frac{dm}{dt} = m\frac{d\mathbf{v}}{dt} \;.$$

Exercises

Discussion Questions

12.1 Golf ball. You hit a golf ball with a heavy golf club with a velocity v. What is the starting velocity of the golf ball?

12.2 Energetic collision. Can a collision between two objects result in zero total kinetic energy?

12.3 A collision paradox. A riddle: Two cars each with speed v_0 collide head on, getting stuck in the collision. If you observe the collision from the side of the road, the change in total kinetic energy is $\Delta K = 2 \cdot (1/2)mv_0^2$. If you instead observe the collision from a system moving with one of the cars, one car has velocity 0 and the other car has velocity $2v_0$. Then the change in total kinetic energy is $(1/2)m(2v_0^2) = 4 \cdot (1/2)mv_0^2$, which is double of what you observed from the side of the road. Is this argument correct? Explain.

Problems

12.4 A bike and a car. You and your bike has a mass of 100 kg.
(**a**) How fast would you have to ride in order to have the same momentum as a car of mass $m = 1200$ kg and a velocity of 50 km/h?

12.5 Kicking a ball. A football is lying at rest on the ground. You kick it. After the kick, it has a horizontal velocity of 20 m/s. You are in contact with the ball for 0.1 s. The mass of the ball is 0.43 kg.
(**a**) What is the change in the momentum of the ball?
(**b**) What is the impulse on the ball during the collision?
(**c**) What is the average force on the ball during the collision?

Assume that you are returning a ball coming toward you at 20 m/s. You kick the ball, staying in contact with the ball for 0.1 s, and return the ball with a velocity of 20 m/s
(**d**) What is the average force on the ball during the collision?

12.6 Stopping a car. During a collision at 60 km/h, a 1200 kg car stops in 0.2 s.
(**a**) What is the average force on the car during the collision?

A crash-test dummy of mass 80 kg are sitting in the car. Thanks to the seatbelt, he stops in 0.4 s.
(**b**) What is the average force on the dummy during the collision?

12.7 Ball reflected from wall. A ball of mass m hits a wall with a velocity $\mathbf{v}_0$ and bounces back. The ball hits the wall with an inclination, so that the velocity forms an

angle θ with the wall surface. When the ball leaves the wall after the collision, the magnitude of the velocity is the same, but its direction has changed. However, only the component of the velocity that is normal to the wall changes during the collision. Thus the velocity forms the same angle with the wall after the collision. The ball is in contact with the wall during a time interval Δt.
(**a**) What is the change in momentum of the ball?
(**b**) What is the impulse on the wall?
(**c**) What is the average force on the ball from the wall?
(**d**) For what angle θ is the average force on the ball largest?

12.8 Snowball on ice. You and your son is throwing snow balls at each other on a slippery (frictionless) frozen lake. Your mass is 80 kg and his mass is 20 kg, and you both start at rest.

You throw a big snow ball (2 kg) towards your son. The snow ball has an initial speed of 20 m/s and you throw it at an angle of 30° with the horizon.
(**a**) What is the momentum of the snow ball?
(**b**) What are you and your sons velocities after you have thrown the snow ball, but before the snow ball reaches him?
(**c**) Your son catches the snow ball. What are your and your sons velocities now?

12.9 Toppling a book. You try to topple a book standing on its shortes end by throwing a ball at it. You have two balls at your disposal. One elastic ball that collides elastically with the book, and one inelastic ball, that sticks to the book during the collision.
(**a**) Which ball should you choose? Explain your reasoning.

12.10 Bullet and a block. You fire a bullet of mass 100 g horizontally into a block of mass 2 kg where it gets stuck. The block lies on a frictionless table, but after the collision, the block enters a rough region of the table with a dynamic coefficient of friction, $\mu = 0.5$. After entering the rough region, the block slides a distance of 10 cm before stopping.
(**a**) What was the velocity of the bullet?
(**b**) What is the loss of energy during the collision?

12.11 Stopping a ball. A ball is hitting the floor with a vertical velocity v_0. The collision between the ball and floor is elastic.
(**a**) If you were able to move the floor up or down (using a vertical accelerator), is it possible to move the floor in a way so that the ball stops? Explain your answer.

12.12 Pendulum and block. A pendulum consisting of a sphere of mass m attached to the end of a massless rope of length L is initially lifted so that the rope is tight and horizontal, and released. At the bottom of its path, the sphere hits a block of mass $M = 2m$ lying at rest on a frictionless table. The collision is elastic.
(**a**) Find the velocities of the sphere and the block immediately after the collision.
(**b**) How high does the pendulum swing after the collision?

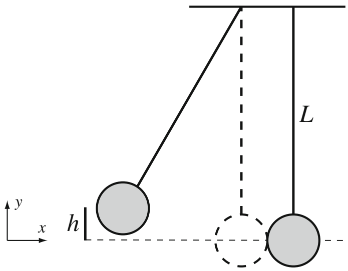

Fig. 12.20 Illustration of Newton's cradle with two balls

12.13 Lifting a chain. You are lifting a chain of mass m and length L from a heap on a table. Show that the force F needed to pull the chain up at a constant velocity v_0 is $F = (m/L)\left(v_0^2 + gy\right)$, where y is the length of the chain that has already been lifted from the table.

Projects

12.14 Newton's cradle. In this project you will learn about collisions and conservations laws by studying the behavior of Newton's cradle. Newton's cradle is a toy consisting of a series of steel balls each suspended by two strings so that the balls form a horizontal line when the cradle is at rest. The balls are initially barely touching each other. You can play with the toy by lifting and releasing a ball on one side. When the moving ball hits the stationary balls, a single ball is ejected on the other side, and the other balls remain stationary.

First, we study a cradle consiting of two balls of identical masses m hanging in thin strings as illustrated in Fig. 12.20. The left ball is lifted to a vertical height h_0 and released. The left ball hits the right ball when the string points directly down.
(**a**) Find the velocity v_0 of the left ball immediately before it hits the right ball.
(**b**) Assume the collision between the balls is elastic. Find the velocities v_1^A and v_1^B of the two balls after the collision. How does your result compare with the behavior of Newton's cradle described above?
(**c**) What is the maximum height, h_1, of the right ball?
(**d**) Assume the collision is perfectly inelastic. Find the maximum height h_1 reached by the right ball after the collision.

Assume the collision is characterized by a coefficient of restitution, r. The relative velocity after the collision is then related to the relative velocity before the colision by $v_1^B - v_1^A = r v_0$.
(**e**) Find the velocities of each of the balls after the collision.

We will in the following study a system with three balls, A, B, and C. We will assume that all forces are conservative, so that all collisions are elastic. Initially, immediately before the collision, ball A has a positive velocity v_0 and the other balls are not moving.

(**f**) Let us assume that the balls are seperated by small distances, so that there are two collisions, first between ball A and B and then between B and C. What are the velocities of the balls after the first collision? And after the second?

Let us now assume that all the balls are initially in contact, so that we cannot assume that there are two seperate, subsequent collision. This is the configuration corresponding to Newton's cradle.

(**g**) Find equations relating the initial and final velocities of all three balls. Can you solve these equations?

In order to understand what happens in Newton's cradle when all the balls are initially in contact, we will develop a simple, numerical model of the process. In the numerical model we will only address the collision itself, and we will assume that the motion of all the balls is one-dimensional along the x-axis during the collision.

We introduce an explicit model for the forces between the balls, and use this to calculate the motion of all the balls throughout the collision using Newton's second law for each of the balls.

The position of the balls are given as x_i, $i = 0, 1, 2$. At the beginning of the collision, at $t = 0$, all the balls are just in contact, so that the distance between them is equal to their diameters, d, $x_i = i\,d$.

The force on ball i from ball $i + 1$ is modelled using a simple, position-dependent force on the form

$$F_{i,i+1} = \begin{cases} -k\,|x_{i+1} - x_i - d|^q & \text{when } x_{i+1} - x_i < d \\ 0 & \text{when } x_{i+1} - x_i \geq d \end{cases} . \tag{12.137}$$

The following program solves the equations of motion from a time $t = 0$ to a time $t = t_1$. You must choose the mass, m, the constant k, and initial conditions for the simulation yourself.

```
from pylab import *
def force(dx,d,k,q): # force function
    if dx<d:
        F = k*abs(dx-d)**q
    else:
        F = 0.0
    return F
N = 2    # nr of balls, <-- Modify from here
m =  ... # kg
k =  ... # N/m
q = 1.0
d =  ... # m
v0 = ... # m/s
time = . # s
dt = ... # s,          <-- to here
n = int(round(time/dt))
x = zeros((n,N),float)
v = x.copy()
t = zeros(n,float)
for i in range(N): # Initial conditions
    x[0,i] = d*i
v[0,0] = v0
for i in range(n-1):
    F = zeros(N,float)
    for j in range(1,N):
        dx = x[i,j] - x[i,j-1]
```

```
        F[j] = F[j] + force(dx,d,k,q)
    for j in range(N-1):
        dx = x[i,j+1] - x[i,j]
        F[j] = F[j] - force(dx,d,k,q)
    a = F/m
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
for j in range(N):
    plot(t,v[:,j]), hold('on')
print 'v/v0 = ',v[n-1,:]/v0
```

(**h**) Test the program and your parameters by direct comparison with your results above for $N = 2$, where N is the number of balls. Your answer to this and the following questions should include plots of the velocities. Hint: You must ensure that the timestep dt is chosen reasonably compared to the values of k and m.
(**i**) Use the program to determine the result of a collision when $N = 3$. What are the velocities of the balls immediately after the collision? Is this result physically reasonable? Does this correspond to the behavior you expect for Newton's cradle?
(**j**) Modify the force law by changing k and q. Can you find parameters that produce a behavior close to what you observe in Newton's cradle, that is, for which the velocity of the middle ball is close to zero after the collision?
(**k**) Can you now explain why only one ball is ejected from the left side when one ball is released from the right side in the toy cradles you can buy?

12.15 Catching an atom. In this project we will study a collision between two identical atoms of mass m that both are affected by forces from a massive particle such as a molecule.

First, we study the behavior of a single atom affected by a force from the molecule. The potential energy for the interaction between the atom and the molecule is:

$$U(x) = \begin{cases} \infty & \text{when} \quad x < b - d \\ \frac{1}{2}k\,(x-b)^2 & \text{when} \quad b - d < x < b + d \\ U_0 & \text{when} \quad x > b + d \end{cases} , \tag{12.138}$$

where b and d are lengths and $d < b$,

$$U_0 = \frac{1}{2}kd^2 , \tag{12.139}$$

and x is the position of the atom. We assume that the molecule is stationary at the point $x = 0$. (The atom cannot enter the region where the potential is infinite. You may instead assume that the energy is very large, $U_1 \gg U_0$, if you find this easier to discuss).
(**a**) Sketch the potential. Draw in an example of the motion of the atom when the total energy is less than U_0, and a motion where the total energy is larger than U_0, and describe the motions briefly.
(**b**) Find the force $F(x)$ on the atom as a function of x.

We will now study a collision between an atom (B) of mass m which start from rest at the point $x_B = b$, and an identical atom (A) which starts at $x_A > b + d$ with

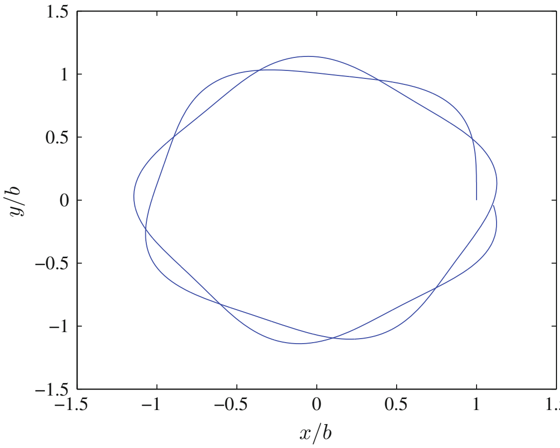

Fig. 12.21 Sketch of simulated motion

an initial velocity $-v_{A,0}$. For each of the atoms, the interaction with the molecule can be described by the potential energy $U(x)$, so that the potential energy for atom A is $U(x_A)$ and the potential energy for atom B is $U(x_B)$. There are no long-range interactions between the atoms. They only interact when they are in the same point, $x_A = x_B$. In that case they collide. After the collision they become attached to each other. You can assume that the atoms have not moved significantly during the collision.

(**c**) Find the velocity of atom A in the point $x_A = b$ immediately before the collision.
(**d**) Find the velocities of atom A and atom B immediately after the collision.
(**e**) How large must $v_{A,0}$ be in order for atom B (and atom A) to detach itself from the molecule after the collision? (An atom is detached if it can move infinitely far away from the molecule).

We will now study the same process, but in two dimensions. The massive molecule is now at rest at the origin, and the potential energy of an atom has the same form as above, but is now a function of the distance $r = \sqrt{x^2 + y^2}$ to the origin (Fig. 12.21):

$$U(r) = \begin{cases} \infty & \text{when} \quad r < b - d \\ \frac{1}{2}k\,(r-b)^2 & \text{when} \quad b - d < r < b + d \\ U_0 & \text{when} \quad r > b + d \end{cases}, \tag{12.140}$$

(**f**) Show that the force on the atom can be written as $\mathbf{F}(\mathbf{r}) = -k(r-b)\frac{\mathbf{r}}{r}$ when $b - d < r < b + d$.
(**g**) The atom starts with velocity $\mathbf{v}_0$ in the position $\mathbf{r}_0$ at the time $t_0 = 0$. Write a program to find the position of the atom as a function of time. Plot the trajectory of the atom.
(**h**) We use the program to simulate the motion of the atom when $\mathbf{r}_0 = (b, 0)$ and $\mathbf{v}_0 = (0, v_0)$. The result is shown in the figure below. Explain the results. How would you measure the period of this motion in your program?

(**i**) You want the atom to follow a circular orbit around the molecule with a constant speed v. How do you have to choose the initial conditions to obtain such an orbit? Can you get a circular orbit for all speeds v? Explain your answer.
(**j**) How would you need to modify your program to model the motion of atom A before and after the collision. (After the collision atom A and atom B moves as one point particle).
(**k**) Atom A starts with velocity $\mathbf{v}_0$ in the position $\mathbf{r}_0$ at the time $t = t_0$ and collides with atom B at the time t_1. Is it possible to get atom B (and atom A) to detach from the molecule afte the collision?
(**l**) How do you have to choose $\mathbf{v}_0$ and $\mathbf{r}_0$ to make atom B follow a circular orbit after the collision.

Chapter 13
Multiparticle Systems

So far we have studied the motion of objects, but we have not been very precise in defining an object. What we really have studied is the motion of mass-points, particles, or extended objects that move as a particle. What does it mean to move as a particle? It means that all the points in the object move with the same velocity and the same acceleration, the whole object is translated as a stiff (rigid) body: The various parts of the object are not moving relative to each other.

But this does not seem to be a good description of everyday phenomena around us. If you are running, you are surely not moving both your arms and your legs with the same velocity and acceleration all the time. And if you throw a ball, it may wobble and spin on its path. Even on the microscopic level objects are not only translated: Molecules may vibrate, spin, or wobble as they move. When the world is so complex, can we still use the simple descriptions and models we have developed so far, or do we have to describe the motion of each small part of the object independently?

Fortunately, we are saved by Newton's third law: If we only define the position of the object in a particular way, by defining the position of the object as its center of mass, Newton's second law is valid for any system of particles, and therefore for any object. This wonderful consequence of Newton's second and third laws allows us to use the force models and the concepts we have developed so far also to address extended objects. The motion of an object is further simplified if the object is a rigid body: An object where the relative positions of any two points do not change, that is, the object is only translated and rotated, but does not change shape, stretch, or otherwise deform during its motion.

In this chapter, we discuss the motion of systems of particles—multiparticle systems. Before we proceed to describe the motion of rigid bodies in Chaps. 15 and 16, we first introduce a general description of rotational motion in Chap. 14.

13.1 Motion of a Multiparticle System

Let us start describing a system of many particles. For most practical purposes, two particles are many particles, but we are ambitious and start with a system of N particles. We number the particles using the index j running from 1 to N as

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Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_13

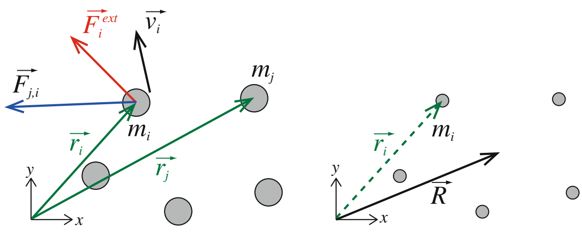

Fig. 13.1 The motion of a system of particles is described by the position $\mathbf{r}_j$ of each of the N particles in the system. Here, the system consists of 5 particles

illustrated in Fig. 13.1. The position of each particle is given in a coordinate system S as illustrated. The position of particle 1 as a function of time is $\mathbf{r}_1(t)$, and its mass is m_1. Similarly, the position of particle j is $\mathbf{r}_j$ and its mass is m_j.

All the quantities we have defined so far are easily extended to each of these particles. For example, the velocity and acceleration of particle j is found from the derivatives of the position vector:

$$\mathbf{v}_j = \frac{d\mathbf{r}_j}{dt} \;, \; \mathbf{a}_j = \frac{d\mathbf{v}_j}{dt} \;, \tag{13.1}$$

and the momentum of particle j is $\mathbf{p}_j = m_j\mathbf{v}_j$.

The motion of each particle is determined from the forces acting on it. In our discussion of momentum, we already discussed that the forces acting on particle j may be either from one of the other particles in the system, or from the environment. Forces from other particles in the system are called *internal forces*, and forces from the environment are called *external forces*. Newton's second law for particle i is therefore:

$$\mathbf{F}_i^{\text{net}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} = \frac{d}{dt}\mathbf{p}_i \;, \tag{13.2}$$

where $\mathbf{F}_{j,i}$ is the force from particle j on particle i.

In order to find the motion of one of the particles inside the system, we need to know both the external forces acting on this particle and the forces from the other particles in a system. This may require a complicated force model, for example, consider the forces acting between two parts of a tennis ball as the ball wiggles and rotates. Often, we are not interested in the detailed motion of particles inside a system, but only in the motion of the system as a whole. We can address this by adding together (13.2) for each particle i, getting:

$$\sum_i \mathbf{F}_i^{\text{net}} = \sum_i \mathbf{F}_i^{\text{ext}} + \underbrace{\sum_i \sum_{j \neq i} \mathbf{F}_{j,i}}_{=0} = \sum_i \frac{d}{dt}\mathbf{p}_i \;, \tag{13.3}$$

As we noticed when we discussed the total momentum of a system of particles, the sum of all the internal forces will always contain both the force $\mathbf{F}_{j,i}$ and the force $\mathbf{F}_{i,j}$. Since these two forces are action-reaction pairs, they are equal, but oppositely directed. Therefore, every such pair will cancel. The sum of all the internal forces is therefore zero!

If we introduce $\mathbf{P} = \sum_i \mathbf{p}_i$ as the total momentum of the system, we find:

$$\sum_i \mathbf{F}_i^{\text{ext}} = \frac{d}{dt}\mathbf{P} \ , \tag{13.4}$$

This looks a lot like Newton's second law, but now for the system of particles. We can make this similarity even stronger by introducing the velocity $\mathbf{V}$ of the system so that:

$$\boxed{\mathbf{P} = M\mathbf{V} = \sum_i \mathbf{p}_i = \sum_i m_i \mathbf{v}_i \ ,} \tag{13.5}$$

where $M = \sum_i m_i$ is the total mass of the system.

Notice that this is a *definition* of the velocity $\mathbf{V}$ of the system. We define it this way to be able to write the total momentum, $\mathbf{P}$, of the system in the intuitive way $\mathbf{P} = M\mathbf{V}$. From (13.5) we find:

$$\boxed{\mathbf{V} = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i} = \frac{1}{M}\sum_i m_i \mathbf{v}_i \ .} \tag{13.6}$$

What is the *acceleration* of the system? It is natural to define the acceleration, $\mathbf{A}$, of the system as the time derivative of the velocity $\mathbf{V}$ of the system:

$$\boxed{\mathbf{A} = \frac{d}{dt}\mathbf{V} = \frac{1}{M}\sum_i m_i \mathbf{a}_i \ .} \tag{13.7}$$

Similarly, we define the position of the system, $\mathbf{R}$, as:

Center of mass: The effective position of the system, or the *center of mass* of the system, is defined as

$$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i \ , \tag{13.8}$$

The velocity and acceleration of the center of mass are defined in the usual way:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \ , \ \mathbf{A} = \frac{d^2\mathbf{R}}{dt^2} \ . \tag{13.9}$$

With these definitions:

Newton's second law for a system of particles: is system as:

$$\sum_i \mathbf{F}_i^{\text{ext}} = \frac{d}{dt}\mathbf{P} = M\mathbf{A} \ . \tag{13.10}$$

where $M = \sum_i m_i$ and $\mathbf{P} = \sum \mathbf{p}_i$.

(where we have assumed that the masses of the particles are constant).

This law, *Newton's second law for a system of particles*, is what we have been looking for. Equation (13.10) shows that if we define the position of the system in this particular way, we can use Newton's law exactly as we are used to, just remembering that we are not describing the motion of each particle separately, but instead we describe the motion of the effective position $\mathbf{R}$ of the system.

This law is powerful and surprisingly beautiful. It is the theoretical justification for why we do not have to care too much about whether we describe the motion of an object as a point or as a system of particles. We can describe the motion of any system of particles as a point: The center of mass $\mathbf{R}$ of the system.

In the next sections we build our intuition about the center of mass $\mathbf{R}$ of a system and we learn to apply Newton's second law for a system of particles.

13.2 The Center of Mass

Why do we call the effective position $\mathbf{R}$ the center of mass? Let us start by address this in a two-particle system. The effective position, $\mathbf{R}$, of the two-particle system is:

$$\mathbf{R} = \frac{1}{M}\left(m_1\mathbf{r}_1 + m_2\mathbf{r}_2\right) \ . \tag{13.11}$$

If the two masses are identical, we see that:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \frac{m\mathbf{r}_1 + m\mathbf{r}_2}{m + m} = \frac{1}{2}\left(\mathbf{r}_1 + \mathbf{r}_2\right) \ , \tag{13.12}$$

which is the midpoint between the two points.

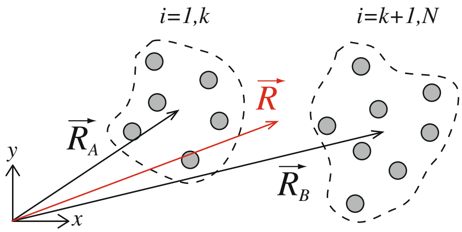

Fig. 13.2 A system consists of two objects A and B. We find the center of mass of the whole system

We get a similar result for the N-particle system: When all the masses are the same, the effective position $\mathbf{R}$ is:

$$\mathbf{R} = \frac{1}{N} \sum_i \mathbf{r}_i \ , \tag{13.13}$$

as illustrated in Fig. 13.1. This is simply the arithmetic mean of the position vectors.

As long as all the masses are the same, the center of mass is what we typically would call the geometric center of the points. What happens when the masses are not equal? In this case, we weigh in the masses in the average, so that the center of mass is the mass-weighed average of the positions of the particles-which is the natural definition of the center of mass of an object.

Notice that the center of mass $\mathbf{R}$ is a **vector**. We can therefore calculate the center of mass for each component, along each axis, independently of the other axes.

The Subdivision Principle

If we combine two systems A and B, where we know the center of mass for each these systems, how can we find the center of mass for the whole system?

The situation is illustrated in Fig. 13.2. Each of the systems could be a rigid body, or just a collection of point masses. System A has total mass M_A and a center of mass at $\mathbf{R}_A$, and system B has a total mass M_B and center of mass at $\mathbf{R}_B$.

Let us enumerate all the particles using the index i. The first k particles are in system A, and the last $N - k$ particles are in system B. What is the center of mass, $\mathbf{R}$, of the whole system with mass $M = M_A + M_B$?

The definition of $\mathbf{R}$ is:

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i = \sum_{i=1}^{k} m_i \mathbf{r}_i + \sum_{i=k+1}^{N} m_i \mathbf{r}_i = M_A \mathbf{R}_A + M_B \mathbf{R}_B \ , \tag{13.14}$$

and therefore:

$$\mathbf{R} = \frac{M_A \mathbf{R}_A + M_B \mathbf{R}_B}{M_A + M_B} \ . \tag{13.15}$$

We can therefore find the center of mass of a system of two objects A and B, by assuming that object A and B are point masses with the whole mass of each object located in the center of mass of each object. This means that if we want to find the center of mass of two solid bodies, we can find the center of mass of each object, and then find the center of mass of the combined object by assuming each object to be a point mass.

Solid Bodies

So far we have only defined the center of mass of a system of a finite number of particles. How can we find the center of mass of a solid body?

The definition of the center of mass for a continuous object follows directly from the definition for a system of many particles. We divide the solid body into small volumes $\Delta V_i = \Delta x_i \; \Delta y_i \; \Delta z_i$ at the position $\mathbf{r}_i$. The center of mass is:

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i \; , \tag{13.16}$$

where the mass of the volume element ΔV_i depends on the local mass density, $\rho(\mathbf{r}_i)$. When the size of the volumes goes to zero, the sum approaches the integral of the mass density of the volume, V, of the solid body:

$$\mathbf{R} = \frac{1}{M} \iiint_V \mathbf{r} \, \rho(\mathbf{r}) \, dV \; . \tag{13.17}$$

In physics, we often write this as an integral over the mass elements dm instead:

$$\mathbf{R} = \frac{1}{M} \int_M \mathbf{r} \, dm \; . \tag{13.18}$$

In order to calculate the integral, we calculate the values for each of the components separately:

$$MX = \iiint_V x \, \rho(x, y, z) \, dx \, dy \, dz \; , \tag{13.19}$$

and similarly for the Y and Z components.

13.2.1 Example: Points on a Line

Problem: Find the center of mass $\mathbf{R}$ for the system of three particles illustrated in Fig. 13.3.

Solution: We want to determine:

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \; . \tag{13.20}$$

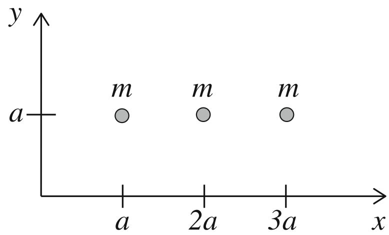

Fig. 13.3 A system of three particles with identical masses, m

Since all the masses are equal, $m_i = m$. We find the x- and y-components independently:

$$X = \frac{1}{3}\sum_i x_i = \frac{1}{3}(a + 2a + 3a) = \frac{1}{3}6a = 2a \ , \tag{13.21}$$

$$Y = \frac{1}{3}\sum_i y_i = \frac{1}{3}(a + a + a) = \frac{1}{3}3a = a \tag{13.22}$$

The center of mass is therefore:

$$\mathbf{R} = 3a\ \mathbf{i} + a\ \mathbf{j} \ . \tag{13.23}$$

13.2.2 Example: Center of Mass of Object with Hole

Problem: Find the center of mass of a homogeneous disk with radius R, with a circular hole of radius r touching the outer edge of the disk, as illustrated in Fig. 13.4.

Solution: This examples demonstrates that the subdivision principle also can be used in reverse to remove a part of an object, such as a circular hole in a circular disk. We

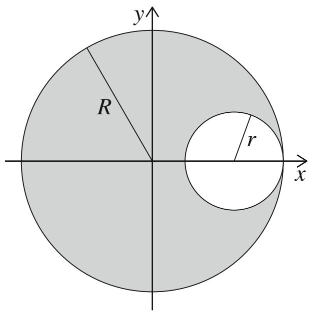

Fig. 13.4 Illustration of the circular disk of radius R with a circular hole of radius r

start from a homogeneous disk, object AB, and remove a smaller circular portion, object B, and is left with a disk with a hole, object A.

The mass of the complete disk is $M_{AB} = \pi R^2 \rho$, where ρ is the mass (area) density, and the mass of the small disk is $M_B = \pi r^2 \rho$. The subdivision principle states that the center of mass of the whole disk (object AB), which is at the origin, $\mathbf{R} = 0$, can be written as:

$$M_{AB} \underbrace{\mathbf{R}}_{=0} = M_A \mathbf{R}_A + M_B \mathbf{R}_B \ . \tag{13.24}$$

We solve this equation to find $\mathbf{R}_A$, the unknown center of mass for object A.

$$\mathbf{R}_A = -\frac{M_B}{M_A}\mathbf{R}_B = -\frac{\pi r^2 \rho}{\pi R^2 \rho}(R - r)\,\mathbf{i} = -\frac{r^2}{R^2}(R - r)\,\mathbf{i} \ , \tag{13.25}$$

Notice the simplicity of this approach. We did not have to perform any integration. This use of symmetries is a characteristic of physics that you will meet many times during your career.

13.2.3 Example: Center of Mass by Integration

Problem: Find the center of mass of a thin, homogeneous triangular plate with sides of length a and b, as illustrated in Fig. 13.5. (You must be able to solve double-integrals to understand this example).

Solution: The center of mass for a continous, homogeneous object is defined as:

$$M\mathbf{R} = \int_m \mathbf{r}\, dm \ , \tag{13.26}$$

where we have written the integral over the mass. Instead, we may integrate over space and use the mass (area) density, ρ:

$$M\mathbf{R} = \iint_A \mathbf{r}\, \rho\, dA \ . \tag{13.27}$$

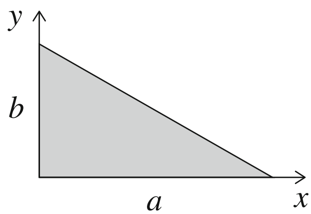

Fig. 13.5 Illustration of a homogeneous *triangle* with sides of length a and b

We need to find both the mass, M, and the position $\mathbf{R}$. Both are found by integration over the area A, which is the area of the triangle. First, we find the mass by integrating over the area A. We integrate x from 0 to a, and y from 0 and up to the line corresponding to the upper boundary of the triangle. This is a line going through the points $x = 0$, $y = b$ and $x = a$, $y = 0$. The straight line through these points has the equation $y = b(1 - x/a)$.

$$M = \iint_A \rho dA = \rho \int_0^a \int_0^{b(1-x/a)} dy\, dx = \rho \int_0^a b(1 - x/a)\, dx = \frac{1}{2}\rho ab \ , \tag{13.28}$$

which, of course, is the well know formula for the area of a triangle multiplied with the mass density ρ of the triangle.

Now, we find the position of the center of mass by calculating the integral for $M\mathbf{R}$ for each component:

$$\begin{aligned} MX &= \iint_A x\rho dA = \rho \int_0^a \int_0^{b(1-x/a)} x\, dy\, dx = \rho \int_0^a xb(1 - x/a)\, dx \\ &= \rho\left(\frac{1}{2}a^2 b - \frac{1}{3}a^3 b/a\right) = \rho\left(\frac{1}{2}a^2 b - \frac{1}{3}a^2 b\right) = \rho\frac{1}{6}a^2 b \ , \end{aligned} \tag{13.29}$$

The center of mass in the x-direction is therefore:

$$X = \frac{MX}{M} = \frac{\rho\,(1/6)\,a^2 b}{\rho\,(1/2)\,ab} = \frac{1}{3}a \ . \tag{13.30}$$

We use the same method in the y-direction:

$$\begin{aligned} MY &= \iint_A y\rho dA = \rho \int_0^a \int_0^{b(1-x/a)} y\, dy\, dx = \rho \int_0^a \frac{1}{2}\left(b(1 - x/a)\right)^2\, dx \\ &= \rho b^2 \frac{1}{2} \int_1^0 u^2(-1/a)\, du = \rho b^2 \frac{1}{2}\frac{1}{3} b^2 a \ , \end{aligned} \tag{13.31}$$

which gives

$$Y = \frac{MY}{M} = \frac{\rho\,(1/6)\,b^2 a}{\rho\,(1/2)\,ab} = \frac{1}{3}b \ . \tag{13.32}$$

The center of mass is therefore:

$$\mathbf{R} = \frac{1}{3}a\,\mathbf{i} + \frac{1}{3}b\,\mathbf{j} \ . \tag{13.33}$$

13.2.4 Example: Center of Mass from Image Analysis

The center of mass is often used to describe the center of an object in an image . It may be because we are taking pictures of an object we want to track, such as the wandering behavior of a small grain of dust dancing through the air or the motion of a asteroid seen on the sky, or it may be to determine the center of mass of an irregularly shaped object.

How can we find the center of mass from an image? First, we need to read the image so that we can access it. The image is taken from a classroom experiment, where we have extracted a smaller part of the image for analysis (see Fig. 7.4). We read the image `ballimage02.png` using:

Let us immediately display it to see if we got the right image:

```
subplot(1,2,1);
imshow(z)
axis('equal')
show()
```

where the `axis` commands are to clean up the plotted image. Notice that Python uses position `[0,0]` for the upper left part of the image, and that the first coordinate is the vertical coordinate and the second coordinate is the horizontal coordinate, so that `[iy,ix]` is position `[ix,iy]` in the image. We call each (x, y) position for a pixel . We find the size of the image using `size`:

```
>> shape(z2)
```

```
(411,559)
```

The image is stored as the matrix $z(y, x, j)$ which contain values of red (R, $j = 1$), green (G, $j = 2$), and blue (B, $j = 3$) . However, we cannot use these color values directly to find the center of mass. Instead, we need to know if a pixel at (x, y) is a part of the object or not. We therefore set a threshold on the image, so that all pixels that are brighter than this threshold is included (set to value 1), and all the rest of the pixels are set to zero (Fig. 13.6):

```
z2 = (z[:,:,0]+z[:,:,1]+z[:,:,2])>1.5
subplot(1,2,2)
imshow(z2)
axis('equal')
```

The resulting images as shown in Fig. 13.7. The left image is the original image and the left image shows the filtered image, where all the pixels that are part of the ball are colored red.

Now, we are ready to find the center of mass:

$$X = \frac{1}{M}\sum_i x_i \ , \ Y = \frac{1}{M}\sum_i y_i \tag{13.34}$$

These formulas can be directly converted into an algorithm: For each pixel i, if the pixel is a part of the object, that is if $z(x_i, y_i) = 1$, we include the positions x_i and y_i in the sum for the center of mass and include the pixel in the sum for the mass.

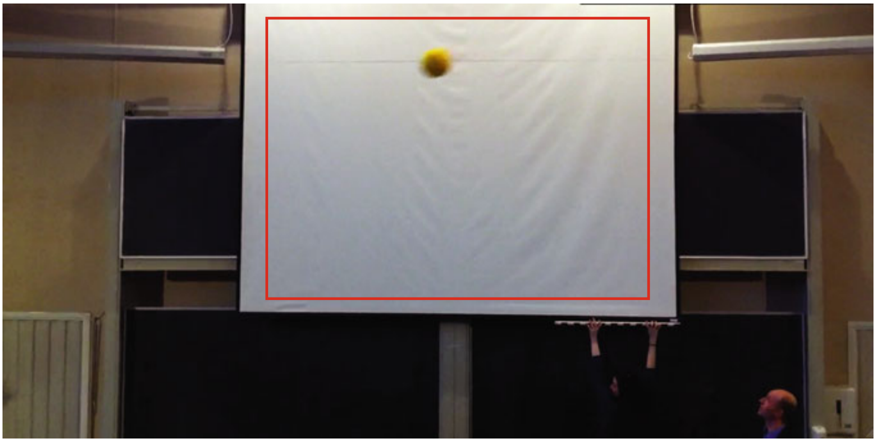

Fig. 13.6 Image from video from classroom demonstration. The inset shows the image used for analysis

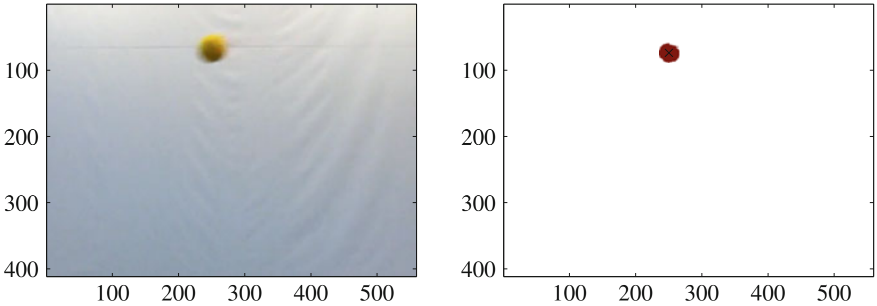

Fig. 13.7 *Left* Image of ball (cut). *Right* Filtered image of ball

```
s = shape(z2)
x = 0
y = 0
m = 0
for iy in range(s[0]):
    for ix in range(s[1]):
        if (z2[iy,ix]==1):
            x = x + ix
            y = y + iy
            m = m + 1
xcm = x/m;
ycm = y/m;
hold('on')
plot(xcm,ycm,'kx');
hold('off')
```

where we also plot the center of mass as an "x". The last three lines ensure that empty pixels, pixels where $z(x, y) = 0$, are shown as white. (We set all three R,G,B values to 1 to generate a white entry in the colormap).

This method is used for motion tracking of an image. If we are able to automatically filter the image so that we only get the object of interest, we can use this method to find the center of mass of the object for each frame in a movie and thereby find the center of mass as a function of time. Usually this requires careful positioning of the camera and a good choice of background for the filming.

13.3 Newton's Second Law for Particle Systems

We have found that if we measure the position of a system of particles using the center of mass, **R**, of the system, the system behaves according to Newton's second law:

$$\sum \mathbf{F}^{\text{ext}} = M\mathbf{A} , \tag{13.35}$$

where **A** is the acceleration of the center of mass of the system of particles, and the sum is over all external forces. This is true for any system of particles, from a galaxy consiting of starts, to the solar system, to a rigid body consiting of a large number of invididual atoms, down to a molecule or even an atom: The acceleration of the center of mass is given by the external forces acting on the system.

It is this law that allows us to use the techniques we have developed so far on any system, a solid body or a system of particles. In the previous chapters we have strictly speaking only discussed the motion of point-particles with a mass. We have always assumed that every part of a solid body has been moving with the same velocity. We have not allowed the object to oscillate, vibrate, change shape, or rotate. We have not allowed it to do any of the things that real objects do. However, we have now been saved by Newton's second law for particle systems: If we measure the position of an object as the center of mass of the object, we can still use Newton's second law to find its motion, even if the object is vibrating, oscillating, rotating, or displaying other types of internal motion.

If I throw a ball through the room, we have previously found that the motion of the ball can be found from Newton's second law for the ball:

$$\sum \mathbf{F} = \mathbf{G} = -mg\,\mathbf{j} = m\mathbf{a} . \tag{13.36}$$

The beauty of Newton's second law for particle systems is that we can use exactly the same analysis for a spinning or oscillating ball. The motion of the center of mass of the ball only depends on the external forces acting on the ball:

$$\sum \mathbf{F} = \mathbf{G} = -mg\,\mathbf{j} = m\mathbf{A} . \tag{13.37}$$

It does not matter what happens internally in the ball—if it is deformed, spinning, or vibrating—the motion of the center of the mass is the same as for a point particle as long as the external forces acting are the same.

However, you might argue that the external forces acting on a spinning ball are different because of air resistance: It is the interaction with the air that causes a spinning ball to move sideways, which is called curving the ball. This is a valid objection. The motion of the center of mass is determined by the external forces acting on the object. But, if we can neglect the effects of air resistance, the motion of the center of mass of a rod when you throw it is the same when it is rotating as when it is not rotating. This may be surprising, but it is a result of Newton's second law for particle systems.

13.3.1 Example: Ballistic Motion with an Explosion

Problem: A projectile is fired from the ground. Its initial velocity in the horizontal direction is v_0. When it reaches its maximum height of h, a charge is set off, splitting the projectile into two equal parts. One part moves forward with the velocity v_1. Find the trajectory of each of the parts, and the trajectory of their center of mass. You may neglect air resistance.

Solution: We have illustrated the process in Fig. 13.8. The process has three stages. In stage one, the projectile moves to its maximum height only under the influence of gravity. In the second stage, the explosion takes place, after which the projectile is split into two projectiles. In the third stage, each part propagates to the ground only affected by gravity.

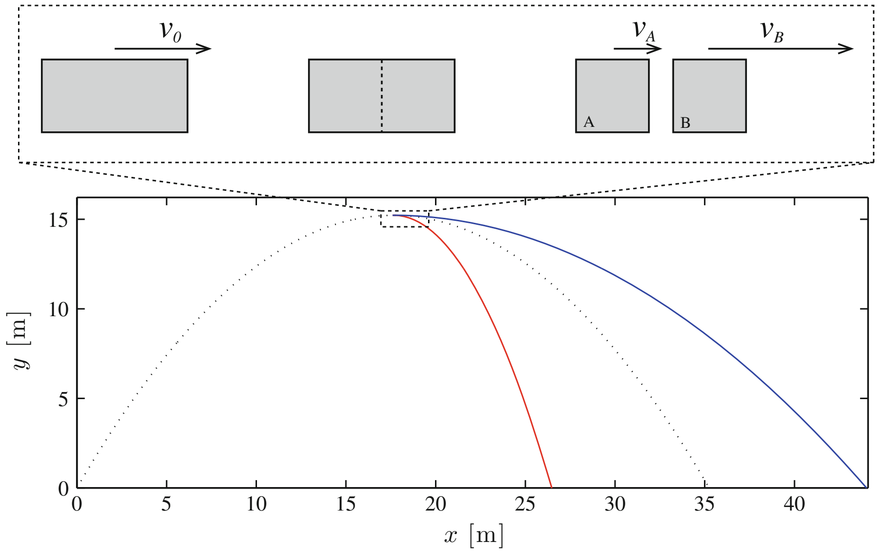

Fig. 13.8 A projectile explodes at the top of its path, splitting into two equal pieces. We track the position of each part until they hit the ground

Model: First, let us address the explosion. During the explosion, each part of the projectile is subject to large forces. But the only external force acting on the parts is gravity. Consequently, there is no external horizontal force acting on the system. The horizontal momentum is therefore conserved at all times. In particular, the momentum is the same immediately before and immediately after the explosion. We use conservation of momentum to determine the horizontal velocities of the parts after the explosion.

Before the explosion, the horisontal momentum of the system is:

$$p_0 = m v_0 \ , \tag{13.38}$$

and after the collision, the horizontal momentum is:

$$p_1 = m_A v_A + m_B v_B = \frac{m}{2} v_A + \frac{m}{2} v_B \ . \tag{13.39}$$

Conservation of momentum gives:

$$v_0 = \frac{1}{2} v_A + \frac{1}{2} v_B \ , \tag{13.40}$$

where $v_B = v_1$ is the velocity of part B after the explosion, and v_A is the velocity of part A, which we need to find:

$$v_A = 2v_0 - v_B = 2v_0 - v_1 \ . \tag{13.41}$$

We therefore know the initial conditions for the motion in the third stage. Each part is affected by gravity alone: $\mathbf{G}_A = -m_A g\,\mathbf{j}$, $\mathbf{G}_B = -m_B g\,\mathbf{j}$.

Finding the motion of part B: First, we find the motion of part B. Newton's second law in the x-direction gives:

$$\sum F_x = 0 = m_B a_B \ . \tag{13.42}$$

The velocity in the x-direction is constant. The position is therefore:

$$x_B(t) = x_B(t_0) + v_B t = v_0 t \ , \tag{13.43}$$

where we have placed the origin at the ground directly below the explosion. Therefore $x_B(t_0) = 0$.

The motion in the y-direction corresponds to the motion of a falling object, hence:

$$y_B(t) = h - \frac{1}{2} g t^2 \ . \tag{13.44}$$

Finding the motion of part A: Similarly, we find the position of part A:

$$x_A(t) = v_A t = (2v_0 - v_1)\,t \ , \ \ y_A(t) = h - \frac{1}{2}gt^2 \ . \tag{13.45}$$

We notice that the motion in the y-direction is the same for the two parts, which is as expected. The two parts will therefore strike the ground at the same time.

Finding the motion of the center of mass: We use these results to find the center of mass:

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \ . \tag{13.46}$$

We find the x- and y-components independently:

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\frac{m}{2}x_A + \frac{m}{2}x_B}{\frac{m}{2} + \frac{m}{2}} = \frac{1}{2}\left(x_A + x_B\right) = \frac{1}{2}\left((2v_0 - v_1)\,t + v_1 t\right) = v_0 t \ . \tag{13.47}$$

This is what we get if we apply Newton's second law for a particle system. There are no external horizontal forces acting on the system, therefore the horizontal component of the center of mass moves with constant velocity:

$$\sum \mathbf{F}^{\text{ext}} = -mg\,\mathbf{j} = m\mathbf{A} \ \Rightarrow \ A_x = 0 \ \Rightarrow \ X = v_0 t \ . \tag{13.48}$$

We see that Newton's second law for particle systems gives the same result as when Newton's second law is applied to each object.

Similarly, we find the y-position of the center of mass:

$$Y = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{\frac{m}{2}y_A + \frac{m}{2}y_B}{\frac{m}{2} + \frac{m}{2}} \frac{1}{2}\left(y_A + y_B\right) \tag{13.49}$$

$$= \frac{1}{2}\left(h - \frac{1}{2}gt^2 + h - \frac{1}{2}gt^2\right) = h - \frac{1}{2}gt^2 \ , \tag{13.50}$$

We could also have found this directly from Newton's second law for particle systems:

$$\sum \mathbf{F}^{\text{ext}} = m\mathbf{A} = -mg\,\mathbf{j} \ \Rightarrow \ A_y = -g \ \Rightarrow \ Y(t) = Y(t_0) - \frac{1}{2}gt^2 = h - \frac{1}{2}gt^2 \ . \tag{13.51}$$

Analyze: We have demonstrated that we can find the motion of the center of mass either by calculating the motion of each of the parts of the system, or we can find the motion of the center of mass by applying Newton's second law for particle systems directly. The results are of course the same. However, there are many questions we only can answer if we know the motion of each part. For example, in order to determine how far apart the two parts are when they hit the ground, we need to find the motion of each of the parts.

13.4 Motion in the Center of Mass System

Newton's law for multiparticle systems gives us the tool to determine the motion of the center of mass of a complex object based on the external forces acting on the system. The law is surprisingly robust. If you throw a rod through the air, and you neglect the effects of air resistance, the motion of the center of mass of the rod does not depend on how the rod is moving relative to its center of mass: The motion of the center of the mass is the same if the rod is moving as a ridig body without rotating, as in Fig. 13.9a; if the rod is rotating around its center of mass, as in Fig. 13.9b; or if the rod is rotating and wobbling. The motion of the center of mass of the rod does not depend on internal forces in the rod. Therefore, the motions of individual parts relative to the center of mass do not affect the motion of the center of mass.

However, in many cases we are interested in both the motion of the center of mass and of the motion of the individual parts relative to the center of mass. For example, we may be interested in the rotation of the rod, or its wobbling, or in how it is vibrating. Then it is useful to split the motion of the system into the motion of the center of mass and the motion of a particle relative to the center of mass.

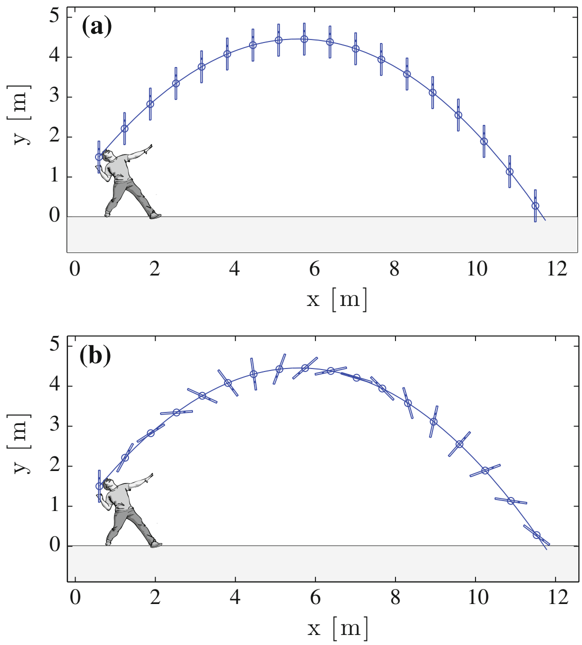

Fig. 13.9 The motion of a rod thrown through the air **a** without rotation, and **b** rotating around its center of mass

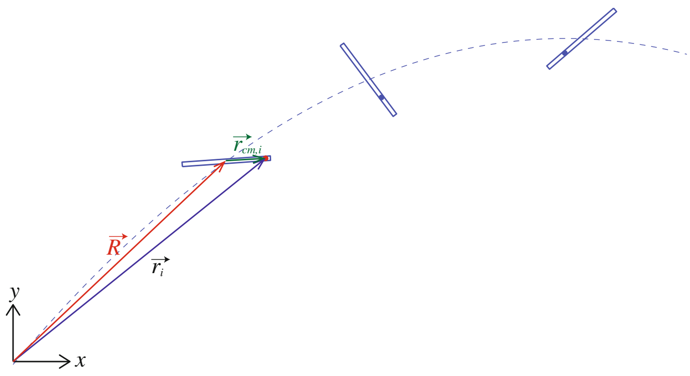

Fig. 13.10 The position of a point on the rod $\mathbf{r}_i$ can be written as a sum of the position of the center of mass, $\mathbf{R}$, and the position of the point relative to the center of mass, $\mathbf{r}_{\text{cm},i}$

Laboratory and Center of Mass Systems

We have already discussed how we always measure the motion of a system relative to some reference system. For example, we may characterize the motion of the rod relative to a point on the ground. We call this system the *laboratory system*. In addition, we introduce a coordinate system located in the center of mass of the system of particles, as illustrated in Fig. 13.10. This system is called the *center of mass system*.

The position of a particle i is $\mathbf{r}_i$ in the laboratory system, and the position of the center of mass of the system is $\mathbf{R}$ in the laboratory system. The position of particle i in the center of mass system is $\mathbf{r}_{\text{cm},i}$:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\text{cm},i} \ , \tag{13.52}$$

Center of Mass in the Center of Mass System

What is the center of mass of the particles measured in the center of mass system? The center of mass is:

$$\mathbf{R}_{\text{cm}} = \frac{1}{M}\sum_i m_i \mathbf{r}_{\text{cm},i} = \frac{1}{M}\left(\sum_i \mathbf{r}_i - \sum_i \mathbf{R}\right) = \frac{1}{M}\sum_i \mathbf{r}_i - \mathbf{R} = \mathbf{R} - \mathbf{R} = 0 \ . \tag{13.53}$$

Not surprisingly, the center of mass measured in the center of mass system, is in the origin of the center of mass system. This was indeed the whole point of the center of mass system.

Total Momentum in the Center of Mass System

What is the total momentum, $\mathbf{P}_{\mathrm{cm}}$, of the system in the center of mass system? The total momentum is defined as:

$$\mathbf{P}_{\mathrm{cm}} = \sum_i m_i \mathbf{v}_{\mathrm{cm},i} = \sum_i m_i \frac{d}{dt}\mathbf{r}_{\mathrm{cm},i} = \frac{d}{dt} \underbrace{\sum_i m_i \mathbf{r}_{\mathrm{cm},i}}_{=M\mathbf{R}_{\mathrm{cm}}=0} = 0 \,. \tag{13.54}$$

The total momentum of the system in the center of mass system is always zero! This result does not depend on the sum of external forces being zero. This is always true. Idenpendently of what is done to the system, it only depends on the definition of the center of mass.

13.5 Energy Partitioning

We can describe the motion of (the center of mass of) multiparticle systems using the concepts we developed in our studies of Newton's laws of point particles. What about the concepts of mechanical energy we used to address the behavior of single particles, can we still use energy concepts for multi-particle systems?

Kinetic Energy of a Multi-particle System

What is the kinetic energy of a system of particles? The total kinetic energy is the sum of the kinetic energy of every particle in the system:

$$K = \sum_{i=1}^{N} \frac{1}{2} m \left(\mathbf{v}_i\right)^2 = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{d\mathbf{r}_i}{dt}\right)^2 \,. \tag{13.55}$$

Now, we want to divide the motion into the motion of the center of mass (the motion of the whole system), and the motion relative to the center of mass:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} \,, \tag{13.56}$$

and similarly for the velocities:

$$\mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i} \ . \tag{13.57}$$

We use this to rewrite the total kinetic energy of the system, getting

$$K = \frac{1}{2} M \left(\mathbf{V}\right)^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 \ . \tag{13.58}$$

(You find a proof in Sect. 1.3). This shows that the total kinetic energy can be divided into two terms: The kinetic energy for the motion of the center of mass, the **external kinetic energy**.

$$\boxed{K_{\mathrm{cm}} = \frac{1}{2} M \mathbf{V}^2 \ ,} \tag{13.59}$$

and the kinetic energy due to the motion of the particles relative to the center of mass, which we call the **internal kinetic energy**:

$$\boxed{K_{\Delta\mathrm{cm}} = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 \ .} \tag{13.60}$$

The total kinetic energy is therefore partitioned into a sum of the *internal* ($K_{\Delta\mathrm{cm}}$) and *external* (K_{cm}) kinetic energies:

$$K = \underbrace{\frac{1}{2} M \mathbf{V}^2}_{K_{\mathrm{cm}}} + \underbrace{\frac{1}{2} \sum_{i=1}^{N} m_i \mathbf{v}_{\mathrm{cm},i}^2}_{K_{\Delta\mathrm{cm}}} \ . \tag{13.61}$$

If the whole object is translated, as illustrated in Fig. 13.9a, there is no motion relative to the center of mass. Then the total kinetic energy is simply the kinetic energy of the center of mass. What we have done previously is therefore correct as long as we assume that the object is translated!

But what if parts of the system are moving relative to the center of mass? Then we must also include the internal kinetic energy: The energy related to the motion relative to the center of mass. For example, if a diatomic molecule is vibrating, we must also include the kinetic energy of the vibrating motion of the atoms. This means that for a system of particles we have more degrees of freedom—there are many possible ways that kinetic energy can be realized inside the system. Hence, the kinetic energy of a system may be conserved, even though the kinetic energy of the center of mass is not conserved.

This means that we need a simplified way to describe the internal motion, the motion relative to the center of mass, of a particle system. For example, as we will see later, for rigid bodies we do not allow vibrations or other deformations of the object. The only motion possible relative to the center of mass is a rotation the whole body. In this case, we need to introduce a kinetic energy term related to the rotation of the body, and this is indeed one of the main focus areas when we discuss the dynamics of rigid bodies.

Potential Energy of a Multi-particle System

For a multi-particle system, the kinetic energy is partitioned into the external and internal kinetic energy. What about the potential energy of a multi-particle system?

First, we need to be more precise. The potential energy related to what force? The rod in Fig. 13.9 has a potential energy due to the gravitational force, which is the sum of the potential energy of every particle in the rod. But in addition, parts of the rod may be compressed or stretched, and as a result the rod has an internal potential energy just as a in a diatomic molecule. We therefore need to discern between **external potential energy**, the potential energy of an external force, and **internal potential energy**, the potential energy due to interactions within the system.

Potential Energy Due to External Forces

If a conservative external force acts on all the particles, we can describe the interaction through a potential energy, $U_i(\mathbf{r}_i)$, of particle i at position $\mathbf{r}_i$ due to the external force. The total potential energy of the system due to the external force is then:

$$U_{\text{TOT}} = \sum_{i=1}^{N} U_i(\mathbf{r}_i) \ . \tag{13.62}$$

If the force is constant, such as for gravity near the Earth's surface, this expression can be further simplified. In this case, the potential energy of particle i is:

$$U_i(\mathbf{r}_i) = m_i g y_i \ , \tag{13.63}$$

and the total potential energy is:

$$U_{\text{TOT}} = \sum_{i=1}^{N} m_i g y_i = g \sum_{i=1}^{N} m_i y_i = M g Y \ , \tag{13.64}$$

where Y is the Y-coordinate of the center of mass of the system. We can therefore use the well-known formula, $U = mgy$, also for the potential energy of a multiparticle system, we only need to use the center of mass position for y. Again, we find that we can use the concepts developed for point particles also for particle systems.

Notice that this conclusion is only true for a constant force. For a force that depends on the position of each small part of the multi-particle system, the total potential energy cannot always be expressed as a function of the position of the center of mass alone. We may have to calculate the full sum (or integral) in (13.62). But if the external force is approximately constant over the multi-particle system we may approximate the potential energy by a function that depends only on the center of mass position.

In addition, a multi-particle system may have internal degrees of freedom and a corresponding internal potential energy.

Potential Energy Due to Internal Forces

The net force acting on a particle i in a multi-particle system includes both external and internal forces:

$$\mathbf{F}_i^{\text{net}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} \ , \tag{13.65}$$

where $\mathbf{F}_i^{\text{ext}}$ is the net external force acting on particle i, external here meaning that it has its origin outside the system. The force $\mathbf{F}_{j,i}$ from particle j on particle i is an internal force.

If all forces, external and internal, acting on the system are conservative, we can introduce a potential energy for every force. The total potential energy of the system is the sum of all the potential energies for each of the forces. We divide the total potential energy into the external potential energy, the potential energy due to external forces, and the internal potential energy, the potential enegy due to internal forces:

$$U_{\text{TOT}} = U_{\text{ext}} + U_{\text{int}} \ . \tag{13.66}$$

The external potential energy was found in (13.62):

$$U_{\text{ext}} = \sum_i U_i \left(\mathbf{r}_i\right) \ . \tag{13.67}$$

The internal potential energy is a sum over all potential energies for all the interactions in the system:

$$U_{\text{int}} = \sum_{i<j} U_{i,j} \left(\mathbf{r}_i, \mathbf{r}_j\right) \ . \tag{13.68}$$

Notice that we include a given interaction, a pair of particles i, j, only once in this sum! Here, we will not develop a general theory for this, but instead illustrate the principle by an example: The conservation of energy in a bouncing dumbbell as discussed in Sect. 13.5.1.

Conservation of Energy in a Multi-particle System

If all the forces, both internal and external, acting on a particle system are conservative, the total mechanical energy of the system is conserved:

$$E_{\mathrm{TOT}} = K_{\mathrm{TOT}} + U_{\mathrm{TOT}} \ , \tag{13.69}$$

where

$$K_{\mathrm{TOT}} = K_{\mathrm{cm}} + K_{\Delta\mathrm{cm}} \ . \tag{13.70}$$

and

$$U_{\mathrm{TOT}} = U_{\mathrm{ext}} + U_{\mathrm{int}} \ , \tag{13.71}$$

which gives:

$$E_{\mathrm{TOT}} = K_{\mathrm{cm}} + U_{\mathrm{ext}} + K_{\Delta\mathrm{cm}} + U_{\mathrm{int}} \ . \tag{13.72}$$

Based on this result we realize that in order to apply the principle of energy conservation when solving problems in mechanics, we need to have expressions for the two terms $K_{\Delta\mathrm{cm}}$ and U_{int} for the system. Unfortunately, these terms are not always simple. For example, if you kick a football, the football will both rotate and wobble during its flight, as illustrated in Fig. 13.11. The kinetic and potential energy associated with the wobbling represent internal degrees of freedom, and we do not know how to quantify these without a detailed model for the deformation of the football. For a detailed study of the motion of the football, energy concepts are therefore of limited value as means for calculation. However, the energy concepts are still useful theoretical techniques that provides us with concepts and methods to discuss the

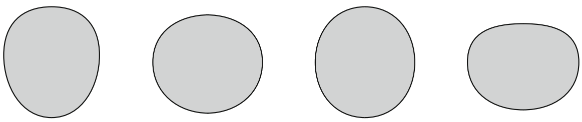

Fig. 13.11 A football in flight filmed using a high-speed camera: The football rotates and wobbles as it moves

motion. In many cases, energy consideration are also the theoretical starting point, for example, for determining the deformation of the football.

For a particular type of object, what we call a rigid body, we assume that the internal deformation and the energies associated with these are negligible, and that the object moves a rigid body. In this case we neglect the internal potential energy of the system, but we still need to develop expressions for the kinetic energy of a rotating rigid body. We return to this in Chap. 15 after a discussion of rotation in Chap. 14.

13.5.1 Example: Bouncing Dumbbell

In this example we will demonstrate the main princples of energy partitioning through a simple model system.

In this example we will address a two-particle system, where the two particles interact through a spring (see Fig. 13.12). This can be a model of a diatomic molecule consisting of two identical atoms, or for an elastic body that deforms and vibrates. Here, we simplify the problem by assuming that the that the particles move along a line, so that the motion is one-dimensional. In the next example, we extend our discussion to two-dimensional motion, opening for rotation in addition to vibrations.

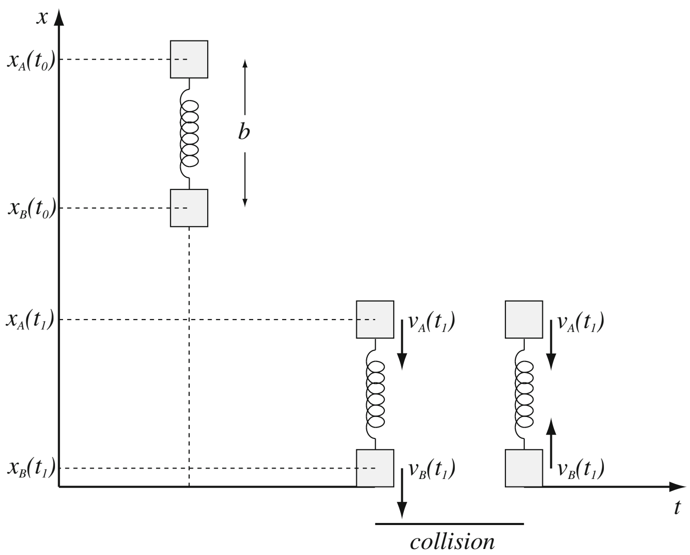

Fig. 13.12 Illustration of the dumbbell. Two identical particles of mass m are connected by a spring with equilibrium length b and spring constant k. The system is lifted to a height h above a flat floor and released. When the *bottom* particle hits the floor its velocity is reversed, as we know happens in an elastic collision with a wall

Identify and Sketch: We address a system of two particles, each with mass m, connected with a spring of equilibrium length b and spring constant k. Particle A starts on top and particle B starts on bottom, their positions are $x_A(t)$ and $x_B(t)$ respectively. We place particle B at a height h_0 above the floor: $x_B(t_0) = h_0$, and particle A is a distance b above particle B:$x_A(t_0) = x_B(t_0) + b$. When a particle hits the floor the collision is elastic, which means that the velocity of the particle is reversed after the collision.

Model: We find the motion of each particle from the forces acting on it. Particle A is affected by two forces: Gravity $\mathbf{G}_A = -mg\,\mathbf{j}$ and the spring force $\mathbf{F}_A$. The spring force depends on the position of both particle A and B:

$$\mathbf{F}_A = -k\,(x_A - x_B - b)\,\mathbf{j}\;, \tag{13.73}$$

Similarly, the forces acting on particle B are $\mathbf{G}_B = -mg\,\mathbf{j}$ and $\mathbf{F}_B$:

$$\mathbf{F}_B = -\mathbf{F}_A = k\,(x_A - x_B - b)\,\mathbf{j}\;. \tag{13.74}$$

Is it sufficient to study the motion of the center of mass alone? No, since we do not have models for the external forces acting on the system, and we do not know when or where they are acting without finding the motion of the individual particles. The center of mass of the system is at:

$$X = \frac{1}{M}\sum_i x_i = \frac{1}{2m}\left(mx_A + mx_B\right) = \frac{1}{2}\left(x_A + x_B\right)\;. \tag{13.75}$$

Since the system collides with the wall when one of the particles hits the wall, and not when the center of mass hits the wall (which it never will), we must find the motion of each particle individually.

Newton's second law: We apply Newton's second law to each particle to find its acceleration:

$$\mathbf{a}_A = -\frac{k}{m_A}\,(x_A - x_B - b)\,\mathbf{j}\;, \tag{13.76}$$

$$\mathbf{a}_B = \frac{k}{m_B}\,(x_A - x_B - b)\,\mathbf{j}\;. \tag{13.77}$$

where we also have to include a possible collision between particle B and the floor.

Numerical solution: We determine the motion of the particles numerically using a Euler-Cromer method. We need to include all forces acting on each particle, and we execute a collision step whenever particle B collides with the wall. When does a collision occur? Since we are determining the positions only at discrete time intervals Δt, we do not expect the position of particle B ever to be exacly at the wall. Instead, the particle will be outside the wall at one time, t, and then inside the wall at a later time, $t + \Delta t$, and the only thing we know is that a collision occurred at some time during the time interval from t to $t + \Delta t$. We could try to improve our estimate of when the collision occurs, but here we will simply assume that the result of a collision is to reverse the velocity at the first time step the particle is "inside" the wall. The most important part of our collision algorithm is that is conserves energy, and this is indeed achieved by this method. Because the particle is not moved during the collision, the potential energy is conserved. And since only the direction of the velocity is changed, the kinetic energy is conserved.

```
from pylab import *
m = 0.1
k = 200.0
b = 0.2
h0 = 1.0
g = 9.8
time = 10.0;
dt = 0.00001;
n = int(round(time/dt))
t = zeros(n,float)
xA = zeros(n,float), vA = zeros(n,float)
xB = zeros(n,float), vB = zeros(n,float)
xA[0] = h0 + b,      vA[1] = 0.0
xB[0] = h0,          vB[1] = 0.0
for i in range(n-1):
    f = k*(xA[i] - xB[i] - b)
    fA = -f - m*g
    fB = f - m*g
    aA = fA/m
    vA[i+1] = vA[i] + aA*dt
    xA[i+1] = xA[i] + vA[i+1]*dt
    aB = fB/m
    vB[i+1] = vB[i] + aB*dt
    xB[i+1] = xB[i] + vB[i+1]*dt
    t[i+1] = t[i] + dt
    if (xB[i+1]<0.0) and (xB[i]>=0.0):
        vB[i+1] = abs(vB[i+1])
xcm = (xA+xB)*0.5, vcm = (vA+vB)*0.5
Kcm = 0.5*(2.0*m)*vcm**2
Kcmdelta = 0.5*m*(vA - vcm)**2+0.5*m*(vB-vcm)**2
Ug = xcm*(2.0*m)*g
Uk = 0.5*k*(xA - xB - b)**2
E = Kcm + Kdeltacm + Ug + Uk
subplot(2,1,1)
plot(t,xA,'-b',t,xB,'-r',t,xcm,':k')
xlabel('t [s]'), ylabel('x [m]');
subplot(2,1,2);
plot(t,Kcm,'-b',t,Kcmdelta,'-r',t,Uk,'-y',t,E,':k')
xlabel('t [s]'), ylabel('E [J]')
```

Visualization of motion: A visualization of the motion of the system for the first two collisions is shown in Fig. 13.13. Here, we have illustrated the positions of the two particles at a few selected times. Before the first collision, the two particles move without relative motion. After the second collision, the relative motion is significant. (Notice that we have drawn the particles as small blocks, but they should be interpreted as point-particles.)

Motion of the center of mass: The dashed line in Fig. 13.13 shows the motion of the center of mass. We know that the motion of the center of mass only depends on the external forces acting on the system. During the collision with the floor, the system

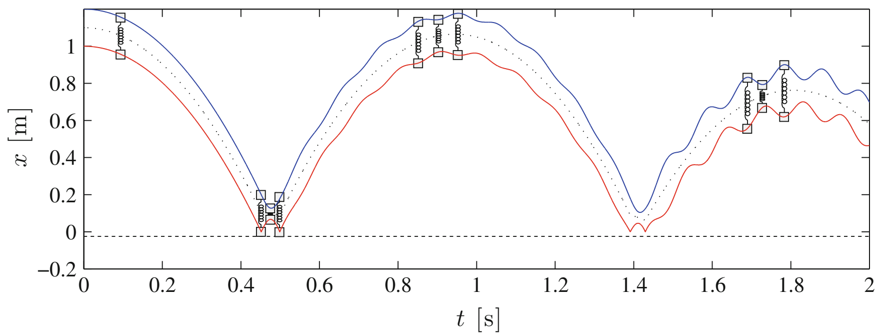

Fig. 13.13 Illustration of the motion of the dumbbell. Here we have illustrated the path of the center of mass (*dashed line*) and the path of each of the particles. We have drawn in a system of two blocks and a spring to help your intuition

is affected by both gravity and the contact force, but between collisions, the system is only affected by gravity. Consequently, we expect the center of mass to behave as a single object of mass $2m$ falling under the effect of gravity. That is, we expect:

$$A = -g \;\Rightarrow\; X(t) = X_0 + V_0 t - \frac{1}{2} g t^2 \;, \tag{13.78}$$

which is seen as the parabolically shaped motion of the center of mass in Fig. 13.13. Even though the particles are oscillating back and forth, the center of mass does not display any oscillations, but shows a smooth, parabolic shape (as a function of time).

Energy partitioning: We notice that after the second bounce, the center of mass does not bounce up to its initial level even though the total energy in the system is conserved. If our system was a rigid ball, and the collision with the floor was elastic (and we neglected air resistance), we would expect the ball to bounce up to its initial level. Why do we expect this? Because at the top of the path, when X is maximum, the velocity is zero. For a rigid ball, this would also imply that the kinetic energy is zero, and that the total mechanical energy is equal to the potential energy, which only depends on the height. For the rigid ball, the height of each bounce must be the same. But for the dumbbell system, there are also internal degrees of freedom. The kinetic energy is not zero at the top of the path, even though the velocity of the center of mass is zero, because the particles are still moving relative to the center of mass. Similarly, the total potential energy is not only equal to the potential energy in the external gravitational field, but also depends on the relative positions of the two particles.

We can use the concept of energy partitioning to analyze the behavior. The total kinetic energy consists of:

$$K = K_{\rm cm} + K_{\Delta \rm cm} \;, \tag{13.79}$$

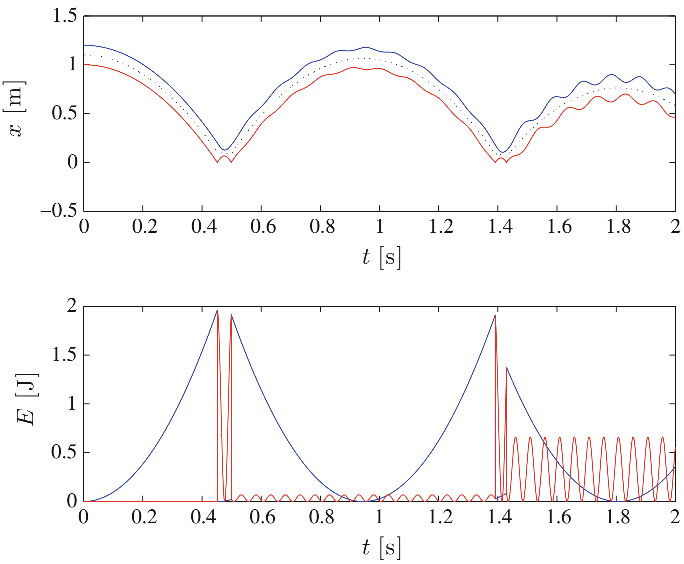

Fig. 13.14 Illustration of the motion of the dumbbell and the kinetic energy of the system, partitioned into the kinetic energy of the center of mass and the internal kinetic energy

where the kinetic energy of the center of mass is:

$$K_{\mathrm{cm}} = \frac{1}{2} M V^2 \,, \tag{13.80}$$

and the kinetic energy due to the motion relative to the center of mass is:

$$K_{\Delta\mathrm{cm}} = \sum_i \frac{1}{2} m_i v_{\mathrm{cm},i}^2 = \frac{1}{2} m \left(v_A - V\right)^2 + \frac{1}{2} m \left(v_B - V\right)^2 \ . \tag{13.81}$$

The kinetic energies are plotted in Fig. 13.14. Notice that before the bottom particle hits the floor, there is no relative motion, and the internal kinetic energy ($K_{\Delta\mathrm{cm}}$) is zero. Each of the particles moves with the same velocity and the center of mass. Immediately after the collision, the internal kinetic energy increases discontinuously—it jumps to a high level. What happened? After the collision, the velocity of the bottom particle is reversed. This has two effects. First, the center of mass now has zero velocity. But the magnitudes of the velocities of each of the particles are unchanged. The result is that each of the particles suddenly has a velocity relative to the center of mass equal to the velocity they have before the collision. As a result, there is a jump in the internal kinetic energy.

Another interesting observation is the oscillation in the internal kinetic energy due to the oscilliation of the particles around the center of mass.

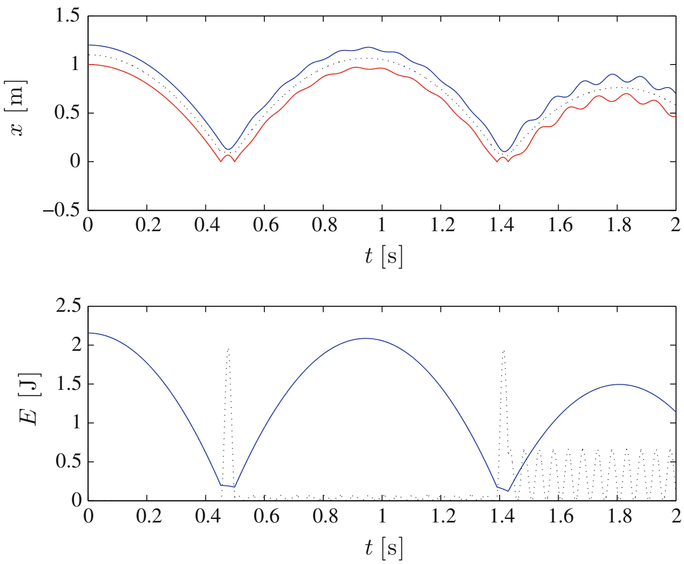

Fig. 13.15 Illustration of the motion of the dumbbell and the potential energy of the system, partitioned into the potential energy of the center of mass and the internal potential energy

The total potential energy of the system is the sum of all the potential energies:

$$U_{\mathrm{TOT}} = \sum_i U_i^{\mathrm{ext}} + \sum_{i<j} U_{i,j}^{\mathrm{int}} = \underbrace{\sum m_i g x_i}_{=U_{\mathrm{ext}}} + \underbrace{\frac{1}{2} k \left(x_A - x_B - b \right)^2}_{=U_{\mathrm{int}}}, \tag{13.82}$$

The two potential energies are plotted in Fig. 13.15. The potential energy of the center of mass only follows the center of mass movement, and the internal potential energy is related to the energy stored in the spring as the spring is compressed. A fine analysis of Figs. 13.14 and 13.15 shows that the internal kinetic and potential energies are oscillating with opposite phases—as we expect—so that when the internal kinetic energy is maximum, the potential energy is at its minimum and vice versa.

Discussion: From this example we learn that mechanical energy is conserved only if we include all the forms it may take. The mechanical energy correspoding to the kinetic and potential energy of the center of mass is generally not conserved. In addition we must take into account the many possible internal modes of motion and associated potential energies. However, if we simplify our system so that vibrational modes are not present, such as for a rigid body that does not deform, is it then sufficient only to use the kinetic and potential energy of the center of mass? No! We have looked at a too simple system! Real, two- and three-dimensional systems can be rotated without deforming. We must therefore also include the effect of rotations. This is the subject of the next chapters.

13.6 Energy Principle for Multi-particle Systems

There are several ways to view energy conservation. So far, we have mainly used energy conservation as a tool to determine the velocity of an object as a function of position. In this case, we include all external and internal conservative forces in the total energy of the object:

$$E_{\text{TOT}} = K_{cm} + K_{\Delta cm} + U_{\text{ext}} + U_{\text{int}} \ , \tag{13.83}$$

As a method for calculation this worked very well as long as we could ignore the internal potentials and motion relative to the center of mass. Fortunately, this reprensents a whole class of problems, the motion of ridig bodies, where there are no internal vibrations and no internal potential energies, although the object may rotate, as we will see further on. This approach is therefore a useful approach from a practical point of view.

However, there is a different view on energy conservation which is very useful from a theoretical point of view, while not that useful for direct calculations. In this view, we consider all conservative interactions to be internal interactions. This is done by including all the interacting objects in the system. For example, if we study the motion of a ball falling towards the Earth under gravity, we include both the ball and the Earth in the system. The potential energy of the ball relative to the Earth is then an internal potential energy in a multi-particle system. Similarly, we would treat the solar system as a system consisting of all the objects in the solar system. We could also take a similar approach on an atomic scale. Indeed, we could consider any object to consist of a large set of atoms, and since all the atoms interact through position-dependent forces, usually central forces, all the internal forces are conservative and the total energy of the system should be conserved. Unless there are other, external forces acting on the system.

Internal Energy

If all the internal forces are conservative, the only way we can change the total energy of the system is by the work done by an external force. For a system of a book lying on the floor, a system of the book and the Earth, we would increase the total energy of the system if we lifted the book by an external force. External work leads to a change in the total energy. We often call the total energy of such a system the *internal energy* of the system. We write this as:

$$W_{\text{ext}} = \Delta E \ . \tag{13.84}$$

The work done by external force is the change in internal energy, ΔE. This formulation is call the first law of thermodynamics, and is considered a fundamental law in physics.

How can we then interpret the work done by an internal force, such as the work done by gravity in the system of the book and the Earth? In this case, the work done by gravity is given as a change in the internal potential energy of the book-Earth system. For example, if we release the book from a height h above the ground and the let it fall, the book has gained a kinetic energy correspondingn to the change in potential energy when book reaches the ground. This is not a change in the total energy. There is no change in internal energy. But it is an energy transfer between different forms of internal energy: It is an energy transfer from potential to kinetic energy inside the system. Since we have assumed that all internal forces are conservative, all internal processes can be considered as transfers of energy.

The Arrow of Time

Hmmm. What about internal forces that are not conservative? How can we treat such processes in this view? If we take the atomic view: All our systems consists of atoms and interatomic interactions are conservative, any system should therefore only have conservative forces. Where does the non-conservative interactions come from? For example, if I take the book and slide it along an inclined surface from its initial height h, there will also be a frictional force, and friction is not a conservative force. Is the total energy still conserved in the book-Earth system in this case? Yes! What happens is that as the book slides, the potential energy of the book-Earth system is transfered to kinetic energy of the book, as well as many internal kinetic and potential energies: The atoms in the book and the floor starts vibrating. This corresponds to an increase in the temperature of both book and Earth, which again corresponds to an increase in the internal energy. Thus, the total energy is conserved, but it is now hidden in different internal energies inside the system. So why do we then call the friction force conservative? Because it is conservative on a microscopic level: The total energy is conserved, but it is not conservative on a macroscopic level: The friction force depends on the relative velocities of the moving objects.

This transfer of energy from macroscopic kinetic energy, the kinetic energy of a sliding book, to microscopic kinetic and potential energies in the various vibrations of the atoms, effectively introduces the arrow of time. If we release the book with an initial velocity downward, it will slide down the slope and eventually come to rest. The total internal energy of the system is conserved (if we ensure that it does not interact with any external forces). It is then fully possible that the book instantaneously starts to slide upwards, back to its original position. If by accident the atomic vibrations every time pushed the book in the right direction, it could happen. None of the individual processes would be against the laws of physics. Such a processes would not change the total energy of the system. But it would be extremely unlikely. And this is the reason why it is not happening. And it is the reason why we laugh when we play movies backwards. You will learn more about this later, when you learn about statistical physics.

Summary

Center of mass: The center of mass $\mathbf{R}$ of a particle system consisting of N particles with masses m_i located at positions $\mathbf{r}_i$ is defined as:

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i \ , \ M = \sum_{i=1}^{N} m_i \ .$$

Center of mass is a vector: The center of mass is defined in a *vector equation* which is valid for each of the coordinates:

$$X = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \ , \ Y = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \ , \ Z = \frac{1}{M}\sum_{i=1}^{N} m_i z_i \ .$$

Velocity of center of mass: The *velocity of the center of mass* is:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{v}_i \ .$$

Acceleration of center of mass: The *acceleration of the center of mass* is:

$$\mathbf{A} = \frac{d^2\mathbf{R}}{dt^2} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{a}_i \ .$$

Center of mass for a solid body: The center of mass of a solid body is:

$$\mathbf{R} = \frac{1}{M}\int_m \mathbf{r}\, dm = \frac{1}{M}\int\int\int \mathbf{r}\, \rho\, dV \ .$$

Newton's second law for a particle system: Newton's second law for a particle system relates the *external forces* to the acceleration of the *center of mass* of the system:

$$\sum \mathbf{F}^{\text{ext}} = M\mathbf{A} \ .$$

Motion in center of mass system: We relate the position $\mathbf{r}_i$ of a particle in the *laboratory system* to a position $\mathbf{r}_{\text{cm},i}$ in the *center of mass system* by:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\text{cm},i} \ .$$

In the center of mass system, the positions are measured relative to the center of mass.

Partitioning of kinetic energy: The *kinetic energy*, K, of a system consist of two terms: The kinetic energy of the center of mass, K_{cm}, and the kinetic energy of the motion relative to the center of mass, $K_{\Delta\text{cm}}$:

$$K = K_{\text{cm}} + K_{\Delta\text{cm}} = frac12MV^2 + \sum_{i=1}^{N} \frac{1}{2} m_i v_{\text{cm},i}^2$$

Partitioning of potential energy: Similarly, the potential energy is partitioned into potential energy due to external forces, U_{ext} and potential energy due to internal forces, U_{int}:

$$U_{\text{TOT}} = U_{\text{ext}} + U_{\text{int}} \ ,$$

Potential energy of a particle system in constant gravity: For a particle system in a homogeneous gravitational field, the potential energy is the same as the potential energy of a point particle with the total mass of the system located in the center of mass of the system:

$$U = MgY \ ,$$

where Y is the vertical position of the center of mass.

Exercises

Discussion Questions

13.1 Balance center. If you want to balance a thin rod on a needle, why should you place the needle at the center of mass?

13.2 Jumping people. If all the people on Earth come together in one place and jump, what happens with the center of mass of the Earth-people system?

13.3 Rectangle. You have to place a rectangle of area A fully inside the first quadrant. How would you place it in order to make the distance from the origin to the center of mass the smallest?

13.4 Thor's hammer. You throw a hammer across the lecture hall (don't try this, please). Discuss its trajectory if you (i) threw it without any rotation or (ii) threw it so that it rotated during its flight.

Problems

13.5 Two-particle system. A 2 kg particle is placed at $x = 2$ m and a 4 kg particle is placed at $x = 6$ m.
(**a**) Where is the center of mass of this two-particle system?

13.6 Center of mass of Earth-Moon system.
(**a**) Estimate the position of the center of mass of the Earth-Moon system. Give your answer in units of the Earth's diameter.

13.7 Carbon-monoxide. For a Carbon-monoxide molecule, the mass of the Carbon atom is 12.0107 g/mol, and the mass of the Oxygen atom is 15.9994 g/mol, and the typical distance from the Carbon to the Oxygen is 112.8 pm.
(**a**) Find the center of mass of a Carbon-monoxide molecule.

13.8 Three-particle system. Three particles of equal mass are placed as shown in Fig. 13.16.
(**a**) What is the center of mass of this system?
(**b**) How can you add another particle to the system without changing the center of mass?

13.9 Tetrahedron. A tetrahedron consists of four points (vertexes) connected by six lines of equal length, with three lines originating from each vertex, as seen in the figure. A particle of mass m is placed on each vertex. The coordinates of the corners are (1,1,1), (−1,−1,1), (−1,1,−1), (1,−1,−1).
(**a**) What is the center of mass of this system?
(**b**) How does the center of mass change if we double the mass of the first particle?

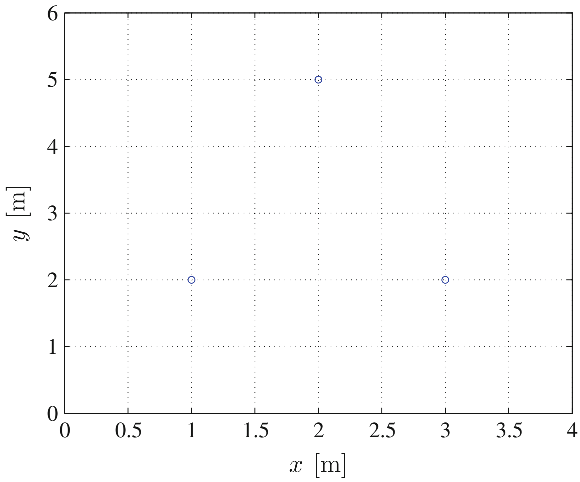

Fig. 13.16 A system of particles

13.10 Cubic hole. You make a cubic hole with sides of length d in the center of one of the sides of a homogeneous cube with sides of length L.
(**a**) Find the center of mass of the cube with the hole.

13.11 Triangle.
(**a**) Find the center of mass of a homogeneous isosceles triangle of base b and height a. (An isosceles triangle have two sides of equal length.)

13.12 Triangle.
(**a**) Find the center of mass of a homogeneous equilateral triangle of base b in Fig. 13.17. (An equilateral triangle has three sides of equal length.)

13.13 A piece of pie. You cut out a piece with an angle θ from a flat, homogeneous pie of radius R.
(**a**) Find the center of mass of the piece.

13.14 Person in a boat. John (80 kg) is standing in a 200 kg boat. He starts at one end and walks 6 m to the other end of the boat. You may neglect drag forces between the boat and the water.
(**a**) How far does the boat move in this process?

13.15 Car on a train. A 1000 kg car is standing on an inclined plane on top of a 2000 kg train cart. The cart rolls without friction on the track. The plane has an inclination of 30° with the horizontal. The car drives a horizontal distance of 10 m from one end of the cart to the other.
(**a**) How far has the cart moved in this process?

Projects

13.16 Pushing the blocks. In this project you will learn about Newton's second law for multi-particle system and energy partitioning in multi-particle systems. We will study the a two-particle system affected by an external force. The system consists of two identical blocks, A and B, sliding on a frictionless, horizontal surface. The blocks have mass m and are attached with a massless spring with spring constant k and equilibrium length d. The blocks start from rest in their equilibrium positions at

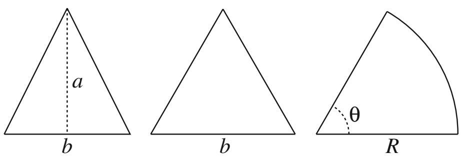

Fig. 13.17 An isosceles triangle, an equilateral *triangle*, and a piece of pie

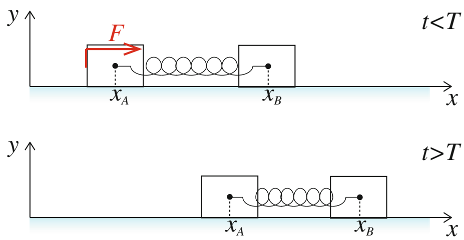

Fig. 13.18 Illustration of two-block system

$x_A(0\text{s}) = 0\,\text{m}$ and $x_B(0\text{s}) = d$. The system is illustrated in Fig. 13.18. The left-most block, block A, is pushed with an external force F for a short time T.
(**a**) Draw a free-body diagram for each of the blocks. Name the force.
(**b**) Introduce force models for all the forces acting on the blocks.
(**c**) Find expressions for the accelerations for each block.
(**d**) Find an expression for the acceleration of the center of the mass of the system.
(**e**) Find the velocity, $V(t)$, and position, $X(t)$, of the center of mass as functions of time.

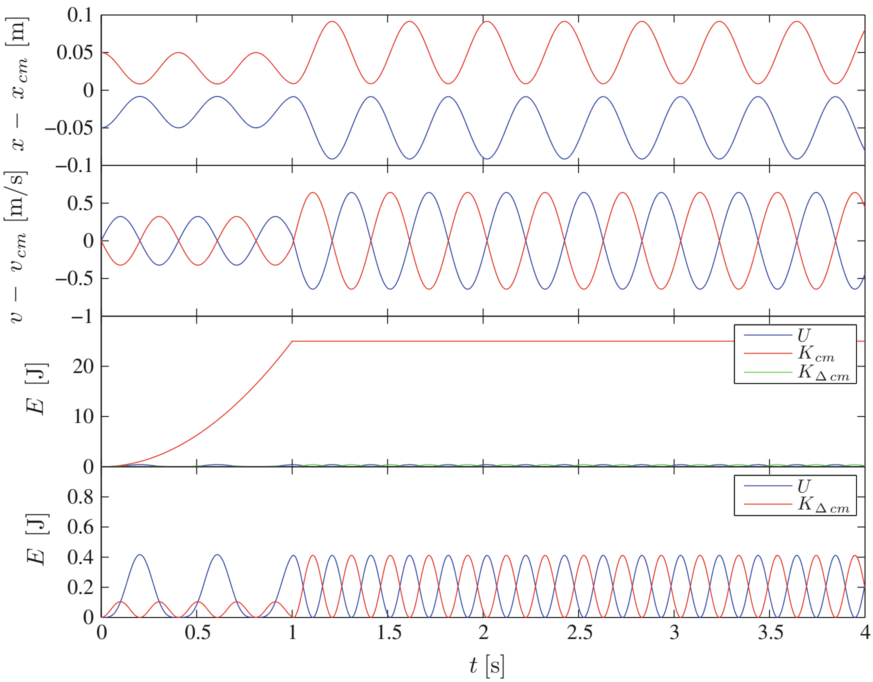

Fig. 13.19 Plots from simulations

In the following we will solve to find the motion of both blocks. It is possible to solve this problem using either analytical or numerical methods. We will here follow a numerical solution, but all results can also be obtained by analytical means. We will study a system where $F = 1000$ N, $T = 1$ s, $k = 5000$ N/m, $d = 0.1$ m, and $m = 0.1$ kg.
(**f**) Write a program to find the positions $x_A(t)$ and $x_B(t)$ as functions of time.
(**g**) Plot the position and velocity of center of mass as calculated by the program and compare with your results from above. Plot the motion of the blocks in the same plot and comment on the results. (See Fig. 13.19 for comparison).
(**h**) Plot the kinetic energy of the center of mass, K_{cm}, the kinetic energy of the motion relative to the center of mass, $K_{\Delta cm}$, and the potential energy, U, as functions of time. Comment on the results.
(**i**) What is the maximum extension of the spring?
(**j**) How much work was done by the external force, F?

Chapter 14
Rotational Motion

How can we describe the rotational motion of the Earth and how can we calculate the velocity of a point on the surface of the Earth due to its rotation?

We can move a rigid body about by moving it, through translation, and by rotating it, through rotation. Up to now we have only discussed translational motion. In this chapter we will introduce the tools to describe rotations.

14.1 Rotational State—Angle of Rotation

How can we describe the motion of the rod shown in Fig. 14.1? We would like to seperate the translational and rotational motion of the rod. In this case, for a rod thrown across the room, the rod rotates around its center of mass. We can therefore use the center of mass, $\mathbf{R}(t)$, to describe the translational motion of the rod. This is a good choice, since the motion of the center of mass is determined from the external forces acting on the rod—we could therefore find the motion of the center of mass by solving the equations of motion. If we study the motion of the rod relative to the center of mass, we get the bottom-right part of Fig. 14.1. How can we describe the rotational state of this system? By the angle θ it has rotated around the center of mass!

Angle and Axis of Rotation

While a freely moving object (such as a rod thrown across the room) usually rotates around its center of mass,[1] objects can also rotate around other points. We could for example nail the rod to the wall in any point along the rod, and the rod would be

[1]You will learn more about conditions for this later, when we discuss the physics of rotational motion.

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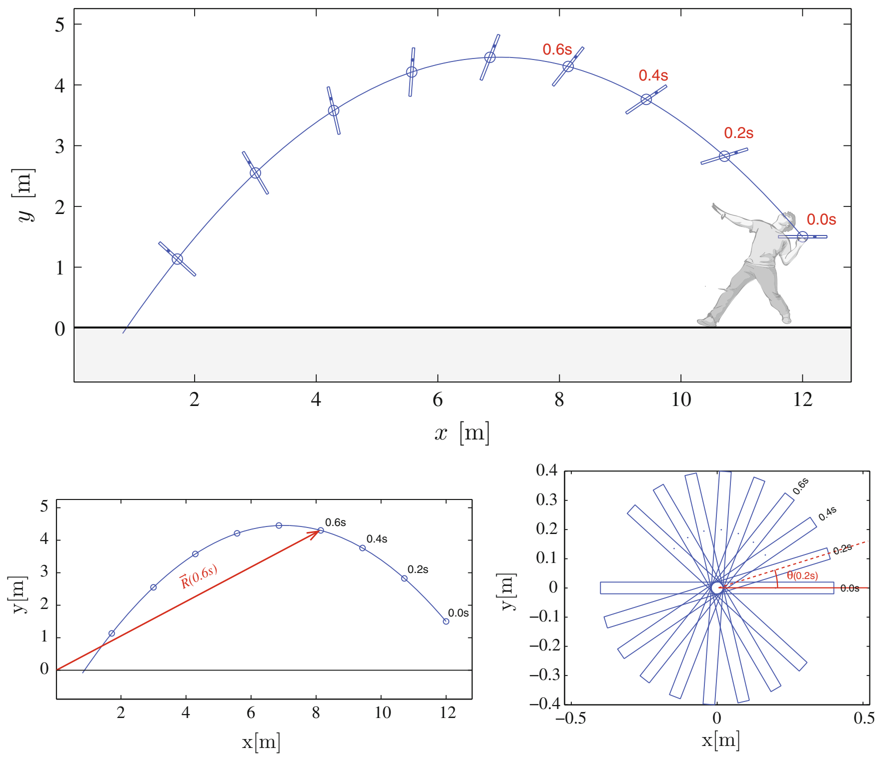

Fig. 14.1 The motion of a rotating rod thrown through the air

forced to rotate around this attachment point. The rotational configuration of the rod is often called the rotational *state* of the rod. In order to uniquely define the rotational state of the rod we need to specify both the attachment point O and the angle θ the rod forms with the horizontal. But if we only specify the point O, we do not really know how the object rotates around this point. We need to specify the **rotational axis** as well as a point on the axis. For rotations in the xy-plane, we say that the rotational axis is normal to this plane, that is, along the z-axis. This description holds for rotations in two dimensions. We describe the three-dimensional case in Sect. 14.6.

> In two dimensions, the rotational configuration of an object is described by: the angle θ; the point O it is rotating around; and the direction of the rotational axis, $\mathbf{k}$.

How do we describe the positive rotational direction? This is customarily determined by the right hand rule. Figure 14.2 shows how the direction of the positive z-axis is determined from the directions of the x- and y-axes. We can also use this rule

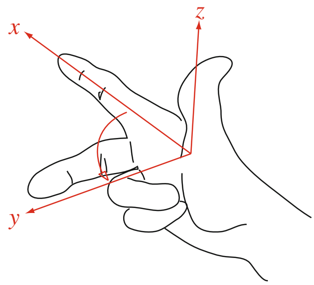

Fig. 14.2 Illustration of the right-hand rule

backwards: Given the direction of an axis, such as the z-axis, we can find the positive rotational direction by pointing the right thumb in the direction of the axis: the positive rotational direction is then in the direction your remaining fingers are curling: from the x- towards the y-axis. In this direction θ increases, in the opposite direction the angle decreases.

A Point on a Rotating Object

Given the angle θ and the rotation axis (including both a point on the axis and the positive direction along the axis), we can unqiuely determine the orientation of a rotating object. But how do we find the position of a particular point on a rotating object from this?

Figure 14.3 shows the motion of a point P on an object rotating around a fixed axis. We describe the position of P using a coordinate system that rotates along with the object. The rotating coordinate system has to unit vectors that rotate with the object: the unit vector $\hat{u}_r$, which is directed radially outwards from the rotation axis, and an axis normal to the radial direction with unit vector $\hat{u}_n$. A point on the rod can be described in this coordinate system by:

$$\mathbf{p} = p_r \hat{u}_r + p_n \hat{u}_n \ . \tag{14.1}$$

When the object has rotated an angle θ both unit vectors have also rotated. The radial unit vector now forms angle θ with the horizontal, and is given as:

$$\hat{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \ , \tag{14.2}$$

as illustrated in Fig. 14.3. The normal vector, $\hat{u}_n$, is obtained by rotating $\hat{u}_r$ 90° in the positive direction:

$$\hat{u}_n = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \ . \tag{14.3}$$

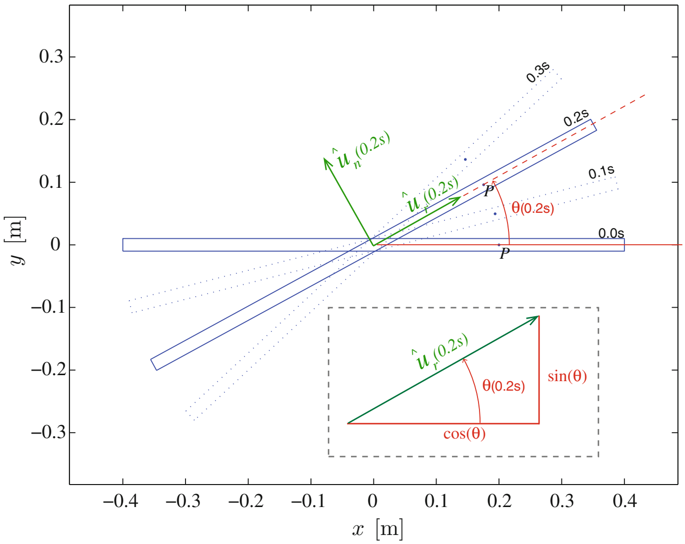

Fig. 14.3 Illustration of the unit vector $\hat{u}_P$, and the position of the point P during rotation of a rod around an axis through the origin

If the object is rotating around a fixed axis, this gives the position of any point on the object. If the object is rotating around a moving axis, such as the rotating rod in Fig. 14.1, we also need to add the position of the axis—here given as the position of the center of mass:

$$\mathbf{p} = \mathbf{R} + p_r \hat{u}_r + p_n \hat{u}_n \ . \tag{14.4}$$

The attentive reader may recognize the decomposition using the unit vector $\hat{u}_r$ and $\hat{u}_n$ as polar-coordinates. This is indeed correct.

Rotational Motion

We can describe the rotational motion of the rod from Fig. 14.1 by a motion diagram for the rod. We have illustrated one such diagram in the bottom-right part of Fig. 14.1, where we show the position of the rod at various times, t_i, taken at constant time intervals Δt. This plot looks like a movie of the motion of the rod, where all the images have been superimposed into one image. A better way to visualize the rotational motion of the rod is to plot the time evolution of the angle, $\theta(t)$. Figure 14.4 shows $\theta(t)$ for the rotational motion in Fig. 14.1.

Test your understanding: Can you sketch two other motion diagrams for a rod that is rotating in the negative direction and for a rod that is rotating faster and faster in the positive direction. Sketch the corresponding diagrams for $\theta(t)$.

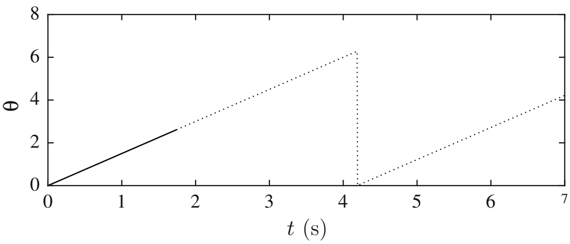

Fig. 14.4 Plot of the angle, $\theta(t)$, for the rotational motion in Fig. 14.1. *Dashed curve* shows how the rod would have continued to rotate if it had not hit the ground—such as if it fell off a cliff

Periodicity of the State $\boldsymbol{\theta(t)}$

The angle, θ, describes a unique configuration of the rod for values from 0 to 2π (measured in radians). What happens when $\theta(t)$ increases beyond 2π? When θ reaches 2π the rod has rotated a full revolution, and the rod is in the same position as it was when θ was equal to 0. We cannot discern these positions: The position of the rod when $\theta = 2\pi$ is exactly the same as when $\theta = 0$. However, it is customary to only use angles between 0 and 2π to describe the rotational position. This means that if the angle is larger than 2π we subtract 2π from the angle. This is seen in Fig. 14.4: When the angle $\theta(t)$ reaches 2π, it continues at $\theta = 0$. Similarly, when the angle decreases below 0, we add 2π to the angle, so that it continues at 2π. You are, of course, free to choose to describe the motion using an angle θ that increases also beyond 2π, but then you have to remember that the motion is periodic so that higher values does not represent new positions.

14.2 Angular Velocity

During rotation, the angle $\theta(t)$ changes with time. How can we characterize how fast the rod rotates? By the angular velocity, which is defined as the rate of the change of the angle in analogy with the (translational) velocity, which is the rate of change of the position. During the time interval from t to $t + \Delta t$, the angle changes from $\theta(t)$ to $\theta(t + \Delta t)$. We define the **average angular velocity** over the time Δt as:

$$\bar{\omega} = \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t} = \frac{\Delta\theta}{\Delta t} \ , \tag{14.5}$$

When the time interval becomes small, we find the *instantaneous angular velocity* for the rotational motion, which we in the following call the **angular velocity**:

Angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta} \ . \tag{14.6}$$

Figure 14.4 shows the angle, $\theta(t)$, and the angular velocity, $\omega(t) = d\theta/dt$ for the rotational motion in Fig. 14.1. Since the angular velocity is the time derivative of the angle, we interpret the angular velocity as the slope of the $\theta(t)$ curve (just as we did for the translational velocity). We see that the motions in Fig. 14.1 has a constant, positive angular velocity.

Test your understanding: Can you sketch $\theta(t)$ and $\omega(t)$ for a rod that is rotating equally fast in the opposite direction?

Velocity of a Point on a Rotating Body

As the rod rotates, every part of the rod moves in a circle around the rotation axis. What is the velocity of a small part of the rotating rod, and how can we relate it to the angular velocity? Let us address the motion of a small part, P, of the rotating body directly. We have illustrated its motion during a small time interval Δt, in Fig. 14.5. The distance from P to the rotation axis is R. The small part P moves along a circular path around the rotation axis with R as the radius. During the small

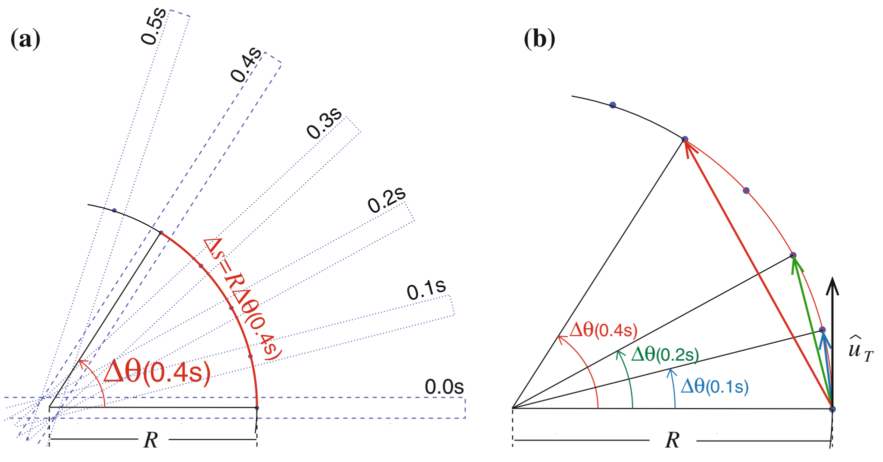

Fig. 14.5 **a** Illustration of the motion of a small part, P, of a rod rotating around an axis through the origin. **b** Illustration of the velocity vector for P as the time interval Δt decreases

time interval Δt, the rod has rotated an angle $\Delta\theta$ from the orientation $\theta(t)$ to the new orientation $\theta(t+\Delta t) = \theta(t) + \Delta\theta$. How far has P moved? It has moved the arc length $\Delta s = R\Delta\theta$ along its circular path. The speed of the small part P is therefore:

$$v = \frac{\Delta s}{\Delta t} = R\,\frac{\Delta\theta}{\Delta t}\ . \tag{14.7}$$

If we let the time interval Δt become infinitesimally small, we find **the speed of the point** P to be:

$$\boxed{v = \frac{ds}{dt} = \frac{d}{dt}\,(R\,\theta) = R\,\frac{d\theta}{dt} = R\,\omega\ .} \tag{14.8}$$

The speed of a point on the rod is therefore proportional to the angular velocity of the rotation, but also proportional to the distance R to the rotational axis: Points further away from the rotation axis rotate with higher speeds.

What is the direction of the velocity vector for P? Fig. 14.5 shows that when the time interval Δt becomes smaller, the change in angle $\Delta\theta$ also becomes smaller, and the direction of the displacement vector from P at time t to P at time $t+\Delta t$ approaches that of a tangent to the circle. Excatly the same result we found earlier when we studied circular motion. The velocity vector is therefore parallel to the tangent to a circle of radius R, and points in the direction of the tangential unit vector $\hat{u}_T$. The velocity of the point P is therefore:

$$\mathbf{v} = R\,\omega\,\hat{u}_T\ . \tag{14.9}$$

Motion with Constant Angular Velocity

If an object rotates with a constant angular velocity, we can find the speed of the point P from the distance traveled during one complete revolution, $s = 2\pi R$, divided by the time of one revolution, call the period T:

$$v = \frac{s}{T} = \frac{2\pi R}{T}\ , \tag{14.10}$$

where R is the distance from P to the rotation axis. We also know that the velocity is $v = R\omega$, therefore we find that:

$$v = \frac{2\pi}{T}R = \omega R \ \Rightarrow\ \omega = \frac{2\pi}{T}\ . \tag{14.11}$$

The angular velocity is often also called the angular frequency.

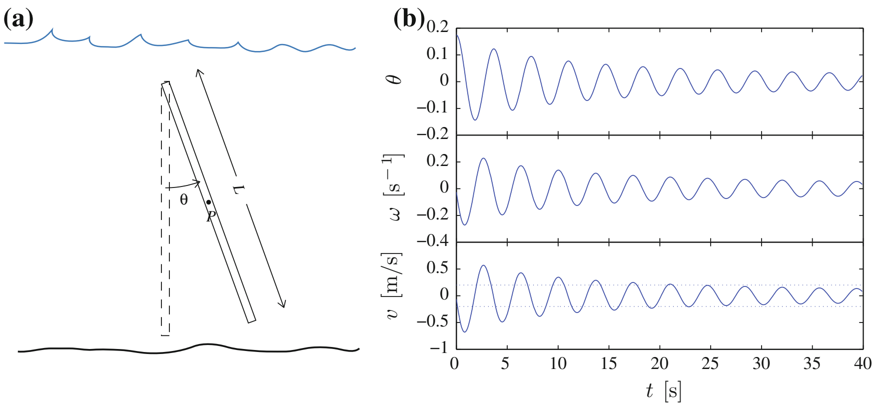

Fig. 14.6 **a** Illustration of the antenna. **b** Plots of the angle, θ, the angular velocity, ω, and the speed of the center of the antenna, v

14.3 Angular Acceleration

The rotation may occur at a constant angular velocity, as in Fig. 14.4, or the angular velocity may vary. By analogy with translational motion, we characterize the rate of change of the angular velocity by *the angular acceleration*, α, defined as:

Angular acceleration:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta} \ . \tag{14.12}$$

14.3.1 Example: Oscillating Antenna

Problem: You are using an underwater antenna to measure the electromagnetic response of the seafloor to search for hydrocarbons. The antenna has a length $L = 5$ m and is suspended from its top point above the seafloor (see Fig. 14.6a). You cannot use the antenna until the speed of its center is less than $v_c = 0.2$ m/s. You have attached a measurement device to the antenna that measures the angle θ the antanna forms with the vertical as a function of time. You measure the behavior after you have lowered it to the bottom, and the resulting data is in antennaangles.dat.[2] How long time does it take before you can use the antenna?

[2]http://folk.uio.no/malthe/mechbook/antennaangles.dat.

Approach: We read the data, find the angular velocity as a function of time by a numerical derivative, use this to find the speed of a point on the antenna, and find when the speed is less than the threshold v_c.

Solution: We read the file, getting a set of angles, $\theta(t_i)$, measured at discrete times, t_i:

```
from pylab import *
t, theta = loadtxt('antennaangles.dat',usecols=[0,1],unpack=True)
plot(t,theta)
```

The resulting angles $\theta(t_i)$ are shown in Fig. 14.6b. The angular velocity is the time derivative of the angle, but since we only know the angles for discrete times, we must calculate the derivative numerically:

$$\omega(t_i) \simeq \frac{\theta t_{i+1} - \theta t_i}{t_{i+1} - t_i} , \tag{14.13}$$

which is implemented as:

```
n = len(theta)
omega = zeros(n,float)
for i in range(n-1):
    omega[i]=(theta[i+1)-theta[i])/(t[i+1]-t[i])
plot(t,omega)
```

The velocity of a point P at the middle of the rod can be found from $v = \omega R$, where $R = L/2$ is the distance from the rotation axis to the point P. We calculate and plot the results:

```
R = 2.5
v = omega*R
plot(t,v)
```

From the figure we find that we need to wait for approximately $t \simeq 25$ s before we can use the antenna.

14.4 Comparing Linear and Rotational Motion

We have now introduced the angle θ, the angular velocity ω, and the angular acceleration α used to describe the rotational motion of an object. Our definitions are clearly similar to the definitions we used to introduce position, x, velocity v, and acceleration a for linear motion, as illustrated in Table 14.1. Notice the analoguous form of these equations: They are identical from a mathematical point of view. It is only our physical interpretation, and therefore also the units, which are different. The mathematical methods we use to determine motion are the same for linear and rotational motion. You can therefore use all the techniques you have developed to study linear motion also to address rotational motion.

Table 14.1 Comparison of linear and rotational motion

Motion	Linear	Rotation
Position	$x(t)$	$\theta(t)$
Velocity	$v(t) = \frac{dx}{dt}$	$\omega(t) = \frac{d\theta}{dt}$
Acceleration	$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

14.5 Solving for the Rotational Motion

The structured problem-solving approach we developed for linear motion is also applicable for rotational motion: We *identify* the quantities to be modeled; we *model* the system, resulting in a set of differential equations; we *solve* the differential equations; and *analyze the results. We will see that for rotational motion, we usually need to solve the equation of motion where the angular acceleration α is given:

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \alpha(t, \theta, \omega) \;, \tag{14.14}$$

with initial conditions $\theta(t_0) = \theta_0$ and $\omega(t_0) = \omega_0$. You should realize that this is *exactly* the same equation we have solved over and over again for linear motion, only with different symbols (and interpretations) for the variables. You can therefore use the machinery you have developed and are comfortable with from linear kinematics. You do not need to learn anything new!

Analytical Integration

For a given $\alpha(t)$ we know how to *solve* to determine the motion: We solve by direct integration. The angular velocity is the time integral of the angular acceleration:

$$\omega(t) - \omega(t_0) = \int_{\omega_0}^{\omega} d\omega = \int_{t_0}^{t} \frac{d\omega}{dt} dt = \int_{t_0}^{t} \alpha(t)\, dt \;. \tag{14.15}$$

And the angle is the integral of the angular velocity:

$$\begin{aligned} \theta(t) - \theta(t_0) &= \int_{t_0}^{t} \omega(t)\, dt = \int_{t_0}^{t} \left(\omega(t_0) + \int_{t_0}^{t} \alpha(t)\, dt \right) dt \\ &= \omega(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \alpha(t)\, dt \right) dt \;. \end{aligned} \tag{14.16}$$

For a constant angular acceleration $\alpha(t) = \alpha_0$ we find

$$\omega(t) - \omega(t_0) = \int_{t_0}^{t} \alpha_0 \, dt = \alpha_0 \, (t - t_0) \ , \tag{14.17}$$

$$\theta(t) - \theta(t_0) = \int_{t_0}^{t} \left(\omega(t_0) + \alpha_0 \, (t - t_0)\right) \, dt = \omega(t_0) \, (t - t_0) + \frac{1}{2}\alpha_0 \, (t - t_0)^2 \ . \tag{14.18}$$

Analytical Solution

If the angular acceleration instead is a function of θ or ω, we cannot integrate, but must instead solve the resulting differential equation. For example, if the angular acceleration is $\alpha = -C\omega$, and $\omega(0) = \omega_0$, then we must solve the equation

$$\frac{d\omega}{dt} = \alpha = -C\omega \ , \ \omega(0) = \omega_0 \ . \tag{14.19}$$

We now recoqnize this equation immediately, knowing from experience that the solution is just the same as we found for motion with air drag. The solution to this equation is on the form $\omega(t) = A \exp(-Ct)$, where A is determined from the initial condition: $\omega(0) = A = \omega_0$.

Symbolic solution: If you do not remember how to solve this equation, you can always use the symbolic solver in Python. Equation (14.19) is solved by

```
>> from sympy import *
>> omega = Function('omega')
>> t = symbols('t')
>> omega0 = symbols('omega0')
>> C = symbols('C')
>> dsolve(Derivative(omega(t),t)+C*omega(t),omega(t))
omega(t) == C1*exp(-C*t)
```

This is the same solution as we found above, we just need to determine the value of `C1` from the initial conditions.

Numerical Integration

These approaches work fine as long as we can solve the problems analytically. However, most problems of practical interest are not solveable, just like for linear motion. However, we can use the same approach and the same numerical methods to solve the equations of motion for the angular motion, as we have done for linear motion.

Euler-Cromer's method for angular motion: Euler-Cromer's method follows exactly the same scheme as for linear motion:

$$\omega(t_i + \Delta t) = \omega(t_i) + \Delta t\, \alpha(t_i) \tag{14.20}$$

$$\theta(t_i + \Delta t) = \theta(t_i) + \Delta t\, \omega(t_i + \Delta t)\ , \tag{14.21}$$

Motion of underwater antenna: We can demonstrate this method by determining the motion for an angular acceleration on the form, $\alpha = -c_1 \sin\theta - c_2\omega^2$, with initial conditions $\theta(0) = 10°$ and $\omega(0) = 0$. (This is a reasonable model for the motion of the antenna in Sect. 14.3.1. You will later learn how to find the parameters for such models.). We solve this problem through the following implementation of Euler-Cromer's method:

```
g = 9.81, L = 5.0, D = 60.0  # Parameters
M = L*(0.10)**2*5.0*1e3
c1 = 1.5*g/L
c2 = 0.75*(D*L)/M
time = 60.0
dt = 0.0001
omega0 = 0.0
theta0 = 10.0*pi/180.0
n = int(time/dt);            # Arrays
theta = zeros(n,float)
omega = zeros(n,float)
alpha = zeros(n,float);9
t = zeros(n,float)
theta[0] = theta0            # Initial conditions
omega[0] = omega0
for i in range(n-1):         # Integration loop
    alpha[i] = -c1*sin(theta[i])-c2*abs(omega[i])*omega[i]
    omega[i+1] = omega[i] + alpha[i]*dt
    theta[i+1] = theta[i] + omega[i+1]*dt
    t[i+1] = t[i] + dt
```

This simulation produces the motion seen in Fig. 14.6b.

14.5.1 Example: Revolutions of an Accelerating Disc

Problem: A DVD is accelerating at a constant rate of 2 rad/s^2 starting from rest. How many times have the disc rotated in 10 s?

Approach: We find the angle as a function of time, use this to find how far the disc has rotated in 10 s, and finally find how many rotations this corresponds to.

Solve: We find the angle as a function of time by first integrating the angular acceleration to find the angular velocity, and then integrating the angular velocity to find the angle. The disc rotates with a constant angular acceleration α starting at rest at $t_0 = 0$ s:

$$\omega(t) - \omega(0\,\mathrm{s}) = \int_0^t \alpha\, dt = \alpha t\ , \tag{14.22}$$

where $\omega(t_0) = 0\,\text{s}^{-1}$ since the disc starts at rest at $t_0 = 0\,\text{s}$. We find the angle θ by integration:

$$\theta(t) - \theta(0\,\text{s}) = \int_{0\,\text{s}}^{t} \omega(t)\,dt = \int_{0\,\text{s}}^{t} (\alpha t)\;dt = \frac{1}{2}\alpha t^2\;. \tag{14.23}$$

We insert $\alpha = 2\,\text{rad/s}^2$ to find the angle after $10\,\text{s}$:

$$\theta(10\,\text{s}) = \frac{1}{2} 2\,\text{s}^{-2}\,(10\,\text{s})^2 = \frac{1}{2}\,2 \times 100 = 100\;. \tag{14.24}$$

The angle θ is related to the number of rotations through $\theta = n2\pi$, where the angle 2π corresponds to one rotation. We find the number of rotations after $10\,\text{s}$ from:

$$n(10\,\text{s}) = \frac{\theta(10\,\text{s})}{2\pi} = \frac{100}{2\pi} \simeq 15.9\;. \tag{14.25}$$

That is, the disc rotates 15.9 times in $10\,\text{s}$.

14.5.2 Example: Angular Velocities of Two Objects in Contact

Problem: Two gears with radius R_A and R_B are rotating and in contact as illustrated in Fig. 14.7. The angular velocity of wheel A is ω_A. Find the angular velocity of wheel B. What is the relationship between the angular acceleration of wheels A and B?

Solution: First, we notice from the figure that the angular velocities must have opposite signs. Second, the condition that the two wheels are rotating without sliding means that their speeds at the point of contact must be equal, but oppositely directed.

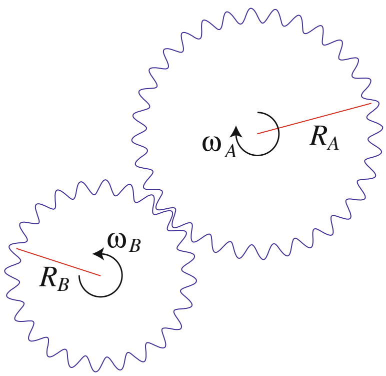

Fig. 14.7 Two gears with radius R_A and R_B are rotating without sliding relative to each other

We call such a condition a *kinematic condition*. We will frequently use such conditions when we solve problems in mechanics.

For each of the wheels, the velocity at the point of contact is related to the angular velocity by:

$$v_A = R_A \omega_A \; , \; v_B = R_B \omega_B \; . \tag{14.26}$$

Since the velocities at the point of contact are equal and oppositely directed, we find that the angular velocities also are related:

$$v_A = -v_B \;\Rightarrow\; \omega_B = -\frac{R_A}{R_B}\omega_A \; . \tag{14.27}$$

This relation is general and does not requires the velocities to be constant. We can therefore find the angular accelerations by taking the time derivatives on each side:

$$\alpha_B = \frac{d}{dt}\omega_B = -\frac{d}{dt}\frac{R_A}{R_B}\omega_A = -\frac{R_A}{R_B}\alpha_A \; . \tag{14.28}$$

14.6 Rotational Motion in Three Dimensions

You now know how to describe rotations in a plane: Rotations around the origin in the xy-plane are described by the rotation angle θ, the angular velocity ω, and the angular acceleration α. We found that a point, P, on the rotating object follows a circular path with a constant distance R from the rotation axis as illustrated in Fig. 14.8, and that the velocity of P is:

$$v = R\omega \; , \tag{14.29}$$

directed along the tangent to the circle. Can we find a simple expression for both the direction and magnitude of the velocity? Yes! Since we know that the tangent

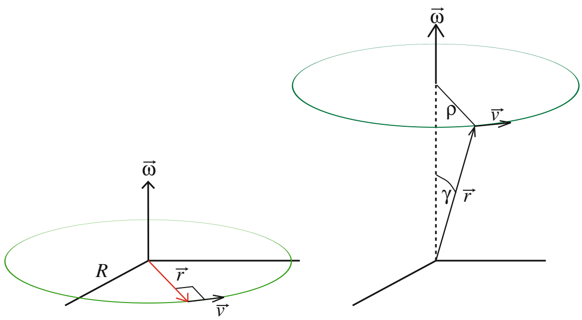

Fig. 14.8 An illustration of rotation around the z-axis. The direction of the velocity vector is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

is normal to the radius vector, $\mathbf{r}$, and since it is in the xy-plane, the tangent is also normal to the unit vector in the z-direction. This allows us to write the velocity vector of the point P as:

$$\mathbf{v} = \omega \mathbf{k} \times \mathbf{r} \ . \tag{14.30}$$

This expression provides the correct magnitude and direction of the velocity vector: It points in the positive rotational direction if ω is positive, and in the negative rotational direction if ω is negative. We can therefore introduce the angular velocity as a vector:

$$\boxed{\boldsymbol{\omega} = \omega \mathbf{k} \ \ \text{(angular velocity vector)}} \tag{14.31}$$

The angular velocity vector, $\boldsymbol{\omega}$, points along the axis of the rotation. For rotation in the xy-plane, the rotation axis is the z-axis.

The expression in (14.30) is valid not only for rotation in the xy-plane, but for any rotation around the axis given by the $\boldsymbol{\omega}$ vector. This is illustrated in the right part of Fig. 14.8, where the velocity of a point at $\mathbf{r}$ going in a circular orbit around the rotation axis is given by the radius of the circle, ρ, and the angular velocity, ω:

$$v_\theta = \omega \rho \ . \tag{14.32}$$

We see from the geometry that the radius ρ is $r \sin(\gamma)$, where γ is the angle between the position vector, $\mathbf{r}$ and the angular velocity vector, $\boldsymbol{\omega}$. Therefore

$$v_\theta = \omega r \ \sin(\gamma) \ . \tag{14.33}$$

The direction of the velocity is tangential to the circular orbit: orthogonal to the position vector, $\mathbf{r}$, and to the angular velocity vector, $\boldsymbol{\omega}$. This means that we can write the velocity as

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \ ,} \tag{14.34}$$

which gives both the correct magnitude: $|\mathbf{v}| = \omega r \sin(\gamma)$, and the correct direction. However, you can only use this expression when the origin of the position vector is *on* the rotation axis.

Let us use (14.34) to find the acceleration of the point P, which moves in a circular orbit with constant radius $\rho = R \sin(\gamma)$ around the rotation axis given by the angular velocity vector $\boldsymbol{\omega}$. The acceleration vector is the time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\boldsymbol{\omega} \times \mathbf{r} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} \ , \tag{14.35}$$

but we know that:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \ , \tag{14.36}$$

and we introduce the angular acceleration vector, $\boldsymbol{\alpha}$ as:

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} , \tag{14.37}$$

which inserted back into (14.35) gives:

$$\boxed{\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \ .} \tag{14.38}$$

For a motion with constant angular velocity, that is with constant speed, the angular acceleration vector is zero: $\boldsymbol{\alpha} = 0$, and the acceleration vector is:

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \ . \tag{14.39}$$

This is the *sentripetal acceleration on vector form*.

- The direction of the vector is correct. If you use the right-hand rule to find the vector product, you realize the vector points orthogonally inwards towards the rotation axis.
- The magnitude of the acceleration is $|\mathbf{a}| = a = \omega^2 \rho = (v/\rho)^2 \, \rho = v^2/\rho$, which is the magnitude of the sentripetal acceleration we found in (14.35).

14.6.1 Example: Velocity and Acceleration of a Conical Pendulum

Problem: A conical pendulum consists of a mass in a string of length L. The mass rotates with angular velocity ω in a circular orbit with radius R, as illustrated in Fig. 14.9. Find the velocity and acceleration of the mass.

Solution: The vector velocity of the conical pendulum is given as $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. We find the velocity when the pendulum crosses the x-axis. In this case, the position vector is

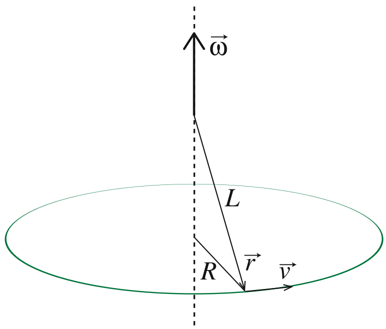

Fig. 14.9 Illustration of a conical pendulum

$$\mathbf{r} = R\,\mathbf{i} + \sqrt{L^2 - R^2}\,\mathbf{k}\ . \tag{14.40}$$

The rotation is about the z-axis. We therefore introduce the angular velocity vector as $\boldsymbol{\omega} = \omega\,\mathbf{k}$.

We find the vector velocity:

$$\begin{aligned}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} &= \omega\,\mathbf{k} \times \left(R\,\mathbf{i} + \sqrt{L^2 - R^2}\,\mathbf{k}\right) \\ &= \omega R\,(\mathbf{k} \times \mathbf{i}) = \omega R\,\mathbf{j}\ .\end{aligned} \tag{14.41}$$

Similarly, we find the acceleration vector:

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = 0 + \omega\,\mathbf{k} \times (\omega R\,\mathbf{j}) = -\omega^2 R\,\mathbf{i}\ . \tag{14.42}$$

The acceleration vector points in towards the axis of rotation, as expected.

Summary

Description of rotation:

- The rotation of an object is described by the angle $\theta(t)$
- The angular velocity of the object is: $\omega(t) = d\theta/dt$
- The angular acceleration of the object is: $\alpha(t) = d\omega/dt = d^2/dt^2$.

Rotation and translation:

- A point on the rotating body at a distance R from the rotational axis has a tangential velocity: $v = R\,\omega$.

Solving rotational motion:

- We solve problems in rotations using the same structured approach as for linear motion.
- In the "Solver" we solve the equation: $d^2\theta/dt^2 = \alpha(t, \theta, d\theta/dt)$ with the initial conditions $\theta(t_0) = \theta_0$ and $\omega(t_0) = \omega_0$.

Numerical solution:

- Numerically, we solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$\begin{aligned}\omega(t_i + \Delta t) &= \omega(t_i) + \Delta t\,\alpha(\theta(t_i), \omega(t_i), t_i) \\ \theta(t_i + \Delta t) &= \theta(t_i) + \Delta t\,\omega(t_i + \Delta t)\end{aligned}$$

Analytical solution:

- When the angular acceleration, $\alpha = \alpha(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$\omega(t) = \omega(t_0) + \int_{t_0}^{t} \alpha(t)\, dt \ , \ \theta(t) = \theta(t_0) + \int_{t_0}^{t} \omega(t)\, dt \ ,$$

  A typical example is motion with constant angular acceleration.
- For motion with **constant angular velocity** the solution is:

$$\theta(t) = \theta(t_0) + \omega\,(t - t_0) \ .$$

- For motion with **constant angular acceleration** the solution is:

$$\begin{aligned} \omega(t) &= \omega(t_0) + \alpha\,(t - t_0) \\ \theta(t) &= \theta(t_0) + \omega(t_0)\,(t - t_0) + \frac{1}{2}\alpha\,(t - t_0)^2 \ . \end{aligned}$$

Rotational motion in three dimensions:

- Generally, rotations occur around an axis, given by the angular velocity vector, $\boldsymbol{\omega}$.
- The velocity of a point on the rotating object at position $\mathbf{r}$ is: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$
- The acceleration of a point on the rotating object at position $\mathbf{r}$ is: $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

Exercises

Discussion Questions

14.1 Flywheel. What is the acceleration of a point at a rim of a rotating flywheel when the flywheel is rotating at a constant rate and when the flywheel is speeding up?

14.2 Spin cycle. Explain the working of the spin cycle of a washing machine in terms of acceleration components.

14.3 Degrees and radians. Why do we use radians and not degrees to describe angles in rotational motion?

Problems

14.4 Flywheel position. The angular position of a flywheel on an engine is given as $\theta = c_1 (t/t_1) + c_2 (t/t_2)^2$, where c_1, and c_2 are dimensionless constants, and t_1 and t_2 are characteristic times.
(**a**) Find an expression for the flywheel's angular velocity.
(**b**) Find an expression for the flywheel's angular acceleration.

14.5 Unbalanced wheel. The angular position of an unbalanced wheel is given as $\theta = 5.0\,\text{rad}\,\sin(t/(2\,\text{s}))$.
(**a**) Find an expression for the wheel's angular velocity.
(**b**) Find an expression for the wheel's angular acceleration.

14.6 Earth and Sun. (**a**) What is the angular velocity of the Earth in its orbit around the Sun?
(**b**) What is the angular velocity of the Earth in its rotation about its own axis?

14.7 Engine. A car engine accelerates from 1000 to 2000 rpm at a constant rate during 15 s.
(**a**) Find the angular acceleration of the engine.
(**b**) Find the number of rotations the engine revolves from it starts at 1000 rpm until it has accelerated to 2000 rpm.

14.8 Spinning down. You start a spinning wheel by rotating it with an angular acceleration of 10 rad/s^2 for 3 s.
(**a**) What is the angular velocity of the wheel as a function of time for the first 3 s.
(**b**) Find the angle of the wheel as a function of time for the first 3 s.
After releasing the spinning wheel, it slows down at a constant rate of 0.1 rad/s^2.
(**c**) What is the angular velocity of the wheel as a function of time after the first 3 s.
(**d**) Find the angle of the wheel as a function of time after the first 3s.
(**e**) How long time does the wheel take to stop?
(**f**) How much longer would the wheel take to stop if you spun it for 6 seconds instead?

14.9 A slippery wheel. You are testing the behavior of a car wheel in a river bed. The angular acceleration of the wheel spinning semi-saturated in water is $\alpha = -k_\omega \omega$, where $k_\omega = 0.1\,\text{s}^{-1}$ for the wheel you are testing.
(**a**) The wheel starts with the angular velocity $\omega_0 = 10$ rad/s when you put the wheel into the water. Find the angular velocity of the wheel as a function of time.
(**b**) How long time does it take until the wheel has $1/10$th of its initial angular velocity?

14.10 Running the curve. A sprinter is running through a circular curve with radius 50 m with a constant speed of 10 m/s.
(**a**) What is the angular velocity of the sprinter?
(**b**) What is the angular acceleration of the sprinter?
(**c**) What is the linear acceleration of the sprinter?

14.11 Rotating Earth. The radius of the Earth is approximately 6378 km.
(**a**) What is the angular velocity of the Earth as it rotates around its own axis?
(**b**) What is the angular velocity of a person on the equator?
(**c**) What is the linear velocity of a person on the equator?
(**d**) What is the angular velocity of a person at 60° North?
(**e**) What is the linear velocity of a person at 60° North?
(**f**) What is the angular acceleration of a person on the equator and at 60° North?
(**g**) What is the linear acceleration of a person on the equator? Compare with $g = 9.8\,\text{m/s}^2$ and comment.
(**h**) What is the linear acceleration of a person at 60° North? (Find both the magnitude and direction of the acceleration.) Comment on the results.

14.12 Rolling wheel. A wheel of radius R is rolling without slipping along a flat surface. The center of the wheel is moving with the constant horizontal velocity v.
(**a**) Show that for the wheel not to slip, the angular velocity of the wheel must be $\omega = v/R$.
(**b**) What is the velocity of the point on the wheel that is in contact with the ground? (Measured in a coordinate system on the ground.)
(**c**) What is the velocity of a point on the top of the wheel?
(**d**) What is the acceleration of the point on the wheel that is in contact with the ground? (Relative to the ground).
(**e**) What is the acceleration of the point on the top of the wheel?

Chapter 15
Rotation of Rigid Bodies

We have found that the motion of the center of mass of a multi-particle system can be determined from the external forces acting on the system. The general form of Newton's second law allows us to find the motion from our knowledge of forces and force models for multi-particle systems, just as we have systematically done for particle systems previously. We therefore have a well developed framework to determine the motion of the center of mass.

But what about the internal motion of the system, the internal motion relative to the center of mass? If you kick a football, it spins and wobbles on its path. How can we determine the configuration of the football—how it is rotated and deformed at a particular point along its path? Unfortunately, we generally cannot determine the configuration without having a detailed model for the particle system. We need to model the internal motion of the various parts of the football to address its deformation and rotation. Even in the simplest multi-particle system we can think of, the diatomic molecule, energy is partitioned in a non-trivial manner, including both kinetic and potential energy contributions for the deformation of the molecule.

Fortunately, in many cases we can simplify the system significantly by assuming that an object behaves as a rigid body that does not deform. A solid sphere, a stiff rod, a ball with little deformation, or a molecule with little internal deformation, may in many cases be approximated as a rigid body—as a body without any internal deformation. This simplifies our description significantly: If the body does not deform, there are no internal energies associated with the deformation either. A rigid body can only be translated or rotated. In this chapter we will introduce the kinetic and potential energy of a rigid body. We find that it depends on the rotational inertia of a rigid body, called the moment of inertia. Rigid bodies with large moments of inertia require more energy to spin than rigid bodies with smaller moments of inertia. This gives us the tools we need to apply energy considerations to systems with rotating parts.

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15.1 Rigid Bodies

A rigid body is a solid body that is not deformed. What does this mean? It means that:

> The distance between any two points in a **rigid body** does not change.

What possibilities does that leave us with?

- We may **translate** the body. If we move all the points by the same amount, we have not changed the distanced between any points.
- We may **rotate** the body, because any rotation does not deform the body—it only changes its orientation in space.

A rigid body can have any shape, and it may also be a very simple object. For example, an object consisting of two point masses at a fixed distance d is a rigid body, and a reasonable model for a diatomic molecule if the internal deformation is negligible. A rigid body is therefore a good approximation for a solid sphere, but not for a soap bubble.

15.2 Kinetic Energy of a Rotating Rigid Body

For a rigid body, we can find a simplified expression for the kinetic energy. Previously (see (13.61)), we found that the kinetic energy of a body can be expressed as:

$$K = \frac{1}{2}MV^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 \ , \tag{15.1}$$

where the first term is related to the motion of the center of mass and describes the kinetic energy for the translational motion. The second term is related to the motion relative to the center of mass. For a rigid body, the only way a part of the body can move relative to the center of mass is by rotation. We therefore interpret the second term as kinetic energy for the rotational motion, and we will here introduce the kinetic energy for rotating objects. An object can rotate either around a fixed axis or around a moving axis such as a moving center of mass. Here, we first address the behavior of a rigid body rotating around a fixed axis, and then generalize to the case of an axis following the center of mass motion.

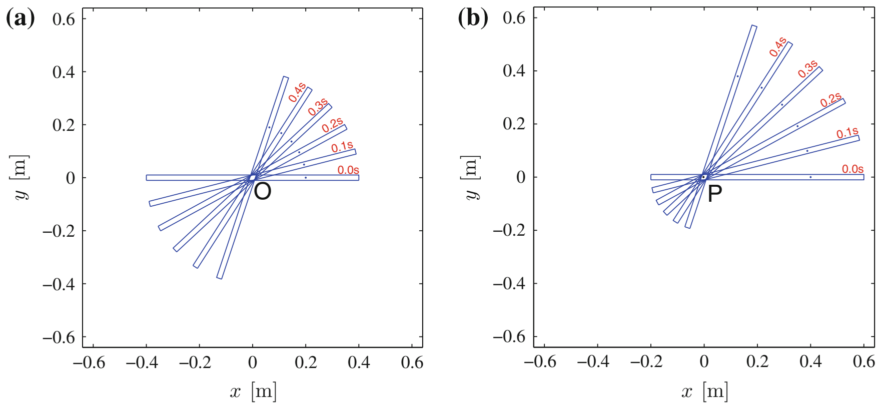

Fig. 15.1 **a** A rod is rotating around an axis O through its center of mass. **b** A rod is rotating around an axis P through another point P along the rod

Rotation Around a Fixed Axis

The rotation of an object around a fixed axis is illustrated by the motion of a rotating rod in Fig. 15.1. As we saw when we discussed rotations, the rod may rotate around an axis through its center of mass (the axis O in the figure), or it may rotate around an axis through some other point along the rod (the axis P in the figure). These two rotations are clearly different. It is therefore important to realize what we mean by a rotation around an *axis*: We mean that the object is rotating around a line. A line can be specified by a point and a vector. It is not sufficient to specify the rotation by a vector alone. In both cases in Fig. 15.1 the object is rotating with the angular velocity $\boldsymbol{\omega}$. The two axes are parallel, but they go through different points. You should therefore always ensure that when you make a sketch of a rotating system, you clearly mark what point or what axis the object is rotating about.

In order to determine the kinetic energy of the rigid body we assume that the body consists of small mass-points with masses m_i and positions, $\mathbf{r}_i$, where the positions are measured relative to an origin placed on the rotation axis. The whole body is rotating with the angular velocity $\boldsymbol{\omega}$ around the fixed axis through the origin, O. We place our coordinate system so that the z-axis points in the direction of $\boldsymbol{\omega}$: $\boldsymbol{\omega} = \omega\,\mathbf{k}$,

What is the velocity of point i on the body? From our discussion of rotations, we know that the velocity of a point on a rotating body at a position $\mathbf{r}_i$ is:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \ . \tag{15.2}$$

We can simplify the description by decomposing the position into two vectors: the vector $\boldsymbol{\rho}_i$, directed normal to the z-axis, and the vector $z_i\,\mathbf{k}$ directed along the z-axis as illustrated in Fig. 15.2:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i\,\mathbf{k} \ . \tag{15.3}$$

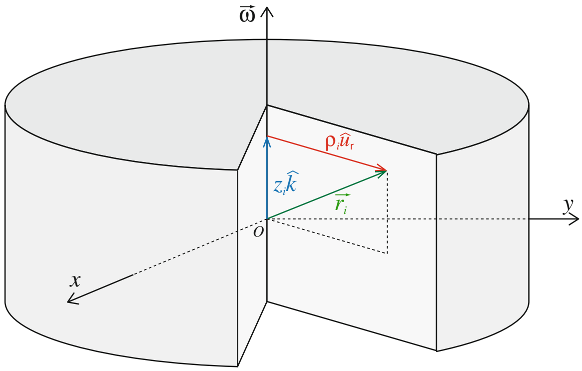

Fig. 15.2 Illustration of an object rotating around the axis O. We choose our coordinate system so that the axis O goes through the origin and it parallel to the z-axis

Using this expression, the velocity is:

$$\mathbf{v}_i = \omega\,\mathbf{k} \times (\boldsymbol{\rho}_i + z_i\,\mathbf{k}) = \omega\,\mathbf{k} \times \boldsymbol{\rho}_i + \omega z_i\,\underbrace{\mathbf{k} \times \mathbf{k}}_{=0} = \omega\,\mathbf{k} \times \boldsymbol{\rho}_i \ . \tag{15.4}$$

We write:

$$\boldsymbol{\rho}_i = \rho_i\,\hat{u}_\rho \ , \tag{15.5}$$

where $\hat{u}_\rho$ is a unit vector pointing radially out from the axis to the point at $\mathbf{r}_i$. The velocity is therefore:

$$\mathbf{v}_i = \omega\rho_i\,\underbrace{\left(\mathbf{k} \times \hat{u}_\rho\right)}_{=\hat{u}_T} = \omega\rho_i\,\hat{u}_T \ , \tag{15.6}$$

where the unit vector $\hat{u}_T$ points in the tangential direction.

In order to find the kinetic energy, we only need the magnitude of the velocity for each point:

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i\,(\omega\rho_i)^2 = \frac{1}{2} \underbrace{\left(\sum_{i=1}^{N} m_i \rho_i^2\right)}_{=I_O} \omega^2 \ , \tag{15.7}$$

where we have introduced the quantity I_O, which we call the *moment of inertia* of the rigid body about the axis O:

$$I_O = \sum_{i=1}^{N} m_i \rho_i^2 \ . \tag{15.8}$$

And we can write the kinetic energy for the rotating body as:

Kinetic energy for rotation about the axis 0:

$$K = \frac{1}{2} I_O \omega^2 \ . \tag{15.9}$$

Notice that the moment of inertia is a property of the object and the axis of rotation: It does not depend on how the object rotates around this axis. We interpret the moment of inertia as the **rotational inertia** of an object around an axis. The rotational inertia tells us how difficult it is to spin an object: The larger the moment of inertia, the larger energy is needed to obtain a certain angular velocity.

The expression $K = (1/2) I_O \omega^2$ in (15.9) is analogous to the expression for the translation kinetic energy for an object, $K = (1/2) M v^2$. The mass M appears in one equation, while I_O, which involves both the mass and how it is distributed around an axis, occurs in the other.

Rotation Around an Axis Through the Center of Mass

Figure 15.3 illustrates the motion of a rod thrown across a room: An example we used to introduce rotational motion in Chap. 14. In this case the object is subject to both translational and rotational motion. The kinetic energy of this object is given by (15.1) as:

$$K = \frac{1}{2} M V^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 \ . \tag{15.10}$$

If the rod is a rigid body, the only possible motion relative to the center of mass is a rotational motion, as illustrated by the motion of the rod seen in the center of mass system in the bottom right part of Fig. 15.3. We can then use exactly the same argument we used above for rotation around a fixed axis, and find the kinetic energy for rotation around an axis through the center of mass is:

$$\sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_{\mathrm{cm},i}^2 = \frac{1}{2} I_{cm} \omega^2 \ . \tag{15.11}$$

The total kinetic energy is therefore:

$$K = \frac{1}{2} M V^2 + \frac{1}{2} I_{cm} \omega^2 \ . \tag{15.12}$$

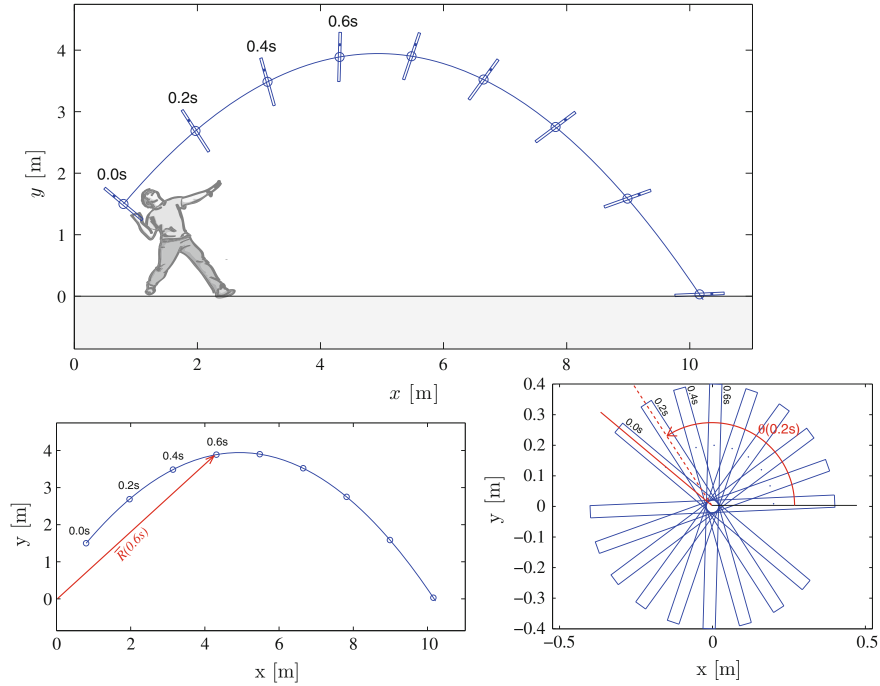

Fig. 15.3 A rod is thrown across the room. The *top* figure shows the motion of the rod. The *bottom left* figure shows the motion of the center of mass of the rod, and the *bottom right* figure shows the motion of the rod relative to the center of mass

15.3 Calculating the Moment of Inertia

The moment of inertia around an axis O of a system of particles with masses m_i at positions $\mathbf{r}_i$ is:

The **moment of inertia around an axis O:** is defined as

$$I_O = \sum_i m_i \rho_i^2 \ . \tag{15.13}$$

where ρ_i is the distance from particle i to the axis O.

Notice that the moment of inertia is not only a property of the object, it also depends on the axis of rotation, and the moment of inertia of a given object around different axes may be different, as illustrated below. A common mistake is to forget that the moment of inertia depends both on the object and on the axis, and that the moment of inertia changes if we change the axis.

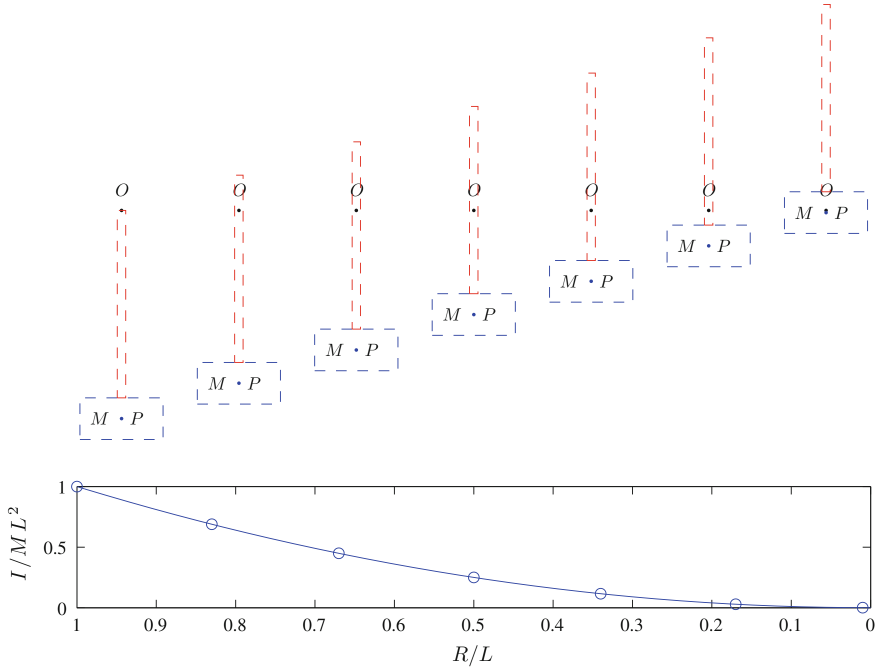

Fig. 15.4 Illustration of a hammer where all the mass is located in the point P in the head of the hammer. The hammer rotates around the point O, and the moment of inertia of a hammer depends on the distance R from the rotation axis to the hammer head

The moment of inertia depends on both the mass and how it is distributed around the rotation axis. Figure 15.4 illustrates a hammer consisting of a heavy head of mass M attached to an effectively massless rod. (You can assume that all of the mass of the head is located in the marked point, P). The moment of inertia of the hammer is

$$I_O = M R^2 , \tag{15.14}$$

where R is the distance from the head, P, to the rotation axis O. We can therefore change the moment of inertia by changing the mass of the head, or by changing the distance from the head to the rotation axis. If you rotate it as a pendulum from your hand this corresponds to changing your grip (point O), since this will change the position of the rotation axis. As you move your grip closer to the head, the moment of inertia becomes smaller, and it becomes easier (requires less energy) to give the hammer a certain angular velocity. Pulling mass in towards the axis of rotation reduces the moment of inertia, pushing it further out from the axis increases it.

Solid Bodies

The mass of the hammerhead is not really located in a point P, it is distributed in space around this point. The moment of inertia of the hammerhead is therefore not zero even if the rotation axis is located at the center of the head. To calculate the moment of inertia of a continuous body—a solid—such as the hammerhead, we must sum all the contributions from a continuum of points corresponding to small volume elements ΔV_i with mass $\Delta m_i = \rho_M \Delta V_i$, where ρ_M is the local mass density of the solid. The sum over all points i in the solid becomes an integral when the size of a each element goes to zero:

$$I_O = \sum_i \Delta m_i \, \rho_i^2 = \sum_i \rho_M \, \Delta V_i \, \rho_i^2 \rightarrow \iiint \rho_M \, \rho^2 \, dV \ , \tag{15.15}$$

where the integral is over the solid body. The moments of inertia around the center of mass of various solid bodies are shown in Fig. 15.5. Examples of how to calculate the moment of inertia of an object using integration are shown below.

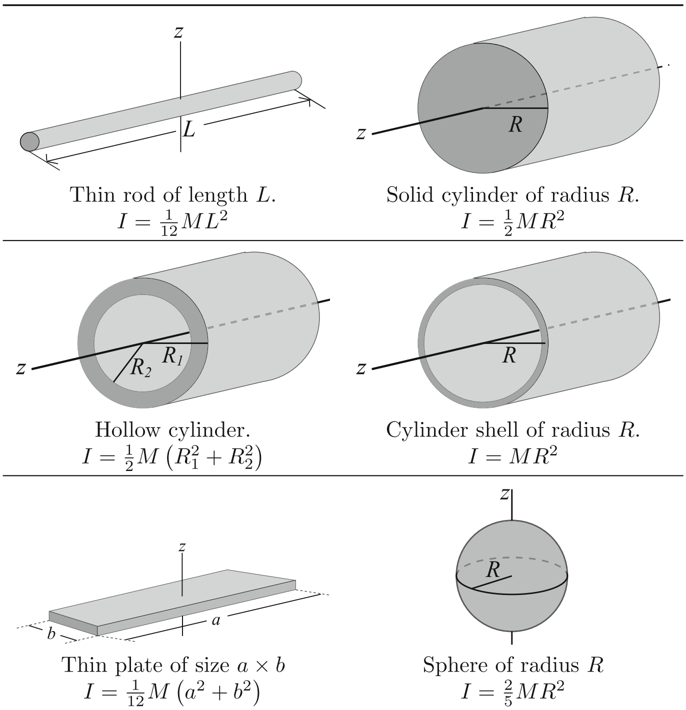

Fig. 15.5 Moments of inertia for various solid bodies

The moment of inertia of a hammerhead of dimensions $h \times w \times d$ (h is height, w is width, and d is depth) around its center O can be found from Fig. 15.5, where we see that the moment of inertia of a plate around its center is

$$I_{cm} = \frac{1}{12} M \left(h^2 + w^2 \right) \; . \tag{15.16}$$

This is therefore the real moment of inertia of the hammer when $R = 0\,\text{m}$ in the example above.

Parallel-Axis Theorem

We have now found a way to calculate the moment of inertia for a continuous object such as the hammer. But what if we now want to know the moment of inertia of the hammerhead for a different rotation axis—what if we want to know it as a function of the distance, R, from the center of the hammerhead to the rotation axis as illustrated in Fig. 15.6?

We could of course calculate the moment of inertia using the integral formulation in (15.15) for each position of the rotation axis. But this is cumbersome. Fortunately, it turns out that it is sufficient to calculate the moment of inertia around an axis through the center of mass. From this moment of inertia, we can find the moment of inertia around any other parallel axis using the parallel-axis theorem:

Parallel-axis theorem: The moment of inertia, I_O, of an object around an axis O is related to the moment of inertia, I_{cm}, of the object around a parallel axis through the center of mass of the object by:

$$I_O = I_{cm} + M s^2 \; , \tag{15.17}$$

where M is the mass of the object, and s is the distance between the axis O and the parallel axis through the center of mass.

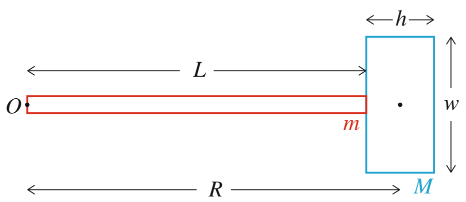

Fig. 15.6 A hammer consisting of a hammerhead (*blue*) of mass M and a thin rod (either massless or of a mass m)

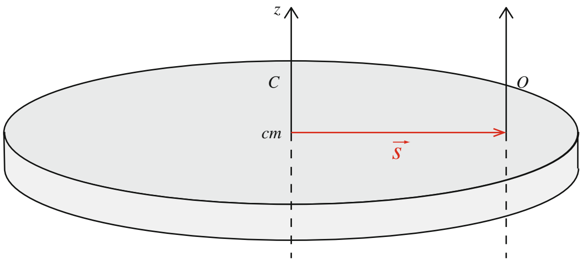

Fig. 15.7 Illustration of the parallel-axis theorem. The moment of inertia around an axis C through the center of mass can be used to find the moment of inertia around an axis O—if the axis O is parallel to the axis C through the center of mass. The vector **s** is perpendicular to both axes, and points from the origin of axis C to the origin of axis O

(You can find a proof in Sect. A.5). The theorem is illustrated in Fig. 15.7. We can use the parallel-axis theorem to find the moment of inertia around any axis if we only know the moment of inertia around the center of mass. This is why you usually only find tabulated the moment of inertia of an object around its center of mass: Given this, you can use the parallel-axis theorem to find the moment of inertia for any other parallel axis.

We can use the parallel-axis theorem to find the moment of inertia of the hammer as a function of the distance R from the center of the hammerhead to the rotation axis—it is simply given by:

$$I_O = I_{cm} + MR^2 , \tag{15.18}$$

as long as the two rotation axes are parallel. This allows us to calculate the moment of inertia for any R—as illustrated in Fig. 15.4.

Superposition Principle

Still we are not satisfied with our characterization of the hammer, because it consists of two pieces: hammerhead and shaft. So far we have assumed that the mass of the shaft is negligible and that we therefore can neglects its moment of inertia. What if cannot neglect it—what to do then? We are saved by a principle called the superposition principle: For an object that consists of several parts, such as the hammerhead and the shaft of a hammer, we can find the moment of inertia around an axis by summing the moments of inertia for each part around the same axis:

The moment of inertia of two systems A and B round the axis O is the sum of the moments of inertia for each part of the systems:

$$I_{O,AB} = I_{O,A} + I_{O,B} \; . \tag{15.19}$$

(You can find a proof in Sect. A.4). We can therefore find the moment of inertia of a compound object by summing the moments of inertia for each of the object. Just be careful to ensure that you sum moments of inertia *around the same axis*.

This allows us to find the moment of inertia of the hammer illustrated in Fig. 15.6 as the sum of the moment of inertia of the hammerhead and the shaft:

$$I_{TOT,O} = I_{h,O} + I_{s,O} \; , \tag{15.20}$$

where we have already found that the moment of inertia of the hammer is:

$$I_{h,O} = \frac{1}{12} M \left(h^2 + w^2 \right) + MR^2 \; . \tag{15.21}$$

The moment of inertia of the shaft around the point O can be found just as for the hammerhead. The moment of inertia around the center of mass for the shaft is found from Fig. 15.5:

$$I_{s,cm} = \frac{1}{12} mL^2 \; , \tag{15.22}$$

and we use the parallel-axis theorem to find its moment of inertia around point O, which is at a distance $L/2$ from the center of mass:

$$I_{s,O} = \frac{1}{12} mL^2 + m \left(\frac{L}{2} \right)^2 = \frac{1}{3} mL^2 \; . \tag{15.23}$$

The total moment of inertia of the hammer around the point O is therefore:

$$I_{TOT,O} = \frac{1}{12} mL^2 + \frac{1}{12} M \left(h^2 + w^2 \right) + MR^2 \; , \tag{15.24}$$

where $R = L + h/2$ for this configuration.

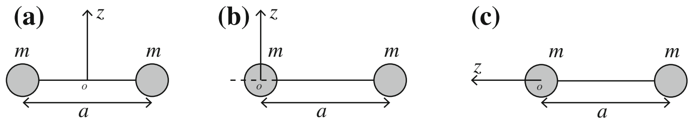

Fig. 15.8 A system of two identical point particles with masses m are attached with a rigid, massless rod of length a. **a** The rotation axis goes through the center of mass. **b** The rotation axis goes through the *left*-most particles. **c** The rotation axis goes through both particles

15.3.1 Example: Moment of Inertia of Two-Particle System

Problem: Find the moment of inertia around the indicated axes in Fig. 15.8.

Solution: In case (**a**) the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m\left(-\frac{1}{2}a\right)^2 + m\left(\frac{1}{2}a\right)^2 = \frac{1}{2}ma^2 \ . \tag{15.25}$$

In case (**b**) the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m0^2 + ma^2 = ma^2 \ . \tag{15.26}$$

Notice that the point on the axis does not give any contribution to I_z, but that the moment of inertia is larger, because the other particle is further away from the origin. In case (**c**), the moment of inertia around the axis z is:

$$I_z = \sum_{i=1}^{2} m_i \rho_i^2 = m0^2 + m0^2 = 0 \ . \tag{15.27}$$

15.3.2 Example: Moment of Inertia of a Plate

Problem: Find the moment of inertia of a homogeneous plate with dimensions a and b and of thickness h around an axis z through the center of mass, as illustrated in Fig. 15.9.

Solution: The moment of inertia of a solid body around the axis O is defined as:

$$I_z = \int_M \rho^2 \, dm = \int\int\int \rho_M \rho^2 \, dV \ . \tag{15.28}$$

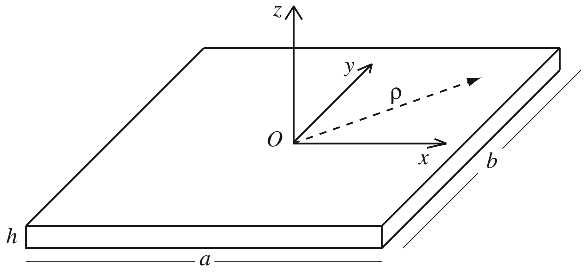

Fig. 15.9 A thin plate with sides a and b and thickness h. The coordinate system is placed at the center of mass, with the axis z in the direction of the thickness of the plate

Since the place is homogeneous, the mass density is uniform— constant in space. The distance ρ from the axis to the point $\mathbf{r}$ is simply the length of the horizontal (xy-plane) component of $\mathbf{r}$:

$$\rho^2 = x^2 + y^2 \ . \tag{15.29}$$

The moment of inertia is found from the integral:

$$I_z = \rho_M \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \int_0^h x^2 + y^2 \, dz \, dy \, dx = \rho_M h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 + y^2 \, dy \, dx \ , \tag{15.30}$$

where we integrate the two parts of the sum inidividually:

$$\begin{aligned} I_z &= \rho_M h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} x^2 \, dy \, dx + \rho_M \, h \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} y^2 \, dy \, dx \\ &= \rho_M hb \int_{-a/2}^{a/2} x^2 dx + \rho_M \, h \, a \int_{-b/2}^{b/2} y^2 \, dy \\ &= \rho_M hb \frac{1}{3} \left[\left(\left(\frac{a}{2}\right)^3 - \left(\frac{-a}{2}\right)^3 \right) + \left(\left(\frac{b}{2}\right)^3 - \left(\frac{-b}{2}\right)^3 \right) \right] \ , \end{aligned} \tag{15.31}$$

which we simplify to:

$$I_z = \frac{1}{3} \rho_M hba \left(\frac{a^2}{4} + \frac{b^2}{4} \right) = \frac{1}{12} \rho_M V \left(a^2 + b^2 \right) = \frac{1}{12} M \left(a^2 + b^2 \right) \ , \tag{15.32}$$

where we used that $V = abh$ and $M = \rho_M V$.

15.4 Conservation of Energy for Rigid Bodies

We have found the kinetic energy of a rigid body that is either rotating around a fixed axis or that is rotating around its moving center of mass. If the solid body is only affected by conservative forces, we may use energy conservation principles to determine the relation between angular velocity and position, just as we have done with translational velocity and position previously. But in order to employ energy conservation, we need to determine the potential energy of a rigid body.

Potential Energy for a Constant Gravitational Force

What is the potential energy for a rigid body due to the gravitational force? We found that for a point particle with mass m_i at the position $\mathbf{r}_i$, the potential energy due to a constant gravitational force directed along the y-axis is:

$$U_i = m_i g y_i \ . \tag{15.33}$$

We often call a constant gravitational force a homogeneous gravity, since gravity is the same everywhere. The total potential energy for multiparticle system of N particles is therefore:

$$U = \sum_{i=1}^{N} U_i = \sum_{i=1}^{N} m_i g y_i = g \underbrace{\sum_{i=1} m_i y_i}_{=MY} = M\,g\,Y \ , \tag{15.34}$$

where Y is the position of the center of mass. This result is general, and is also valid for a rigid body:

> Potential energy of a rigid body for a constant gravitational force
>
> $$U = M\,g\,Y \ . \tag{15.35}$$

Potential Energy Due to a Spring Force

A rigid body may also be affected by a force modelled by a spring force. This force may have several origins. It may represent a contact force due to an elastic contact; a contact force due to a rope or string attachment; an approximation for a more complex force acting on a single part of the rigid body; or as a force acting on all parts of the rigid body.

First, we start from the simplest case, where a spring is acting on a particular point on the rigid body. We illustate this situation by the contact between a sphere and an elastic floor for example while the rod is bouncing off the floor (see Fig. 15.10). We can then model the force from the floor on the sphere using a spring model, where

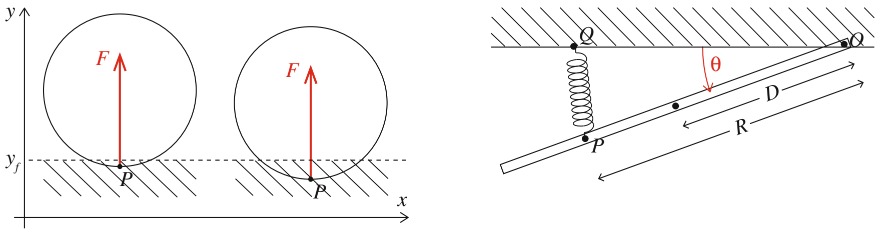

Fig. 15.10 *Left* Illustration of contact between a sphere and a surface. *Right* Illustration of a rod that can rotate around point O, but it is attached with a spring at point P. The other end of the spring is attached at the point Q

we assume that the force depends on the position of a particular point, the point P, on the sphere. If the surface is horizontal at $y = y_f$, the spring models a vertical normal force from the floor:

$$\mathbf{F} = -k(p_y - y_f) \ , \tag{15.36}$$

when the sphere is in contact with the floor, that is when $p_y < y_f$, where p_y is the vertical position of the point P. The potential energy of the sphere due to this spring force is then:

$$U(p_y) = \frac{1}{2}k\left(p_y - y_f\right)^2 \ . \tag{15.37}$$

For the sphere, we see that the position of point P is related to the center of mass of the sphere, $p_y = Y - R$, where Y is the vertical position of the center of mass, and R is the radius of the sphere.

This result is not only valid for a sphere, but for another body affected by a spring force. The potential energy of a spring force is related to the elongation of the spring. We therefore need to relate the motion of the rigid body to the elongation of the spring by relating for example the position of the center of mass of the body or the rotation angle θ of the body to the elongation of the spring. This is illustration by the rod on the right in Fig. 15.10: The rod can rotate freely around the point O, but is affected by a spring attached to the rod at the point P and to the wall at the point Q, so that when the rod is parallel to the wall, corresponding to $\theta = 0$, the spring is in its equilibrium position. The potential energy in the spring then depends on the distance between P and Q, which for small rotation angles θ can be approximated as $\Delta QP \simeq R\theta$, where R is the distance from P to O. The potential energy due to the spring force affecting the rotating rod is therefore:

$$U_s(\theta) = \frac{1}{2}k\,(\Delta QP)^2 = \frac{1}{2}kR^2\theta^2 \ . \tag{15.38}$$

If the rod also is subject to gravity, we must also add the potential energy due to gravity:

$$U_g(\theta) = -MgD\sin\theta \ , \tag{15.39}$$

where D is the distance from O to the center of mass of the rod. The total potential energy of the rod is then $U = U_s + U_g$.

If you want to use a spring force model for a rigid body, you must therefore consider how to describe it in detail. We will introduce several ways to model such interactions in the following, but generally you are left to your own ingenuity.

Conservation of Energy for a Rigid Body

How can we now put the pieces together and use conservation of energy of a rigid body relate its velocity (angular and translational) to its position (rotational and translational)? As long as the system is only subject to conservative forces, the total energy of a rigid body is:

$$E = K + U = \frac{1}{2}MV^2 + \frac{1}{2}I_{cm}\omega^2 + U \ . \tag{15.40}$$

For a *rigid* body the potential energy can only depend on the translational and rotational motion of the body and not on the internal deformation. We have therefore only included one term for the potential energy: The potential energy due to external forces.

If the object is rotating around a fixed axis—an axis that does not move—we can simplify the kinetic energy, getting

$$E = \frac{1}{2}I_O\omega^2 + U \ . \tag{15.41}$$

These two equations ((15.40) and (15.41)) forms the basis for using energy conservation to solve problems with rigid bodies.

15.4.1 Example: Rotating Rod

Problem: Find the angular velocity for a rod that rotates without friction about an attachment point at one of its ends. The rod starts in a horizontal position. You can neglect air resistance.

Approach: We use an energy argument to relate the angular velocity of the rod to its position based on the kinetic energy of a rotating rigid body and the potential energy of a solid body affected by gravity.

Solution: How can we address this problem with what we have learned so far? Can we use Newton's laws of motion? Not really, because we do not know the forces acting in the attachment point. Normal forces will be present in this point—but we do not have a model to determine their magnitude.

What about energy conservation? We know how the kinetic energy of the rod depends on the angular velocity, and we know the potential energy of the rod in the gravity field. But is the mechanical energy conserved in this case? There are no frictional forces and no air resistance. The only external forces acting are the normal force in the attachment point and gravity. The force acting on the attachment point does no work, because this point is not moving! And we know that the work done by gravity can be expressed as a potential energy. We can therefore use energy conservation to solve this problem!

Identify: We start from a sketch of the system in Fig. 15.11. The rod has a length L. Its position is given by the angle θ, and the rod starts at $\theta = 0$. Our task is to find the angular velocity as a function of θ. We choose a positive rotational direction, which means that the angle θ in Fig. 15.11 is negative!

Kinetic energy: The kinetic energy of a rigid rod rotating around the z-axis through O is given as:

$$K = \frac{1}{2} I_z \omega^2 \;, \tag{15.42}$$

where I_z is the moment of inertia for the rod around the axis z through O. For a rod rotating around a *fixed* axis, this is the only term for the kinetic energy.

Caution: Notice that for an object rotating around a fixed axis you use the term $K = (1/2)I\omega^2$ for the kinetic energy, where ω is the angular velocity around the fixed axis. If you instead want to seperate the kinetic energy into the motion of the

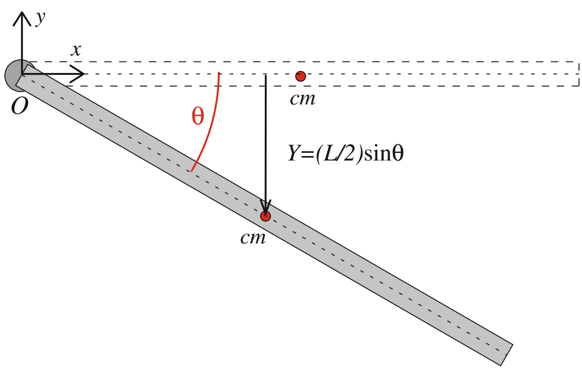

Fig. 15.11 A thin rod of length L and mass M attached at the point O. The orientation of the rod is given by the angle θ. The position of the center of mass of the rod is found from the geometry to be $Y = -(L/2)\sin\theta$

center of mass and the rotation around the center of mass, you have to figure out what the angular velocity around the center of mass is, which may not be the same as the angular velocity around O. A frequent mistake is the confuse the two cases of rotation around a fixed axis and rotation around the center of mass.

Potential energy: From (15.35) we found that the potential energy of a solid body due to gravity is

$$U = M\,g\,Y\ , \tag{15.43}$$

where Y is the vertical position of the center of mass of the object. For a homogenous rod, the center of mass is located at its geometric center, that is at a distance $L/2$ from the end. What is the vertical coordinate of this position? From Fig. 15.11 we see that of the center of mass is:

$$Y = \frac{L}{2}\sin\theta\ . \tag{15.44}$$

This is a negative number as long as θ is negative (and larger than $-\pi$). The potential energy of the rod is therefore:

$$U = -Mg\frac{L}{2}\sin\theta\ . \tag{15.45}$$

Solve: The total energy of the rod is conserved since all the forces are conservative (or not doing any work on the body), the total energy, $E = K + U$, is conserved. The total energy is therefore the same in the initial position, 0, when $\theta = \theta_0 = 0$, and in the position 1, when the angle is θ:

$$E_0 = K_0 + U_0 = E_1 = K_1 + U_1\ , \tag{15.46}$$

where

$$E_0 = K_0 + U_0 = \frac{1}{2}I_z\omega_0^2 + MgY_0 = 0\ , \tag{15.47}$$

and

$$E_1 = K_1 + U_1 = \frac{1}{2}I_z\omega_1^2 + MgY_1 = \frac{1}{2}I_z\omega^2 - \frac{L}{2}Mg\sin\theta\ . \tag{15.48}$$

Energy conservation, $E_0 = E_1$, gives:

$$\frac{1}{2}I_z\omega^2 - \frac{L}{2}Mg\sin\theta = 0\ \Rightarrow\ \frac{1}{2}I_z\omega^2 = \frac{L}{2}Mg\sin\theta\ \Rightarrow\ \omega = \pm\sqrt{\frac{MgL}{I_z}\sin\theta}\ , \tag{15.49}$$

where we get two solutions, a positive and a negative solution, depending which way the rod is swinging. Both directions are possible solutions: For every angle θ the rod will first swing one way, then swing the other way. Since there are no damping it will continue swinging back and forth indefinitely.

Analyze: We have found a general solution, valid for the rotation of any type of homogeneous staff of length L. The solution depends on the moment of inertia for the staff. If the staff is slim, we know that the moment of inertia around the center of mass is:

$$I_{\text{cm}} = \frac{1}{12} M L^2 \; , \tag{15.50}$$

but the rod is not rotating around the center of mass, instead we need to find the moment of inertia around the end-point of the rod, that is, around the axis z. We use the parallel-axis theorem to find this, since the rod is rotating around an axis parallel to an axis through the center of mass of the rod. The distance s from the center of mass to the axis z in point O is $L/2$. The parallel-axis theorem therefore gives:

$$I_z = I_{\text{cm}} + M s^2 = \frac{1}{12} M L^2 + M \left(\frac{L}{2}\right)^2 = \frac{1}{3} M L^2 \; . \tag{15.51}$$

For this rod, the angular velocity is:

$$\omega = \sqrt{\frac{MgL}{I_z} \sin\theta} = \sqrt{\frac{MgL}{\frac{1}{3} M L^2} \sin\theta} = \sqrt{3 \frac{g}{L} \sin\theta} \; . \tag{15.52}$$

Test your understanding: How would your results change if the rod was not homogeneous, but instead had all its mass located in a point at a distance L from the attachment point?

15.5 Relating Rotational and Translational Motion

For a wheel rolling along a surface or for a rope running over a spinning wheel, the rotational and translational motion of the objects are related: If the wheel is to roll without slipping the center of the wheel must move a distance equal to the path "rolled out" by the wheel, and for the rope to run over the spinning wheel without slipping the length of rope pulled from the wheel must correspond to the length a point on the wheel has moved (see Fig. 15.14). How do we introduce such relations between the rotation and the motion of the object?

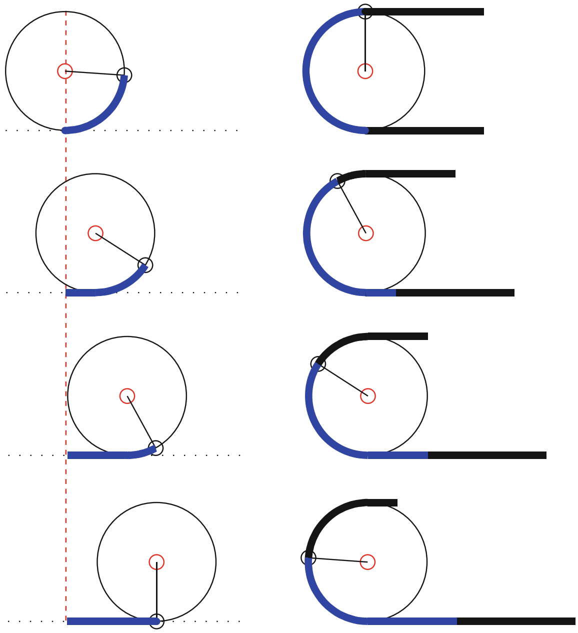

Fig. 15.12 Illustration of a wheel rolling without slipping (*left*) and a rope running over a spinning wheel without slipping (*right*)

Rolling Motion

The left part of Fig. 15.12 illustrates the motion of a wheel that is rolling without slipping. What does this mean? That it is rolling wihtout slipping? It means that the distance moved by the center of the wheel during a time interval Δt must correspond to the distance "rolled out"—the distance moved by a point on the surface of the wheel in the same time interval. Alternatively, we could say that the point P on the wheel that is in contact with the floor, does not move relative to the floor (see Fig. 15.13). This point must therefore have zero velocity relative to the floor. Let us find an expression for the point P. We do this by first finding the velocity of the point

Fig. 15.13 Illustration of rolling condition: The point P is at a position $\mathbf{p}_{cm} = -R\,\mathbf{j}$ relative to the center of mass

relative to the center of mass, and then adding the velocity of the center of mass to find the velocity relative to the ground. In the center of mass system (a system with an origin that moves with the center of mass), the wheel is rotating around the origin. The velocity of P relative to the center of mass is therefore the velocity of a point moving in a circle around the origing with angular velocity $\boldsymbol{\omega}$:

$$\mathbf{v}P, cm = \boldsymbol{\omega} \times \mathbf{p}_{cm} \ , \tag{15.53}$$

where $\mathbf{p}_{cm}$ is the position of point P relative to the center of mass. If we place the origin at the center of mass, we find that

$$\mathbf{p}_{cm} = -R\,\mathbf{j} \ , \tag{15.54}$$

as illustrated in Fig. 15.13. The angular velocity is:

$$\boldsymbol{\omega} = \omega\,\mathbf{k} \ , \tag{15.55}$$

and therefore

$$\mathbf{v}_{P,cm} = \omega\,\mathbf{k} \times (-R)\,\mathbf{j} = \omega R\,\mathbf{i} \ . \tag{15.56}$$

The velocity of point P relative to the ground is found by adding the velocity of the center of mass relative to the ground (using the Galilei-transformations):

$$\mathbf{v}_P = \mathbf{v}_{cm} + \mathbf{P, cm} = \mathbf{V} + \omega R\,\mathbf{i} \ . \tag{15.57}$$

The wheel is rolling if this is zero:

$$\mathbf{v}_P = \mathbf{V} + \omega R\mathbf{i} = 0 \Rightarrow \mathbf{V} = -\omega R\,\mathbf{i} \ . \tag{15.58}$$

Are the signs in this equation correct? Yes. If V_x is positive we see that ω must be negative, as expected.

Rolling condition: If the condition $V_x = -\omega R$ is satisfied, the wheel is rolling without slipping. This relation is called the *rolling condition*. If this condition is not satisfied, the contact between the wheel and the ground is moving relative to the ground, and we say that the wheel is *sliding*. This condition is often used to determine when a wheel starts rolling without sliding.

Caution: A common mistake is to use the rolling condition also in the case when the wheel is slipping. The rolling condition is only valid in the case of rolling without slipping.

15.5.1 Example: Weight and Spinning Wheel

Problem: A weight of mass m is hanging from a massless rope that is wound around a spinning wheel of mass M and radius R. The spinning wheel can rotate without friction around its attachment point at the center of the wheel. The weight is released from rest. Find the velocity v of the weight as a function of its height h. You may neglect air resistance.

Approach: We plan to use energy conservation for the (wheel+weight) system. Both the wheel and the weight has a kinetic energy, but only the potential energy of the weight changes as it descends. The velocity of the weight and the angular velocity of the wheel is related because they are connected by the rope. We solve to find how the velocities depend on the position of the weight.

Solution: This problem may be addressed in several ways: We could apply Newton's laws of motion, but then we need a law for rotational motion. We learn that in Chap. 16, and we return to this example then. The problem can also be addressed using energy conservation—energy is conserved for the complete system consisting of the spinning wheel, the rope, and the weight since the only external forces are gravity, which is a conservative force, and the force acting at the attachment point of the spinning wheel—and this force does no work since the spinning wheel does not move.

Identify: The system consists of the weight, the rope, and the spinning wheel (see Fig. 15.14). The weight of mass m has the position y and velocity v, with the positive direction of y chosed upwards. The spinning wheel rotates with an angular velocity ω around its attachment point at its center (of mass). Positive rotational direction is chosen according to the right hand rule, and is shown with the arrow indicating the angular velocity ω. The weight starts at the position $y_0 = 0$ with the velocity $v_0 = 0$.

Model: Because all the external forces acting on the system are either conservative (gravity) or not performing any work (the normal force on the axis of the spinning

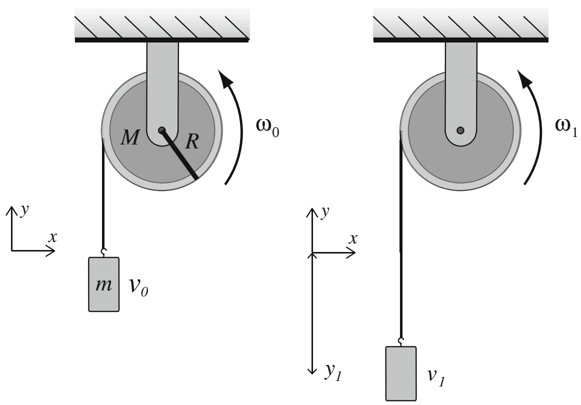

Fig. 15.14 A weight of mass m is attached to a massless rope wrapped around a spinning wheel of radius R and mass M

wheel), we can use energy conservation. The total energy is $E = K_m + K_W + U_m + U_w$, where $K_m = (1/2)mv^2$ is the kinetic energy of the weight, $K_w = (1/2)I\omega^2$ is the kinetic energy of the spinning wheel, $U_m = mgy$ is the potential energy of the weight (where $y < 0$ since we assume the weight to start at $y = 0$), and $U_w = Mgy_w$ is the potential energy of the spinning wheel (which is constant since center of mass of the spinning wheel is not moving). Since the rope does not have any mass and does not stretch, we do not need to include the kinetic or potential energy of the rope.

Solve: The total energy is calculated for two configurations: the initial configuration 0 where the velocity of the weight is $v_0 = 0$ and $\omega_0 = 0$, and the position 1 where the angular velocity of the spinning wheel is ω and the velocity of the weight is v:

$$\begin{aligned} E_0 &= K_{m,0} + U_{m,0} + K_{w,0} + U_{w,0} \\ &= \frac{1}{2}mv_0^2 + mgy_0 + \frac{1}{2}I\omega_0^2 + Mgy_w \\ &= 0 + 0 + 0 + Mgy_w \ , \end{aligned} \tag{15.59}$$

and

$$\begin{aligned} E_1 &= K_{m,1} + U_{m,1} + K_{M,1} + U_{M,1} \\ &= \frac{1}{2}mv^2 + mgy + \frac{1}{2}I\omega^2 + Mgy_w \ . \end{aligned} \tag{15.60}$$

These equations involve both v and ω. However, the motions of the weight and the spinning wheel are related because they are connected by a massless rope. In order for the rope not to slip along the wheel, the rope must follow the motion of the wheel at point P. This means that the velocity of point P on the wheel must be the same as the velocity of the rope at this point:

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r} = \omega\,\mathbf{k} \times (-\mathbf{R}\,\mathbf{i}) = -\omega\,R\,\mathbf{j} \ . \tag{15.61}$$

In order for the rope to remain tight and not stretch, the velocity of the rope at point P must be the same as the velocity of the weight. We therefore have:

$$\mathbf{v}_P = -\omega R\,\mathbf{j} = \mathbf{v} \ , \tag{15.62}$$

which is the velocity of the weight. We insert $v = -\omega R$ into (15.60):

$$E_1 = \frac{1}{2}mv^2 + mgy + \frac{1}{2}I\left(-\frac{v}{R}\right)^2 + Mgy_w = \frac{1}{2}\left(m + \frac{I}{R^2}\right)v^2 + mgy + Mgy_w \ . \tag{15.63}$$

What is the moment of inertial I of the spinning wheel around its center of mass? If the mass is homogeneously distributed, the wheel is a cylinder with mass M

and radius R, and the moment of inertia of a cylinder around its center of mass is $I = (1/2)MR^2$, which we insert into (15.63):

$$E_1 = \frac{1}{2}\left(m + \frac{MR^2}{2R^2}\right)v^2 + mgy + Mgy_w = \frac{1}{2}\left(m + \frac{M}{2}\right)v^2 + mgy + Mgy_w \; . \tag{15.64}$$

Applying energy conservation $E_0 = E_1$ we find:

$$Mgy_w = \frac{1}{2}\left(m + \frac{M}{2}\right)v^2 + mgy + Mgy_w \;\Rightarrow\; -\frac{1}{2}\left(m + \frac{M}{2}\right)v^2 = mgy \; , \tag{15.65}$$

$$v = \sqrt{2mg(-y)/\left(m + (M/2)\right)} \; , \tag{15.66}$$

where we have chosen the negative solution for v, since we know that the weight is moving downwards. We recall that y is negative since the weight is falling down.

Analyze: Let us test this result by checking what happens when the spinning wheel is massless, that is when $M \ll m$. In this limit we find $v = \sqrt{2g(-y)}$, which is the result we expected. Since the velocity of the weight for finite masses of the spinning wheel is smaller than this, the effect of the spinning wheel is to slow down the acceleration of the weight.

15.5.2 Example: Rolling Down a Hill

Problem: You are arranging a rolling competition. Various symmetrical, round objects of mass M, radius R, and moment of inertia I about an axis through the center of mass, are rolled down a hill of vertical height h. Find the velocity v of the object at the end of the hill. You may neglect the effects of air resistance.

Approach: We plan to use energy conservation: The initial potential energy is converted to translational and rotational energy of the object. The rolling condition relates translational and rotational motion.

Solution: We may use the conservation of total mechanical energy if all the forces are conservative. A rolling object is affected by gravity, which is conservative, by the normal force, which does no work, and by a friction force from the surface. What, you say, friction? How can we then use energy conservation? It turns out that for an object that is rolling without slipping, the friction force acts in the point of contact between the rolling object and the surface, and the velocity of this point is zero. The work done by the friction force on the object is therefore zero:

$$W = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v}\, dt = 0 \; . \tag{15.67}$$

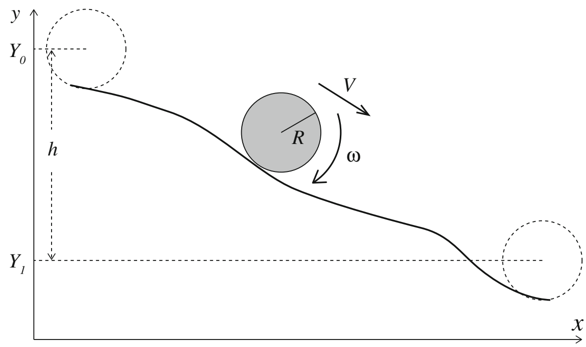

Fig. 15.15 A round object rolling *down* a hill

In this particular case we can therefore use energy conservation. Notice that you cannot use energy conservation if the object is slipping!

Identify: The rolling object has mass M, radius R, and moment of inertia I as illustrated in Fig. 15.15. The object starts at Y_0 with the velocity $V_0 = 0$ and the angular velocity $\omega_0 = 0$. What is the velocity (of the center of mass) at $Y = Y_1 = Y_0 - h$?

Model: The total energy of the object is $E = K_T + K_R + U$, where $K_T = (1/2)MV^2$ is the translational kinetic energy, related to the motion of the center of mass, and $K_R = (1/2)I\omega^2$ is the rotational kinetic energy, related to the motion relative to the center of mass. The potential energy is due to gravity $U = MgY$.

Solve: Conservation of energy gives:

$$\begin{aligned} E_0 &= E_1 \\ \frac{1}{2}MV_0^2 + \frac{1}{2}I\omega_0^2 + MgY_0 &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgY_1 \\ 0 + 0 + MgY_0 &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgY_1 \\ Mg\underbrace{(Y_0 - Y_1)}_{=h} &= \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 \,. \end{aligned} \tag{15.68}$$

We relate the translational and rotational motion using the rolling condition. As long as the object is rolling without slipping:

$$V = R\omega \;\Rightarrow\; \omega = V/R \,, \tag{15.69}$$

which gives

$$Mgh = \frac{1}{2}MV^2 + \frac{1}{2}I\left(\frac{V}{R}\right)^2 = \frac{1}{2}M\left(1 + \frac{I}{MR^2}\right)V^2 \,. \tag{15.70}$$

Analyze: The result depends on the ratio I/MR^2. Let us introduce the number c for this:

$$c = \frac{I}{MR^2} , \tag{15.71}$$

The number c characterizes how the mass is distributed around the center of mass. Large values of c means that the mass is placed far from the center of mass. As more mass is pulled towards the center of mass, c is reduced. We can find c for various ordinary shapes. For a cylinder shell $c = 1$, for a cylinder $c = 1/2$, and for a sphere $c = 2/5$. We interpret $c = 0$ as the case where a block is sliding without rolling on a frictionless surface. We can find the velocity V expressed using c:

$$gh = \frac{1}{2} V^2 (1 + c) , \tag{15.72}$$

$$V = \sqrt{\frac{2gh}{1 + c}} . \tag{15.73}$$

We see that smaller c gives larger velocities, and the object with the largest velocity wins the race. Hence the object with the smallest value of c would win. Notice that the results do not depend on the mass, only on how the mass is distributed around the axis of rotation.

Test your understanding: How would you construct a rolling object with a small value of c?

Summary

A rigid body: A rigid body can be **translated** or **rotated**, but the distance between any two points in the body does not change.

Moment of inertia:

- The moment of inertia of a multiparticle system around an axis O is: $I_O = \sum_i m_i \rho_i^2$, where m_i is the mass of particle i, and ρ_i is the distance from particle i to the rotation axis O.
- The moment of inertia of a solid body with mass density ρ_M around an axis O is: $I_O = \iiint \rho_M \, \rho^2 \, dV$, where ρ is the distance from the element dV to the axis O.
- Moments of inertia are added according to the **superposition principle**: The moment of inertial of two systems A and B around the axis O is the sum of the moments of inertia for each of the systems: $I_{O,AB} = I_{O,A} + I_{O,B}$
- The **Parallel-axis theorem**: The moment of inertia I_O of an object around an axis O can be found from the moment of inertia of the object around an *parallel* axis through the center of mass of the object, I_{cm}: $I_O = I_{\mathrm{cm}} + Ms^2$, where s is the distance between the axis O and the center of mass.

Energy of a rotating body:

- The **kinetic energy** of a rigid body rotating around the axis O is: $K = (1/2)I_O\omega^2$, where ω is the angular velocity around the axis O.
- The **potential energy** of a rigid body in a **homogeneous gravity field** is: $U = MgY$, where Y is the vertical position of the center of mass of the system.

Relating translational and rotational motion:

- For two objects connected by a thin, non-elastic rope, the point of contact with the rope must have the same speeds for both objects.
- An object is **rolling without slipping** along a surface if the point in contact with the surface has zero velocity relative to the surface.

Exercises

Discussion Questions

15.1 Moment of inertia. Can you find an object where the moment of inertia is larger around the center of mass than around an axis directed in the same direction but going through a different point?

15.2 Symmetries. Are there any objects for which the moment of inertia around an axis through the center of mass is the same for all possible axes? For all possible axes in a plane?

15.3 Dumbbell. A dumbbell consists of two spheres connected by a thin rod. Around which axis is the moment of inertia minimal for the dumbbell?

Problems

15.4 Three-particle system. Three particles of mass m are placed at $(-a, -a)$, $(a, -a)$, and $(0, a)$.
(**a**) Find the center of the mass of this system.
(**b**) Find the moment of inertia I_{cm} around the center of mass of the system for an axis along the z-axis.
(**c**) Find the moment of inertia $I_{0,z}$ for an axis along the z-axis through the origin.
(**d**) Find the moment of inertia $I_{0,x}$ for an axis along the x-axis through the origin.
(**e**) Find the moment of inertia $I_{0,z}$ for an axis along the y-axis through the origin.

15.5 Compound system. A dumbbell consists of two spheres of radius R and mass M connected by a rigid rod of mass m and length L, as illustrated in Fig. 15.16.

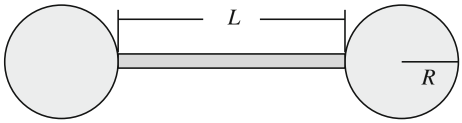

Fig. 15.16 A dumbbell system

(**a**) Find the moment of inertia for the system around an axis normal to the direction of the rod and through the center of mass of the system.
(**b**) Find the moment of inertia for the system around an axis along the direction of the rod and through the center of mass of the system.
(**c**) Find the moment of inertia for the system around an axis normal to the direction of the rod and through the center of one of the spheres.
(**d**) Find the moment of inertia for the system around an axis along the direction of the rod and through the center of one of the spheres.

15.6 Water molecule. A water molecule consists of an oxygen atom of mass 16u and two hydrogen atoms of mass 1u each. The two hydrogen atoms are placed at a distance a from the center of the oxygen atom, and the angle between the lines from the center of the oxygen atom to each of the hydrogen atoms is 105°. You can assume that each atom is a point particle with all its mass at its center.
(**a**) Find the moment of inertia for the molecule around an axis normal to the plane with all three atoms and through the center of mass of the molecule.
(**b**) Find the moment of inertia for the molecule around an axis normal to the plane with all three atoms and through the oxygen atom.

15.7 Compound system. A dumbbell consists of two spheres of radius R and mass M attached together so that the spheres are just touching. The dumbbell can rotate freely without friction about an attachment point through one of the spheres. The system starts at rest in a horizontal position and is released.
(**a**) Find the moment of inertia for the dumbbell around an axis normal to the plane of the dumbbell through one of the spheres.
(**b**) Find the angular velocity ω of the system as a function of its rotation angle θ.

15.8 Atwood's fall machine. Atwood's fall machine consists of two weight of mass m_1 and m_2 attached with a massless rope running around a spinning wheel of mass M and radius R without slipping. The spinning wheel is attached at its center and rotates around an axis through its center without friction.
(**a**) Find velocity of each of the weights as a function of their vertical positions.
(**b**) Find angular velocity of the spinning wheel as a function of the vertical positions of the weights.

15.9 Triangular pendulum. In this exercise we study a pendulum consisting of two point masses, each of mass, m attached to the ends of a massless rod of length, L as illustrated in Fig. 15.17. Two massless strings of length, L, are attached to each of the mass points and to the point, O, so that the pendulum forms an equilateral

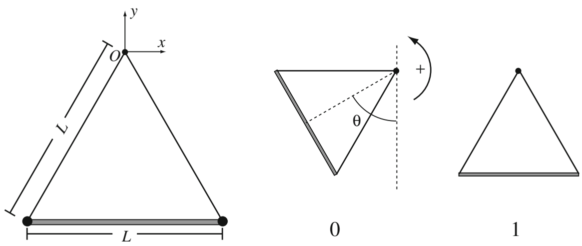

Fig. 15.17 A triangular pendulum

triangle. The pendulum is lifted to the position 0 and released. You may ignore air resistance.
(**a**) Find the moment of inertia, I_O about the point O for the pendulum.
(**b**) Show that the angular acceleration of the pendulum about the point O when it is at the position θ is:

$$\alpha = -\left(\sqrt{3}\right)(2)\ (g/L)\ \sin\theta\ . \tag{15.74}$$

(**c**) Find the angular velocity of the pendulum about the point, O, when the pendulum is in its lowest position.
(**d**) When the pendulum reaches its lowest position, both strings break. Describe the subsequent motion of the rod. Justify your answer.

15.10 Spinning toy car. In this exercise we study a simplified model for a toy car. The car consists of two identical wheels with mass m and radius R conected by a massless rod as illustrated in Fig. 15.18. The wheels rotate without friction around their attachment points on the rod. Initially, the trailing wheel (wheel A) starts with an angular velocity ω_0, with positive direction as illustrated in the figure. The leading wheel (wheel B) starts from rest. You put the car down on a flat, horizontal floor, and release it. We assume that the trailing wheel (wheel A) initially slides on the floor, whereas the leading wheel (wheel B) is rolling without slipping. You can also assume that the case remains horizontal throughout the motion.

The dynamic coefficient of friction between wheen A and the floor is μ, the acceleration of gravity is g ,and the moment of inertia for each of the wheel around their respective center of masses are I.

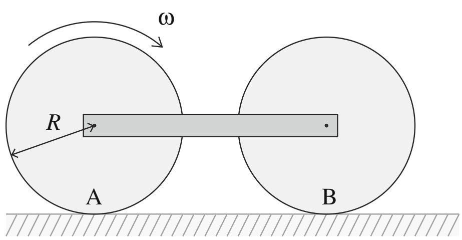

Fig. 15.18 Illustration of a model toy car

(**a**) Draw a free-body diagram for each of the three bodies (wheel A, wheel B, and the rod), showing the forces acting on each of the objects.
(**b**) Show that the acceleration of the car immediately after it is put down on the floor is $a = g\ (\mu) / (2 + c)$ where $c = I/\left(mR^2\right)$.
(**c**) Find the angular velocity for wheel A as a function of time. (The expression is only valid until the trailing wheel starts rolling without sliding).
(**d**) How long time does it take before both wheels roll without sliding? Describe the motion after this.

Projects

15.11 Micro-electromechanical system. In this project you will learn about the moment of inertia, and the potential and kinetic energy of a rotating object, and we will use this to study a simple electromechanical system, similar to a micro-mirror used in most modern projectors.

Modern production techniques for microscopic systems allows us to construct small mechanical elements made of silicon. For example, we can construct small silicon cantilevers with dimensions down to a few micrometers. In this project we will address the motion of a thin, microscopic beam using energy techniques.

First, let us consider a small, square mechanical element of dimensions $L \times L \times h$ and mass M. We assume that the thickness h is so small that we can neglect the finite thickness of the square. The square is attached with a hinge at one of the ends of length L. The hinge follows the y-axis as show in Fig. 15.19.
(**a**) Find the position $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j}$ of the center of mass of the square. The origin is in one of the corners of the square.
(**b**) Show by integration that the moment of inertia, $I_{cm,y}$, for rotations around an axis parallel with the y-axis going through the center of mass is $I_{cm,y} = ML^2/12$.
(**c**) Find the moment of inertia, I_y, for rotations around the y-axis.
The micromechanical cantilever we want to study is a bit more complicated than a single square. We can construct the cantilever from four identical squares, each with dimensions $L \times L$ and masses M, as illustrated in Fig. 15.20. The squares are rigidly attached to each other, so that they move as a single body. At the edge, along the

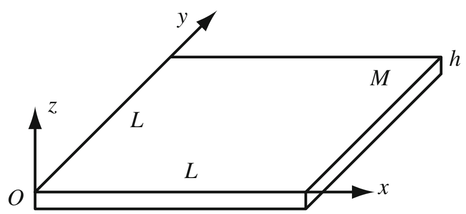

Fig. 15.19 Illustration of a square mechanical element. Each side has a length L, and the mass of the square is M. The thickness h is small compared to L

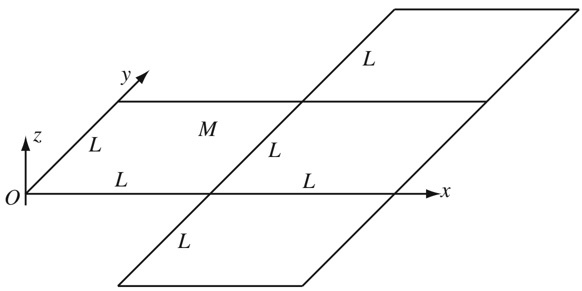

Fig. 15.20 Illustration of a cantilever constructed from four squares

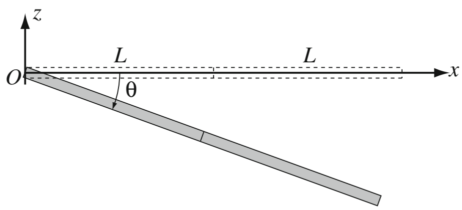

Fig. 15.21 Illustration of a cantilever in the xz-plane. The cantilever rotates around the y-axis to an angle θ

y-axis, the cantilever is attached with a hinge so that the cantilever can rotate freely about this axis.

(**d**) Show that the center of mass of the cantilever is $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} = (5/4)\,L\,\mathbf{i} + (L/2)\,\mathbf{j}$.

(**e**) Show that the moment of inertia, I_y for the whole object is $I_y = (22/3)\,ML^2$. Hint 1: You can calculate the moment of inertia for each part of the object independently and sum the results. (This is called the superposition principle). Hint 2: Use the parallel-axis theorem to find the moment of inertial around the y-axis for each part of the object.

Even though we are considering a microscopic system, where the effect of gravity typically will be negligible, let us first consider the motion of the cantilever when it is affected by gravity. For a real microscopic cantilever the effects of electrostatic forces is typically more important. Often such interactions results in a constant electrostatic force on the cantilever. The results from studying the behavior of the cantilever when affected by gravity as we do here can therefore easily be translated into the behavior of a cantilever affected by electrostatic forces.

The gravitational force acts in the negative z-direction. As a result of gravity, the cantilever rotates and angle θ around the y-axis as illustrated in Fig. 15.21.

(**f**) Show that the potential energy of the cantilever due to the gravitational force is $U_G = -5MLg\sin\theta$, where the potential energy is zero when the cantilever is horizontal.

(**g**) Assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.

In the following, we will no longer assume that the cantilever rotates freely around the y-axis. Instead, we will assume that it bends around a hinge along the y-axis, and

that the bending is like the bending of an elastic body. This means that there is an potential energy associated with the bending. The potential energy of the cantilever when it is bent an angle θ due to the stiffness of the hinge is $U_h = (1/2)\,\kappa\,\theta^2$ where κ is a constant that depends on the material properties (and the size) of the hinge.
(**h**) Again, assume that the cantilever starts with an initial angular velocity $\omega_0 = 0$ when $\theta = 0$. Find the angular velocity of the cantilever, $\omega(\theta)$, when it has reached the angle θ.
(**i**) Describe the motion of the cantilever.
For small θ we can approximate $\sin\theta \simeq \theta$. We will use this approximation in the following.
(**j**) Find the maximum angle θ of the cantilever when it is released as described above.
(**k**) Draw an energy diagram in the form of the total potential energy of the cantilever as a function of θ. Can you find any equilibrium points for the cantilever?
Small cantilevers are used for many technological applications. For example, projectors using the DLP technlogy consists of a vast number of micromirrors, small cantilevers that reflect light. When an electrical field is applied to the cantilever, the cantilever is affected by an electrostatic force. We can describe this in the same way as we described gravity above, but the electrical field can be turned on or off. As a result, the cantilever can be bent, and the light is reflected in a different direction. You can look at other interesting applications by searching for MEMS in your favourite search engine.

Chapter 16
Dynamics of Rigid Bodies

How do you spin a ball? And how do you jump-spin on skates? In this chapter you learn what causes changes in rotational motion using Newton's second law for rotational motion.

You know how to describe the rotation of a wheel around a fixed or moving axis, using the angle, the angular velocity, and the angular acceleration of the wheel. And you know how to find the kinetic energy of a rotating rigid body. But what causes changes in rotational motion? For translational motion we can use Newton's second law to determine the change in the translational state, in the translational momentum, from the external forces acting on a body. We use this both to find the acceleration of a body, and from the acceleration we can calculate the motion, and to find conservation laws for the translational momentum. Can we find a similar law for rotational motion? In this chapter we will introduce the rotational analogue to translational momentum: rotational momentum or angular momentum; the rotational analogue to force: torque; and the rotational analogue to Newton's second law: Newton's second law for rotational motion. Armed with these tools you will see that you are ready to solve any problem of moving and rotating rigid bodies, such as figuring out what causes a ball to spin or how you jump-spin on skates.

16.1 Motivating Example—Spinning a Wheel

Increasing the Angular Velocity of a Spinning Wheel

Figure 16.1 shows how a wheel is spun by a force **F** applied to a pedal, which is attached to a lever, which again is attached to the wheel. The wheel accelerates—it increases its angular velocity from $\omega_0 = 0$ rad/s to ω as the wheel rotates an angle θ. We can find the angular velocity ω as a function of θ from the work-energy theorem. The work done by **F** is:

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_16

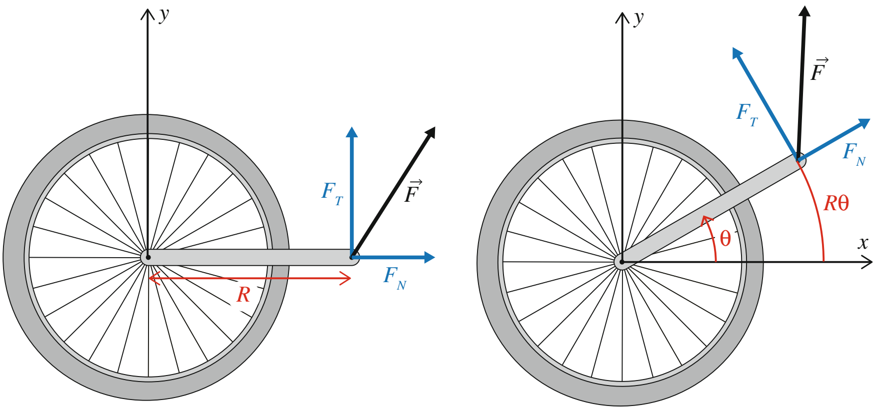


Fig. 16.1 We spin up a bicycle wheel by applying a force **F** of constant magnitude to an arm attached to the axis of the wheel. The wheel rotates around a fixed axis at its center, and the distance from the axis to the point where the force is applied is R

$$W = \int_{t_0}^{t} \mathbf{F} \cdot \mathbf{v} \, dt \ , \tag{16.1}$$

where **v** is the velocity of the point where the force is acting, that is, the velocity of the pedal. The pedal rotates around a fixed axis, the axis of the wheel, and therefore moves in the tangential direction:

$$\mathbf{v} = v \hat{u}_T \ . \tag{16.2}$$

We can similarly decompose the force in the tangential and normal direction

$$\mathbf{F} = F_T \hat{u}_T + F_N \hat{u}_N \ , \tag{16.3}$$

as illustrated in Fig. 16.1. It is only the tangential component that contributes to the work. The normal component does no work since there is no motion in this direction. If we apply a constant tangential force, F_T, and this is the only force acting, the work-energy theorem gives:

$$W = F_T \, s = F_T \, R\theta = K_1 - K_0 \ , \tag{16.4}$$

where the distance $s = R\theta$ moved by the pedal depends on the distance R from the rotation axis to the point where the force is acting. The change in kinetic energy of the rotating wheel is:

$$K_1 - K_0 = \frac{1}{2} I \omega^2 - \frac{1}{2} I \omega_0^2 \ , \tag{16.5}$$

If we insert this into (16.4) and assume that we start from rest, $\omega_0 = 0$, we get:

$$F_T\, R\theta = \frac{1}{2} I\omega^2 \;\Rightarrow\; \omega^2 = \frac{2F_T\, R\,\theta}{I} \; . \tag{16.6}$$

How does this compare with our intuition? We see that the rotational inertia, I, plays an important role: If we increase I but keep everything else fixed, the resulting angular velocity gets smaller: It becomes more difficult to get the wheel started if I is larger.

In addition, the angular velocity depends on how large the force F_T is and how far, R, from the rotation axis it is applied. First, we notice again what we already observed: It is only the tangential component of **F** that matters: We cannot accelerate the rotation of the rod by pulling at it in the radial direction. Second, we see that it is the combination $\tau = F_T R$ that matters. This combination is often called the *torque* of the force **F**. We can therefore increase the final angular velocity by increasing the force F_T or by increasing the distance R from the axis where the force is applied.

Angular Acceleration of a Spinning Wheel

Can we use this approach to find the angular acceleration? Yes! By applying the method to a very short time interval. As the interval becomes smaller and smaller, we effectively introduce the time derivative of both sides of (16.4). For the rotational system in Fig. 16.1, the work done by the constant, tangential force F_T is

$$W = F_T\, R\Delta\theta = \frac{1}{2} I\omega^2(t + \Delta t) - \frac{1}{2} I\omega^2(t) \; . \tag{16.7}$$

when the wheel rotates an angle $\Delta\theta$ during the short time interval Δt. We divide by Δt on both sides:

$$F_T R \frac{\Delta\theta}{\Delta t} = \frac{\frac{1}{2} I\omega^2(t + \Delta t) - \frac{1}{2} I\omega^2(t)}{\Delta t} \; , \tag{16.8}$$

This becomes the time derivative as Δt becomes small:

$$F_T R \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right) = I\omega \frac{d\omega}{dt} \; , \tag{16.9}$$

where we have applied the chain rule and assumed that I is constant. Finally, by dividing by ω on both sides:

$$F_T R = I \frac{d\omega}{dt} = I\alpha = \frac{d}{dt} I\omega \; , \tag{16.10}$$

we have found the angular acceleration! This equation looks very much like Newton' second law for translational motion:

$$F = m\frac{dv}{dt} = ma = \frac{d}{dt}mv \; . \tag{16.11}$$

For rotational motion, we replace:

The force by the torque	$F \rightarrow F_T R$
The translational inertia (mass) by the rotational inertia	$m \rightarrow I$
The acceleration by the angular acceleration	$a \rightarrow \alpha$

(We can derive Newton's second law from the work-energy theorem in exactly the same way.) Equation (16.10) is indeed Newton's second law for rotational motion. This law is general, even though it was here derived for a special situation.

Interpreting Newton's Second Law for Rotations

From Newton's second law for rotational motion in (16.10), we interpret the torque $RF_T = \tau$ as the *cause* of the angular acceleration, just as we interpreted the force as the cause of acceleration for translational motion. We see that I plays the role of a rotational inertia. For a given torque, $\tau = F_T R$, a larger value of I means a smaller angular acceleration. Also, we see that the torque $\tau = F_T R$ depends on both the tangential force, F_T and the distance to the rotation axis, R: If we apply the same force F further out from the rotation axis, we get a larger torque and a larger angular acceleration.

The top figures in Fig. 16.2 illustrates how we must increase the force as the distance R changes in order to keep the same torque and therefore the same angular acceleration: If we increase R we have to decrease F_T by the same factor to keep the acceleration the same. This is illustrated in the figure, where we have shown arms of length $R/2$, R and $2R$ and the corresponding forces, $2F$, F, and $F/2$.

The bottom figures in Fig. 16.2 shows what happens when we keep the torque the same, but change the moment of inertia. If we assume that the mass is concentrated in two weights attached to a thin rod, the moment of inertia is $I = 2Mr^2$, where r is the distance from the center of the rod to the center of each of the masses. From Newton's second law for rotational motion in (16.10), we find the angular acceleration:

$$\alpha = \frac{F_T R}{I} \; . \tag{16.12}$$

If the torque, $F_T R$ is constant, the acceleration depends inversely on I. We illustrate this by showing the orientations of the object at three subsequent (but not equally spaced) timesteps. All these systems have the same mass. It is how the mass is distributed around the rotation axis that matters. And as evident, changing the distribution has a significant effect on the acceleration.

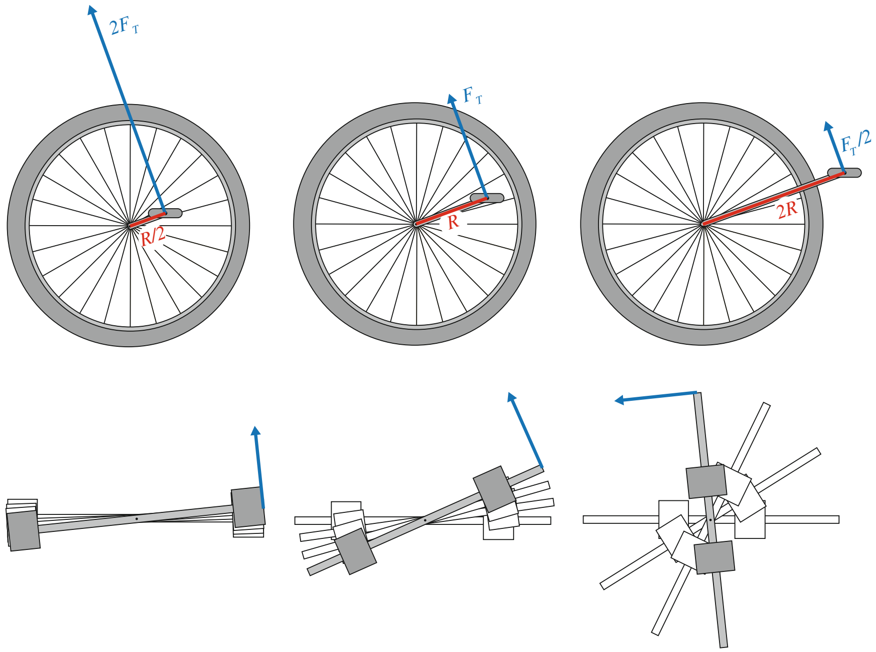


Fig. 16.2 *Top* All three wheels have the same torque, $\tau = F_T r$, where r is the distance from the axis to where the force acts, and F_T is the tangential component of the force. *Bottom* The same torque is applied to three systems with different moment of inertia—changed by moving the positions of the masses. The position of the object are shown at four times, t_0, t_1, $t_2 = \sqrt{2}t_1$ and $t_3 = \sqrt{3}t_1$, which are the same for all systems: The differenced are due to differences in angular acceleration due to differences in the moment of inertia

While Newton's second law for rotational motion was introduced in the special situation of a constant force, we will in the rest of this chapter see that the law is general, and you will learn how to apply it to determine the rotational motion of a system.

16.2 Newton's Second Law for Rotational Motion

In the introductory example we introduced the following law for the rotational motion of an object subject to a single, constant force:

$$\tau = F_T\, R = I\alpha \ , \tag{16.13}$$

where $\tau = F_T R$ is called the torque for the force $\mathbf{F}$. The torque depends only on the tangential component of $\mathbf{F}$. This is the component that is normal to the vector from the rotation axis to the point where the force acts, $\mathbf{r}$, as illustrated in Fig. 16.3. You

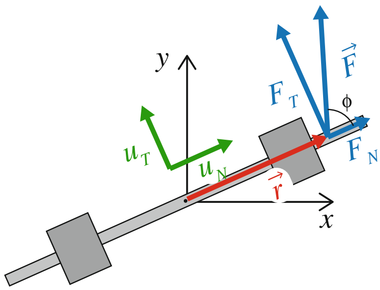


Fig. 16.3 Illustration of how the torque of the force **F** is calculated: Only the tangential component F_T contributes

may recall to have seen this before. We recognize the torque as the magnitude of the cross product between **r** and **F**. This is easily seen by decomposing **r** and **F** in the normal (radial) and tangential directions: $\mathbf{r} = r\hat{u}_N$ and $\mathbf{F} = F_T\hat{u}_T + F_N\hat{u}_N$. The cross product is then:

$$\mathbf{r} \times \mathbf{F} = r\hat{u}_N \times F_T\hat{u}_T + r\hat{u}_N \times F_N\hat{u}_N = rF_T\,\mathbf{k} \,, \tag{16.14}$$

where $\hat{u}_N \times \hat{u}_N = 0$, and $\hat{u}_N \times \hat{u}_T = \mathbf{k}$ is a unit vector that point out of the plane, in the z-direction. This allows a more general definition of the (vector) torque of the force **F** around the point O:

Definition of torque:

$$\boldsymbol{\tau}_O = \mathbf{r} \times \mathbf{F} \,, \tag{16.15}$$

where **r** is the vector from O to the point where **F** is acting.

The torque, $F_T R$, in (16.13) is therefore the z-component of the vector torque:

$$\tau_z = F_T R = I\alpha \,. \tag{16.16}$$

In the introductory example we discussed the effect of a single force, producing a single torque. However, an object may be subject to several forces, all acting in separate points, giving rise to separate torques. The work-energy theorem used in the example depends on the *net* force. Hence we must insist on using the **net torque** when formulating Newton's second law for rotational motion.[1]

[1]Notice that the torque $\boldsymbol{\tau}$ points in the z-direction, which is also the direction of the rotation vector, $\boldsymbol{\omega} = \omega\,\mathbf{k}$. This suggests a vector formulation of Newton's second law for rotational motion: $\sum \boldsymbol{\tau}_j = I\boldsymbol{\alpha}$. Unfortunately, this is generally not correct. We will return to a vector formulation later.

Newton's second law for rotational motion (N2Lr): for a rigid body rotating around a **fixed axis** (the z-axis) is:

$$\sum_j \tau_{z,j} = \tau_z^{\text{net}} = I_z \alpha_z \ , \tag{16.17}$$

where $\boldsymbol{\tau}_j = \mathbf{r}_j \times \mathbf{F}_j$ is the torque of force j, $\mathbf{r}_j$ is the position of the point where force j is applied, and I_z is the moment of inertia (the rotational inertia) of the object around the rotation axis.

Structured Problem-Solving Approach

This law allows us to determine the rotational motion of a rigid body, just like we previously have found the translational motion of an object using Newton's second law. We can follow the same structured problem-solving approach as we used for translational motion, but with some modifications as illustrated in Fig. 16.4.

There are a few, but important differences between how we address problems with translational and rotational motion. When we *identify* the relevant systems and variables for rotational motion around a fixed axis, we must of course describe the configuration of the object using the angle θ and the angular velocity ω. When we *model* the system, we still start from a free-body diagram of the rotating object, but we must now carefully specify the rotation axis and where each of the forces are acting, because this is essential in order to calculate the torque. We find the torque for each of the external forces acting, and add them all to get the net torque, which

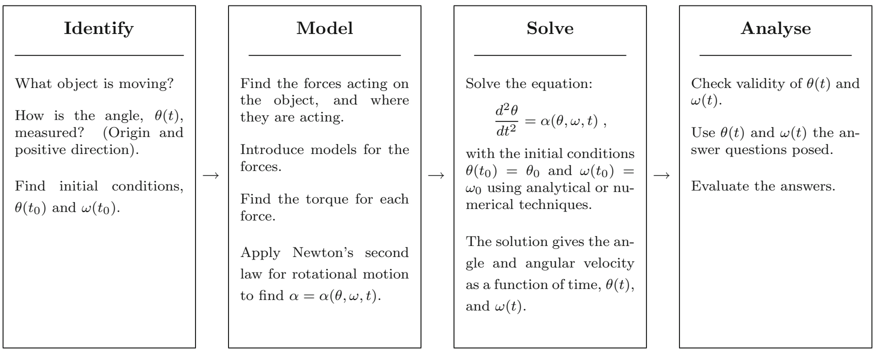


Fig. 16.4 Illustration of structured problem-solving approach for rotating objects

we use in Newton's second law for rotational motion. When we *solve* and *analyze* the system, we follow the same structure as we have done before, now working with angular coordinates.

Torque

In order to apply Newton's second law for rotation and solve rotational problems, we must know how to calculate the torque:

$$\boldsymbol{\tau}_O = \mathbf{r} \times \mathbf{F} \,, \tag{16.18}$$

of the force $\mathbf{F}$ around the point O. What are the properties of torque?

- The torque of a force depends on the point it is taken relative to—the origin. We say that the torque is around the point O to show where the origin is when we calculate the torque.
- The torque of a force $\mathbf{F}$ depends both on the force and on the position $\mathbf{r}$ where the force is acting. For example, the torque around point O of the force $\mathbf{F}$ on the hammer in Fig. 16.5 varies from $\boldsymbol{\tau} = FL\,\mathbf{k}$ when the force is applied at the end of the hammer to $\boldsymbol{\tau} = 0$ when the force is applied at the rotation axis.

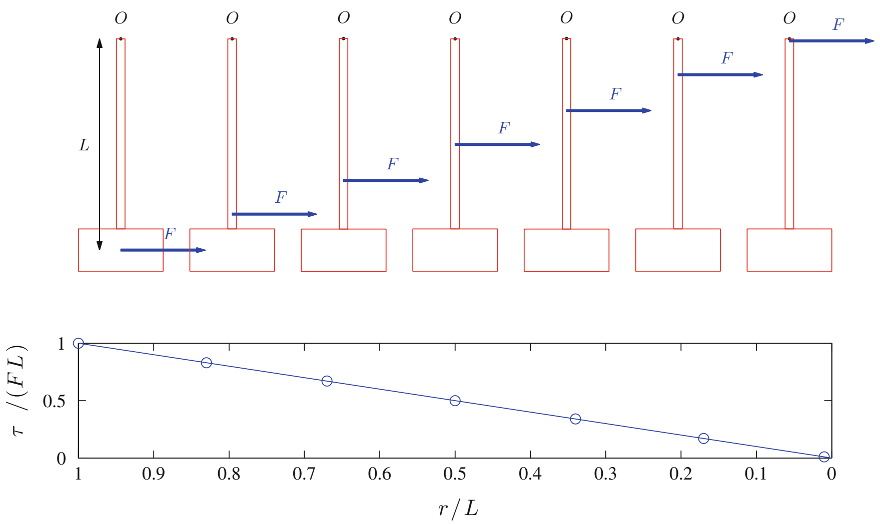


Fig. 16.5 Illustration of the torque $\boldsymbol{\tau}$ for a rigid hammer of length L for a force $\mathbf{F} = F\mathbf{i}$ applied at various position, $\mathbf{r} = y\mathbf{j}$, where $y < 0$

- The torque is given as a cross product: The torque is therefore normal to both the position vector **r**and the force **F**. We find the direction of the torque using the right-hand rule.
- For a two-dimensional system in the xy-plane, the torque is always directed along the z-axis. We often use the z-component τ_z of the torque instead of the full vector notation for the torque. The sign of τ_z indicates if the torque is in the positive or negative z-direction.
- The direction of the torque is normal to the position and force vectors, and its magnitude is:

$$|\tau| = |F|\,|r|\ \sin\phi \ , \tag{16.19}$$

where ϕ is the angle between the position vector **r** and the force vector **F** as illustrated in Fig. 16.3.
- If the force is parallel to the position vector, the torque is always zero for this force. For example, for a weight in a rope whirled in a circular path, the torque of the rope tension around the center of the path is always zero, since the rope tension acts in the direction of the position vector. Similarly, the torque of the gravitational force from the Sun on the Earth (taken around the Sun) is also always zero.
- If the position vector **r** is zero, that is, if the force acts in the origin, the torque of the force is zero. This is commonly used "trick" to solve problems: If we place the origin at a force we do not know, its torque is zero independently of the force.
- Torques obey the **superposition principle**. The total torques of a force $\mathbf{F}_1$ acting at the point $\mathbf{r}_1$ and the force $\mathbf{F}_2$ acting at the point $\mathbf{r}_2$ is:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 \ . \tag{16.20}$$

Notice that you can only add torques that all are taken around the same point!
- To calculate the **torque of a the gravitational force** acting on a rigid body (when gravity is a constant) you assume that the total gravitational force acts in the center of mass:

$$\boldsymbol{\tau} = \mathbf{R} \times M\mathbf{g} \ , \tag{16.21}$$

where **R** is the position of the center of mass. This is found by summing all the torques acting on each small element i of the rigid body:

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times m_i\mathbf{g} = \left(\sum_i m_i\mathbf{r}_i\right) \times \mathbf{g} = M\mathbf{R} \times \mathbf{g} = \mathbf{R} \times M\mathbf{g} \ . \tag{16.22}$$

Test your understanding: Why does it require less force to close a door if you push far from the hinge than if you push near it?

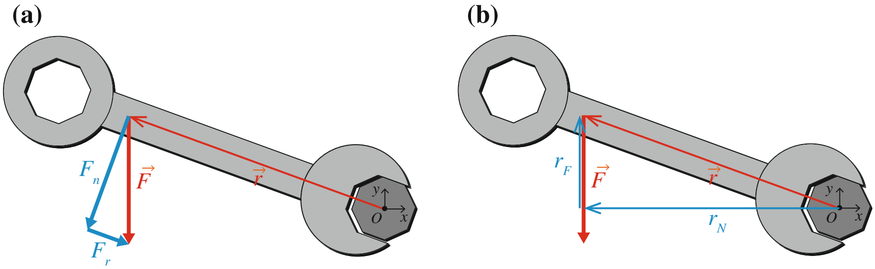


Fig. 16.6 **a** Torque of **F** by decomposing **F**. **b** Torque of **F** by decomposing **r**

16.2.1 Example: Torque and Vector Decomposition

Figure 16.6a illustrates a force **F** applied to a wrench, at a point **r** relative to the rotation axis, which is placed in the origin. While we can always find the torque directly from

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \, , \tag{16.23}$$

if we know **r** and **F**. However, it is often practical to decompose either the force **F** or the arm, **r**, in order to simplify the calculation of the torque.

If you decompose the force, **F** into a component $\mathbf{F}_r = F_r \hat{u}_r$ in the direction along **r** and a component $\mathbf{F}_n = F_n \hat{u}_n$ in the direction normal to **r**, the torque is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left(F_r \hat{u}_r + F_n \hat{u}_n \right) = \mathbf{r} \times F_r \hat{u}_r + \mathbf{r} \times F_n \hat{u}_n = r F_n \, \mathbf{k} \; . \tag{16.24}$$

since the force component along **r** does not contribute to the torque - it is only the component of **F** normal to the arm that contributes.

Alternatively, you could decompose the arm **r** into a component $\mathbf{r}_F = r_F \hat{u}_F$ in the direction along **F**, and a component $\mathbf{r}_N = r_N \hat{u}_N$ in the direction normal to **F** as illustrated in Fig. 16.6b. The torque is then:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \left(r_F \hat{u}_F + r_N \hat{u}_N \right) \times \mathbf{F} = r_N F_{\cdot} , \tag{16.25}$$

which shows that it is only the component of the arm that is normal to the force that contributes to the torque. With experience you will learn to use the method that is simplest in a given situation.

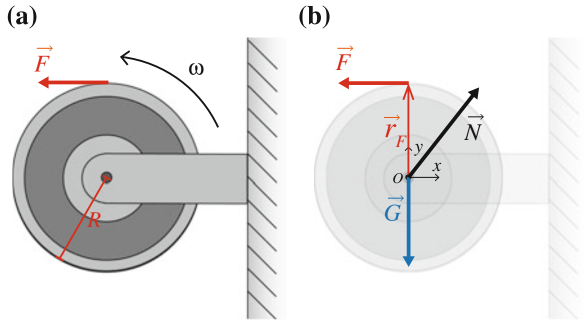


Fig. 16.7 A wheel is affected by a force F acting on its rim. **a** A sketch of the system. **b** A free-body diagram for the wheel

16.2.2 Example: Pulling at a Wheel

Problem: A force F is applied tangentially at the rim of a wheel made of a homogeneous cylinder of mass M and radius R. (You may think of this as pulling at a rope with a constant force, where the rope is wrapped around the wheel and unwinds as you pull). The wheel starts at rest and is attached to a frictionless axis at its center. Find the angle θ of the wheel as a function of time.
Approach: We use Newton's second law for rotational motion since the wheel rotates around a fixed axis: We find the forces and where they are acting, then we find the torques, calculate the angular acceleration, and integrate to find the motion.
Identify: The rotational position of the wheel is described by the angle θ, with positive direction shown in Fig. 16.7. The wheel starts at $\theta_0 = 0$ with angular velocity $\omega_0 = 0$ at $t_0 = 0$.
Model: The external forces acting on the wheel are: gravity, $\mathbf{G}$, acting in $\mathbf{r}_G = 0$ at the center of mass of the wheel; the contact force $\mathbf{N}$ from the axis on the wheel acting in $\mathbf{r}_N = 0$ at the axis (notice that $\mathbf{N}$ has to balance the two other forces, because the center of mass of the wheel does not move), and the force $\mathbf{F} = -F\,\mathbf{i}$, acting at $\mathbf{r}_F = R\,\mathbf{j}$ at the rim of the wheel.

The torques of both gravity and the normal force are zero, since $\mathbf{r}_G$ and $\mathbf{r}_N$ are zero. The torque of $\mathbf{F}$ around O is:

$$\boldsymbol{\tau}_F = \mathbf{r}_F \times \mathbf{F} = R\,\mathbf{j} \times -F\,\mathbf{i} = RF\,\mathbf{k} \; , \tag{16.26}$$

which is the net torque around the origin. The z-component, the component along the rotation axis, is $\tau_z = RF$. Newton's second law for rotatational motion around a fixed axis gives:

$$\tau_z^{\text{net}} = RF = I_z \alpha \; , \tag{16.27}$$

where I_z is the moment of inertia of the wheel for rotation around the axis, which goes through the center of mass of the wheel. We find I_z from Fig. 15.5. We insert this into (16.27), getting:

$$\alpha = \frac{RF}{\frac{1}{2}MR^2} = \frac{2F}{MR} , \tag{16.28}$$

which is a constant. The initial conditions for the motion are: $\omega(0) = \omega_0 = 0$ and $\theta(0) = \theta_0 = 0$.

Solve: We find the angular velocity by integrating the (constant) angular acceleration from (16.28).

$$\omega(t) - \underbrace{\omega(t_0)}_{=0} = \int_0^t \underbrace{\alpha}_{=2F/MR} dt = \frac{2F}{MR} t . \tag{16.29}$$

Similarly, we find the angle by integrating the angular velocity:

$$\theta(t) - \underbrace{\theta(t_0)}_{=0} = \int_0^t \underbrace{\omega}_{=(2F/MR)t} dt = \frac{1}{2}\frac{2F}{MR} t^2 , \tag{16.30}$$

which is the solution.

Analyze: Notice that the method used is exactly the same as the method we used to solve problems with translational motion, but now we use Newton's second law for rotational motion instead of Newton's second law for translational motion.

16.2.3 Example: Blowing at a Pendulum

Problem: A pendulum consists of a sphere of radius R and mass M. The center of the sphere is attached at the end of a thin, massless rod of length L. The other end revolves around a bolt, providing a fixed, friction-free axis for the pendulum. The sphere is affected by air resistance, which you for simplicity can assume to act at the center of the sphere. You can use a quadratic law for the air drag. Develop a method to find the motion of the pendulum when affected by a horizontal wind, $\mathbf{w}(t) = w(t)\,\mathbf{i}$, which may vary in time, and explore the behavior of the pendulum for various types of winds.

Approach: The pendulum is rotating around a fixed axis. We can therefore solve the problem using Newton's second law for rotational motion: From the forces acting we introduce force models, find the torques, find the angular acceleration from the net torque, and solve to find the motion.

Identify: We measure the position of the pendulum by the angle, θ, with the vertical as shown in Fig. 16.8. We need to define initial conditions for the motion, and assume that the pendulum starts at $\theta_0 = 0$ and with initial angular velocity $\omega(t_0) = \omega_0$ at $t_0 = 0$ s.

Model: We find the forces acting on the pendulum from the free-body diagram (see Fig. 16.8). The pendulum is affected by gravity $\mathbf{G} = -Mg\,\mathbf{j}$. To calculate the torque of gravity, we assume it acts ate the center of the mass, which is at the center of the

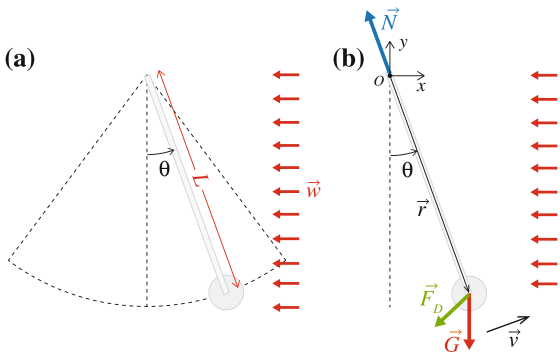


Fig. 16.8 **a** Sketch of the motion of the pendulum. **b** A free-body diagram for the pendulum

sphere, since the rod is without mass. The gravitational force is therefore drawn in the center of the sphere, at $\mathbf{r}_G = \mathbf{r}$, where $\mathbf{r}$ is a vector pointing from the rotation axis O to the center of the sphere.

From the figure, we see that

$$\mathbf{r} = L\sin\theta\,\mathbf{i} - L\cos\theta\,\mathbf{j}\ . \tag{16.31}$$

The torque of gravity is:

$$\boldsymbol{\tau}_G = \mathbf{r}\times\mathbf{G}\ , \tag{16.32}$$

which we find by inserting the vector for $\mathbf{G}$ and $\mathbf{r}$:

$$\boldsymbol{\tau}_G = (L\sin\theta\,\mathbf{i} - L\cos\theta\,\mathbf{j})\times -Mg\,\mathbf{j} = -MgL\sin\theta\,\mathbf{k}\ . \tag{16.33}$$

The air resistance, $\mathbf{F}_D = -D|\mathbf{v}-\mathbf{w}|(\mathbf{v}-\mathbf{w})$, also acts at the center of the sphere, $\mathbf{r}_D = \mathbf{r}$. Here, $\mathbf{v}$ is the velocity of the center of the sphere and $\mathbf{w}$ is the velocity of the air. When there is no wind, $\mathbf{w} = 0$, and the air resistance reduces to the well know form, $\mathbf{F}_D = -Dv\mathbf{v}$. What is $\mathbf{v}$? We can find the velocity of any point on a rotating rigid body using $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$. The torque of $\mathbf{F}_D$ is:

$$\boldsymbol{\tau}_D = \mathbf{r}_D\times\mathbf{F}_D\ , \tag{16.34}$$

where $\mathbf{r}_D = \mathbf{r}$. For the general case where an external wind $\mathbf{w}$ is applied, we need to calculate this based on the wind. However, it is useful to see that we get a simplified result when $\mathbf{w} = 0$ and $\mathbf{F}_D$ consequently is directed along $\mathbf{v}$. Then

$$\boldsymbol{\tau}_D = \mathbf{r}\times -Dv\mathbf{v}\ . \tag{16.35}$$

Because $\mathbf{r}$ is normal to $\mathbf{v}$, we get $\mathbf{r}\times\mathbf{v} = Lv\,\mathbf{k}$. (Ensure you get the same by applying the right-hand rule). If we insert $v = \omega L$ we get

$$\tau_{z,D} = -DL^3|\omega|\omega \ . \tag{16.36}$$

This expression also ensures that we get the right sign for the torque both when $\omega > 0$ and when $\omega > 0$. Check that you also get this!

The pendulum is also affected by a contact force, **N**, from the attachment bolt. In reality, the contact force is distributed along the contact between the bolt and the rod, but here we assume that the contact force acts so close to the rotation axis, that the distance $\mathbf{r}_N \simeq \mathbf{0}$, and the torque of the force is zero.

For no wind ($\mathbf{w} = 0$), the z-component of the net torque is therefore:

$$\tau_z^{\text{net}} = \tau_{z,G} + \tau_{z,N} + \tau_{z,D} = -MgL \sin\theta - DL^3|\omega|\omega \ . \tag{16.37}$$

And Newton's second law for rotational motion gives:

$$I\alpha = \tau_z^{\text{net}} = -MgL \sin\theta - DL^3|\omega|\omega \ . \tag{16.38}$$

The moment of inertia, I, can be related to the moment of inertia of a sphere around its center of mass, $I_{cm} = 2/5MR^2$ (from Fig. 15.5) using the parallel-axis theorem: $I = I_{cm} + ML^2$.

Solve: While you may be able to solve the motion of the pendulum for a limited range of motion for small θ, we concentrate here on the general numerical approach.

Without Wind

When there is no wind, the motion is determined from (16.38), giving

$$\alpha = -\frac{MgL}{I}\sin\theta - \frac{DL^3}{I}|\omega|\omega \ , \tag{16.39}$$

with initial conditions $\theta(t_0) = \theta_0 = 0$, and $\omega(t_0) = \omega_0$. We solve using Euler-Cromer's method (see (14.21)), implemented by:

```
from pylab import *
rho = 7000.0 # kg/m^3
R = 0.02 # m
L = 1.0 # m
M = 4/3*pi*R**3*rho
I = 2/5*M*R**2 + M*L**2
g = 9.8 # m/s^2
rhoair = 1.293 # kg/m^3
D = 12*rhoair*R**2
omega0 = 8.0 # rad/s
time = 20.0 # s
dt = 0.001 # s
# Numerical variables
n = int(round(time/dt))
theta = zeros(n,float)
omega = zeros(n,float)
t = zeros(n,float)
# Initialize
theta[0] = 0
```

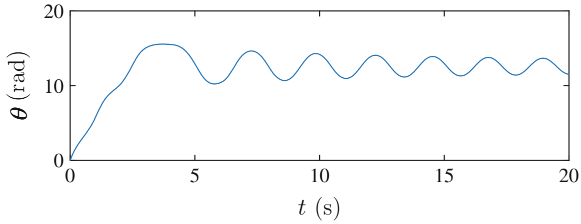


Fig. 16.9 Plot of $\theta(t)$ for a simulation of the pendulum with no applied wind

```
omega[0] = omega0
# Integration loop
for i in range(n-1):
    tauz = -M*g*L*sin(theta[i])-D*L**3*omega[i]*abs(omega[i])
    alpha = tauz/I
    omega[i+1] = omega[i] + alpha*dt
    theta[i+1] = theta[i] + omega[i+1]*dt
    t[i+1] = t[i] + dt
```

The resulting motion is shown in Fig. 16.9. Here, you see that the system first rotates completely around the axis, before damping limits the motion to an oscillation. For the presented simulation we used $L = 1\,\mathrm{m}$, $R = 0.01\,\mathrm{m}$, mass density $\rho_m = 7000\,\mathrm{kg/m}^3$, and $D = 12\rho_{\mathrm{air}}R^2$ (from (5.20)) as seen in the program.

With Wind

What happens if we want to include an external wind field, $\mathbf{w} = w(t)\,\mathbf{i}$, which may vary with time? Then the torque of the air resistance force becomes more complicated:

$$\boldsymbol{\tau}_D = \mathbf{r}_D \times \mathbf{F}_D \ , \tag{16.40}$$

where $\mathbf{r}_D = \mathbf{r}$, which is given in (16.31), and

$$\mathbf{F}_D = -D|\mathbf{v} - \mathbf{w}|(\mathbf{v} - \mathbf{w}) \ , \tag{16.41}$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, and $\boldsymbol{\omega} = \omega(t)\,\mathbf{k}$. Fortunately, it is simple to implement this numerically by calculating the cross-product directly in the numerical implementation. We must only remember to extract the z-component of the calculated torque and use this to calculate the angular acceleration. Let us start by assuming that the wind is described by the function `wind(t)`, which returns the value of $w(t)$ for a given time. We start by assuming that the wind blows with a constant velocity, $w(t) = w_0 = 20.0\,\mathrm{m/s}$, and let the pendulum start at rest, $\omega(t_0) = 0\,\mathrm{rad/s}$. This is implemented by:

```
def wind(t):
    #WIND Returns the wind veocity w(t) at the time t
    w0 = 20.0; % m/s
    return w0
```

and the integration loop of main program is now:

```
G = array([0,-M*g,0])
for i in range(n-1):
    r = array([L*sin(theta[i]),-L*cos(theta[i]),0.0])
    tauG = cross(r,G)
    # F_D = - D|v - w|(v - w)
    omegavec = omega[i]*array([0,0,1])
    v = cross(omegavec,r)
    w = wind(t[i])*array([1,0,0])
    dv = v - w
    FD = -D*dv*linalg.norm(dv)
    tauFD = cross(r,FD)
    tau = tauG + tauFD
    tauz = tau[2]
    alpha = tauz/I
    omega[i+1] = omega[i] + alpha*dt
    theta[i+1] = theta[i] + omega[i+1]*dt
    t[i+1] = t[i] + dt
    if (mod(i,10)==0):
        plot(array([0,r[0]]),array([0,r[1]]),'-o')
        xlabel('x [m]')
        ylabel('y [m]')
        axis([-L,L,-L,L])
        axis('equal')
```

where we also have included an animated plot to show the motion of the pendulum. The test for `mod(i,10)==0` is a test to check if the counter `i` is divisible by 10. This means that what is inside the `if`-statement will only be executed every time `i` is increased by 10. This is a standard way to speed up the animation by only plotting some of the results. You need to select an appropriate number here depending on your computer to get a smooth motion. If it is moving too slowly you need to increase the number 10 to a large value. If it is moving too fast you should lower the number. The resulting behavior is illustrated in Fig. 16.10, where we see that the pendulum now does not return to $\theta = 0$, but rather reaches a stationary situation with $\theta =$, which corresponds to the equilibrium configuration, that is the configuration where the net torque is zero. You may solve to find the value of θ corresponding to zero torque for yourself.

What happens if we now blow in a more complicated way, such as by blowing periodically, but always in the positive direction. We can model this using the cosine:

$$w(t) = \frac{w_1}{2}\left(1 + \cos(2\pi t/T)\right) \ , \tag{16.42}$$

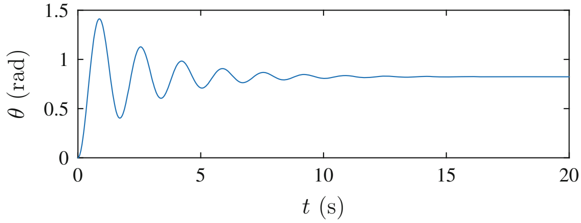


Fig. 16.10 Plot of $\theta(t)$ for a simulation of the pendulum with a applied wind velocity that is constant

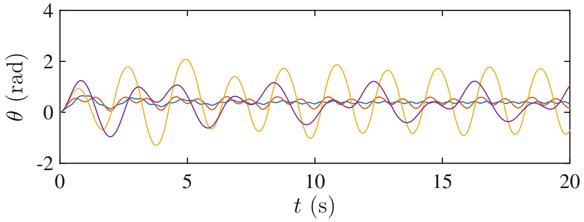


Fig. 16.11 Plot of $\theta(t)$ for a simulation of the pendulum with a applied, periodic wind velocity, $w(t) = (w_1/2)\,(\cos(2\pi t/T))$, with $w_1 = 20$ m/s and $T = 0.5$ s, 1.0 s, 2.0 s, and 4.0 s

where w_1 is the maximum wind velocity and T is the period of the wind. This is now implemented by changing the `wind(t)` function:

```
def wind(t):
    %WIND Returns the wind veocity w(t) at the time t
    w1 = 20.0 # m/s
    T = 1.0 #s
    w = 0.5*w1*(1.0 + cos(2*pi*t/T))
    return w
```

Now, we start the pendulum from $\theta(t_0) = 0$ and $\omega(t_0) = 0$rad/s, corresponding to the case where it hang straight down at rest before you start blowing periodically at it. The resulting behavior for $T = 0.5$ s, 1.0 s, 2.0 s and 4.0 s is shown in Fig. 16.11. What happens here? Play around with the model yourself to find out what happens.

16.3 Rotational Motion Around a Moving Center of Mass

Figure 16.12 shows a rod being thrown across the lecture hall. After it has been thrown it is just affected by gravity and air resistance. We know that the motion of the center of mass of the object only depends on the external forces acting on the object—its motion is determined from Newton's second law of motion. But what about the rotational motion around the center of mass? The rotational motion around the center of mass for a ridig body (rotating around a fixed axis) is determined from Newton's second law for rotational motion around the center of mass. (A proof is found in Sect. A.11)

Newton's Second Law of Translational Motion

The motion of the center of mass of a rigid body, such as the rotating rod in Fig. 16.12 is determined from Newton's second law for translational motion:

$$\boxed{\sum_j \mathbf{F}_j^{\text{ext}} = M\mathbf{A}\,.} \tag{16.43}$$

This is true irrespective of how the system moves relatively to the center of mass. This means that the motion of the center of mass of the rod in Fig. 16.12 would be the same if the rod rotated or if it moved with the same angle all the time. But only if

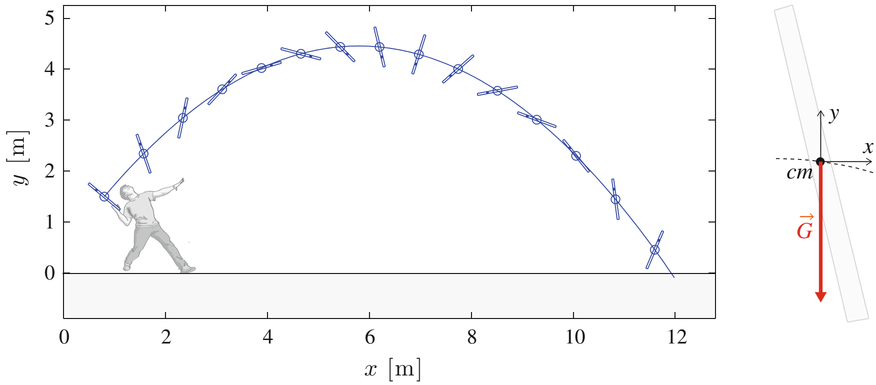


Fig. 16.12 *Left* The motion of a rigid rod thrown through the air, showing the motion of the center of mass and the rotational motion around the center of the mass. *Right* Free-body diagram for the rod

the external forces are the same! If we include the effects air resistance, the external forces depend on whether the rod is rotating or not, and the motion would not be exactly the same in the two situation.

Newton's Second Law for Rotational Motion Around the Center of Mass

For a system rotating around a fixed axis through the center of mass, the rotational motion is determined by Newton's second law for rotational motion around the center of mass, which is just like Newton's second law for rotation around a fixed axis, but now all the torques must be taken around the center of mass:

$$\boldsymbol{\tau}_{cm} = \mathbf{r}_{cm} \times \mathbf{F} \ , \tag{16.44}$$

where $\mathbf{r}_{cm}$ is a vector from the center of mass to the point where the force $\mathbf{F}$ acts. Using this notation, we get:

Newton's second law for a rigid body rotating around **a fixed axis through the center of mass** (the z-axis) is:

$$\sum_j \tau_{z,cm,j} = \tau_{z,cm}^{\text{net}} = I_{z,cm}\alpha_z \ , \tag{16.45}$$

where $\boldsymbol{\tau}_{cm,j}$ is the torque of force j around the center of mass, and $I_{z,cm}$ is the moment of inertia (the rotational inertia) of the object around the fixed rotational axis through its center of mass.

It may be confusing to call the axis fixed if it is moving along with the center of mass. What we mean is that the axis has a fixed direction: The direction of the rotation axis

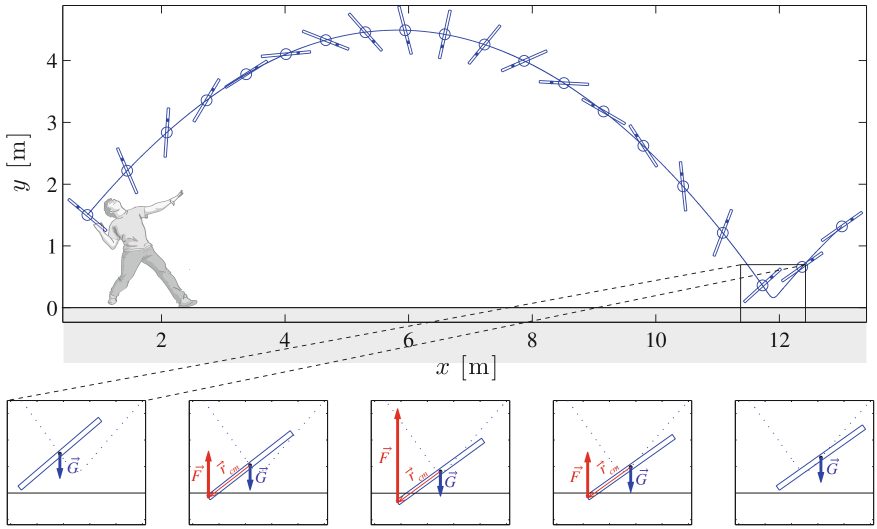


Fig. 16.13 *Top* The motion of a rigid rod thrown through the air, showing the motion of the center of mass and the rotational motion around the center of the mass. The rod hits the floor, and bounces back up. *Bottom* Free-body diagram for the rod at five different times. When the rod is not in contact with the floor (the first and the last inset) only gravity acts. When the rod is in contact with the floor (middle three insets) both gravity and a contact force from the floor act

does not change during the motion, but the position of the axis follows the center of mass. The application of the law is illustrated in Fig. 16.13. Here, we follow a rod thrown across the lecture hall also during its collision with the floor. The small insets at the bottom shows the forces acting on the rod at five different time steps. When the rod is not in contact with the floor, the only external force acting is gravity (we ignore air resistance), and since gravity acts in the center of mass, the torque of gravity around the center of mass is zero, and the angular acceleration is therefore zero: The rod rotates with a constant angular velocity. However, when the rod is in contact with the floor, as illustrated in the middle three insets, the contact force $\mathbf{F}$ from the floor on the rod gives rise to a net torque around the center of mass, which leads to an angular acceleration during the contact. The force $\mathbf{F}$ acts at the point $\mathbf{r}_{cm}$ relative to the center of mass of the rod, giving rise to a torque: $\boldsymbol{\tau} = \mathbf{r}_{cm} \times \mathbf{F}$ around the center of mass. We address the motion of a bouncing rod in Sect. 16.3.3.

16.3.1 Example: Kicking a Ball

Problem: You are kicking a stationary football lying on the ground. The ball is spherical, with a mass M, radius R, and moment of inertia I around the center of mass. You kick the ball with a constant force $\mathbf{F} = F_x\,\mathbf{i}$ for a short time interval Δt.

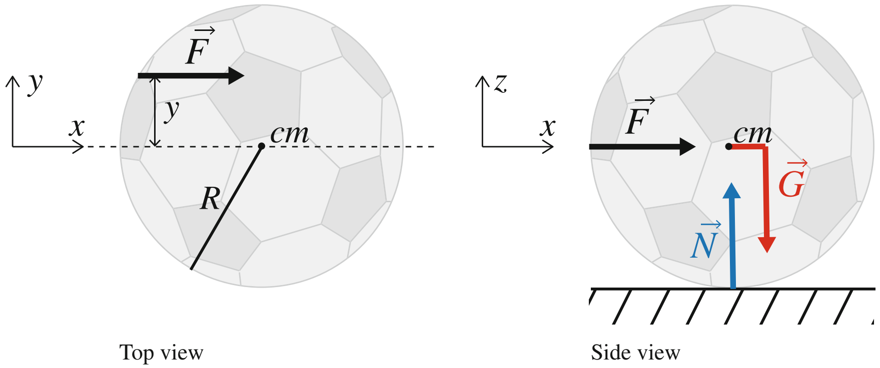


Fig. 16.14 You are kicking a ball—exerting a constant force $\mathbf{F} = F_x\,\mathbf{i}$ for a small time interval Δt. The force is acting at a distance y from the center of mass

hitting the ball at a distance y from its center, as illustrated in Fig. 16.14. Find the motion of the ball during and after the kick. You may neglect the effects of friction and air resistance.

Approach: We plan to use Newton's second law of motion to find the motion of the center of mass from the external forces acting, and Newton's second law for rotational motion around the center of mass to find the rotational motion around the center for the ball during and after the kick.

Identify: We assume that the ball behaves as a rigid body and describe its motion by the position, $\mathbf{R}(t)$ of its center of mass and its angle $\theta(t)$ around the z-axis. The ball starts from rest: $\mathbf{R}(0\,\mathrm{s}) = \mathbf{0}$ and $\theta(0\,\mathrm{s}) = 0$.

Model: The translational and rotational motion depends on the external forces acting. The ball is affected by gravity, $\mathbf{G} = -Mg\,\mathbf{k}$, acting at the center of mass, hence $\mathbf{r}_{G,cm} = 0$; the normal force $\mathbf{N} = N\,\mathbf{k}$ from the ground, acting at $\mathbf{r}_{N,cm} = -R\,\mathbf{k}$; and the force $\mathbf{F} = F\,\mathbf{i}$ from the foot, acting at $\mathbf{r}_{F,cm} = -x\,\mathbf{i} + y\,\mathbf{j}$, where x and y are given. All the positions are relative to the center of mass, since the acting points of the forces will move with the center of mass as the ball starts moving. The forces are illustrated in Fig. 16.14.

We apply Newton's second law to find the motion of the center of mass:

$$\sum_j \mathbf{F}_j^{\mathrm{ext}} = -Mg\,\mathbf{k} + N\,\mathbf{k} + F\,\mathbf{i} = M\mathbf{A}\,. \tag{16.46}$$

Since the applied force $\mathbf{F}$ only acts in the x-direction, we assume that the ball does not move in the y or z-directions. Therefore, the acceleration in the z-direction is zero and $N = Mg$. The motion in the x-direction is given by:

$$MA_x = F_x \;\Rightarrow\; A_x = \frac{F_x}{M}\,, \tag{16.47}$$

starting from $V_x = 0\,\text{m/s}$ and $X = 0\,\text{m}$ at $t = 0\,\text{s}$. Notice that the motion of the center of mass does not depend on the rotational motion of the ball, since (16.47) and the time, Δt, the force is acting does not depend on either the angle θ or the angular velocity ω of the ball!

The rotational motion of the ball is found from Newton's second law for rotational motion around a fixed axis around the center of mass. (How do we know the ball rotates around a fixed axis? We know, since the direction of the torque does not change during the motion—this is an advanced point we will return to later in the chapter. For now you should consider this an assumption):

$$\sum_j \tau_{z,cm,j} = I\alpha \ , \tag{16.48}$$

where the net torque around the center of mass is:

$$\begin{aligned}\sum_j \boldsymbol{\tau}_{cm,j} &= \mathbf{r}_{G,cm} \times \mathbf{G} + \mathbf{r}_{N,cm} \times \mathbf{N} + \mathbf{r}_{F,cm} \times \mathbf{F} \\ &= \mathbf{0} \times \mathbf{G} - R\mathbf{k} \times N\mathbf{k} + (-x\mathbf{i} + y\mathbf{j}) \times F\mathbf{i} \\ &= \mathbf{0} - NR\underbrace{(\mathbf{k} \times \mathbf{k})}_{=\mathbf{0}} - xF\underbrace{(\mathbf{i} \times \mathbf{i})}_{=\mathbf{0}} + Fy\underbrace{(\mathbf{j} \times \mathbf{i})}_{=-\mathbf{k}} \\ &= -yF\mathbf{k} \ ,\end{aligned} \tag{16.49}$$

which inserted in (16.48) gives:

$$\tau_{z,cm}^{\text{net}} = -yF_x = I\alpha \ \Rightarrow \ \alpha = -\frac{yF_x}{I} \ , \tag{16.50}$$

where $\theta(0\,\text{s}) = 0$ and $\omega(0\,\text{s}) = 0\,\text{rad/s}$.

Solve: Both the translational and the rotational motion occur with constant accelerations. We can therefore solve by direct integration to find the positions and velocities. First, we find the velocities:

$$V_x(t) = \underbrace{V_x(0)}_{=0} + \int_0^t A_x\,dt = \int_0^t \frac{F}{M}\,dt = \frac{F}{M}t \ , \tag{16.51}$$

and

$$\omega(t) = \underbrace{\omega(0)}_{=0} + \int_0^t \alpha\,dt = \int_0^t -\frac{yF}{I}\,dt = -\frac{yF}{I}t \ . \tag{16.52}$$

We integrate once more to find the positions:

$$X(t) = \underbrace{X(0)}_{=0} + \int_0^t V_x(t)\,dt = \int_0^t \frac{F}{M} t\,dt = \frac{1}{2}\frac{F}{M}t^2 , \tag{16.53}$$

and

$$\theta(t) = \underbrace{\theta(0)}_{=0} + \int_0^t \omega(t)\,dt = \int_0^t -\frac{yF}{I} t\,dt = -\frac{1}{2}\frac{yF}{I}t^2 . \tag{16.54}$$

However, this solution is only valid as long as the ball is affected by the force **F**, which only is for a time interval Δt. After that, the ball is not affected by the force **F**, and the net force in the x-direction is zero and the net torque is zero. Therefore the translational and the rotational accelerations are zero, and the ball continue with the same translational and rotational velocities:

$$V(t) = V(\Delta t) = \frac{F}{M}\Delta t \text{ when } t > \Delta t , \tag{16.55}$$

and

$$\omega(t) = \omega(\Delta t) = -\frac{yF}{I}\Delta t \text{ when } t > \Delta t . \tag{16.56}$$

Since the motion for $t > \Delta t$ occurs with constant translational and angular velocities, it is easy to find the position at a time $t > \Delta t$ by integration:

$$X(t) = X(\Delta t) + V(\Delta t)(t - \Delta t) \text{ when } t > \Delta t , \tag{16.57}$$

and

$$\theta(t) = \theta(\Delta t) + \omega(\Delta t)(t - \Delta t) \text{ when } t > \Delta t . \tag{16.58}$$

Analyze: We notice that the motion of the center of mass does not depend on y—the position where the force is acting. For any choice of y, the effect on the motion of the center of mass is the same. But the rotational motion depends on y. In particular, we notice that when $y = 0$, that is when the force acts on an axis through the center of mass, the net torque is zero, and the ball does not start to rotate. We also notice that if we kick the ball on the left side, with $y > 0$, the ball rotates in the negative direction, whereas if we kick the ball on the right side, with $y < 0$, the ball rotates in the positive direction, which is consistent with our experience with kicking balls. This is how you give them spin.

Comment: Notice that the translational and rotational motions are independent in this case. Why? Because the force acts at a constant distance y from the center of mass throughout the motion. Therefore the torque of **F** does not depend on the position of the ball, and similarly, the direction or magnitude of **F** does not depend on the rotation of the ball. This is not always the case: In many cases the two motions are coupled because the force acting on the object depend either on the position of the object or on its rotation. But in this particular case—as well as in many cases in typical mechanics exam problems—the two motions are not coupled.

16.3.2 Example: Rolling down an Inclined Plane

(*This problem is a classic in mechanics*.)

Problem: A round object is placed on an inclined plane. The object has radius R, mass M, and moment of inertia I around the rotation axis through the center of mass. Find the acceleration of the object and the friction force on the object. Discuss various angles of inclination ϕ, various objects, and various initial conditions for the object.

Approach: We plan to find the external forces and use Newton's second law of translational and rotational motion to find the acceleration, using the rolling condition as long as the object is rolling.

Identify: We choose a coordinate system with the x-axis oriented along the inclined plane, so that there is no motion in the y-direction. (See Fig. 16.15). The object is described by the position $X(t)$ of its center of mass, and the rotational angle, $\theta(t)$, around the center of mass. The object starts at the position $X(0\,\text{s}) = 0\,\text{m}$, and $\theta(0\,\text{s} = 0$ with initial velocities $V_x = V_0$, and $\omega(0\,\text{s}) = \omega_0$.

Model: The object is affected by a normal force, $\mathbf{N} = N\,\mathbf{j}$, acting at $\mathbf{r}_{N,cm} = -R\,\mathbf{j}$; a friction force, $\mathbf{f} = -f\,\mathbf{i}$, acting at $\mathbf{r}_{f,cm} = -R\,\mathbf{j}$; and a gravitational force $\mathbf{G}$ acting at the center of mass, $\mathbf{r}_{G,cm} = \mathbf{0}$. We decompose the gravitational force in the chosen coordinate system:

$$\mathbf{G} = Mg\sin\phi\,\mathbf{i} - Mg\cos\phi\,\mathbf{j}\ . \tag{16.59}$$

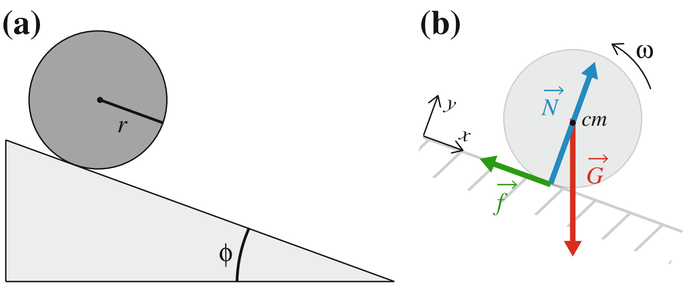


Fig. 16.15 **a** Illustration of a round object moving along an inclined plane. **b** Free-body diagram for the object

We apply Newton's second law to determine the translational acceleration:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \mathbf{G} + \mathbf{N} + \mathbf{f} = M\mathbf{A} \ . \tag{16.60}$$

In the x-direction:

$$\sum \mathbf{F}_x = Mg \sin\phi - f = MA_x \ . \tag{16.61}$$

Since there is no motion in the y-direction, we get

$$\sum \mathbf{F}_y = N - Mg\cos\phi - f = MA_y = 0 \ , \tag{16.62}$$

which gives $N = Mg\cos\phi$.

Newton's second law for rotation around the center of mass gives:

$$\sum_j \tau_{z,cm,j} = I\alpha \ , \tag{16.63}$$

where the net torque around the center of mass is:

$$\begin{aligned} \sum_j \boldsymbol{\tau}_{cm,j} &= \underbrace{\mathbf{0} \times \mathbf{G}}_{=0} + \underbrace{\mathbf{r}_{N,cm}}_{=-R\,\mathbf{j}} \times \underbrace{\mathbf{N}}_{=N\,\mathbf{j}} + \underbrace{\mathbf{r}_{f,cm}}_{=-R\,\mathbf{j}} \times \underbrace{\mathbf{f}}_{=-f\,\mathbf{i}} \\ &= -RN \underbrace{\mathbf{j} \times \mathbf{j}}_{=0} - R(-f) \underbrace{\mathbf{j} \times \mathbf{i}}_{=-\mathbf{k}} = -Rf\,\mathbf{k} \ , \end{aligned} \tag{16.64}$$

which inserted in (16.63) gives:

$$\tau_z^{\text{net}} = -Rf = I\alpha \ . \tag{16.65}$$

We now have 2 equations, (16.61) and (16.65), but 3 unknowns, A_x, α, and f. How do we proceed?

The main challenge in this problem lies in the friction force. In the case where the friction is dynamic, that is, if the object is sliding relative to the surface, the magnitude of the friction force is simply proportional to the normal force, and the direction is determined from the local relative velocity of the two surfaces in contact. However, if the object does not slide, the friction force is a static friction force, and its magnitude must be determined from other principles.

Let us first assume that the object is not sliding relative to the surface. This means that the object is rolling without slipping. The rolling condition is that there is no relative velocity at the contact point, P, $\mathbf{v}_P = \mathbf{0}$, where

$$\mathbf{v}_P = \mathbf{V} + \mathbf{v}_{P,cm} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_{P,cm} = V_x\,\mathbf{i} + \omega\,\mathbf{k} \times (-R\,\mathbf{j}) = (V_x + \omega R)\,\mathbf{i} \ , \tag{16.66}$$

and the condition $\mathbf{v}_p = \mathbf{0}$ therefore gives $V_x = -\omega R$, which is usually called the rolling condition. Taking the time derivative gives a similar expression for the accelerations: $A_x = -R\alpha$, which we insert this into (16.65), getting:

$$f = -\frac{I}{R}\alpha = -\frac{I}{R}\frac{-A_x}{R} = \frac{I}{R^2}A_x \; . \tag{16.67}$$

The static friction force must have this value to ensure that there is no slipping between the object and the surface. We insert this result into (16.61), giving:

$$\begin{aligned} Mg\sin\phi - \underbrace{f}_{=(I/R^2)A_x} &= MA_x \\ Mg\sin\phi &= \left(1 + \frac{I}{MR^2}\right)MA_x \; , \end{aligned} \tag{16.68}$$

and

$$A_x = \frac{g\sin\phi}{1 + \frac{I}{MR^2}} \; . \tag{16.69}$$

We introduce $c = I/(MR^2)$ to simplify the expression:

$$A_x = \frac{1}{1+c}g\sin\phi \; . \tag{16.70}$$

The number c depends on the distribution of mass around the rotation axis. For a sphere, $c = 2/5$, for a cylinder $c = 1/2$, and for a ring, $c = 1$.

The friction force is:

$$f = \frac{I}{R^2}A_x = Mg\sin\phi\frac{\frac{I}{MR^2}}{1 + \frac{I}{MR^2}} = Mg\sin\phi\frac{c}{1+c} \; . \tag{16.71}$$

It is clear that the friction force f increases with ϕ, and at the same time N decreases with ϕ. When will the friction force reach the static friction threshold? The objects starts to slip at the critical angle ϕ_c when $f = \mu_s N$, where we now insert the values we found for f and N, getting:

$$Mg\sin\phi_c\frac{c}{1+c} = \mu_s Mg\cos\phi_c \;\Rightarrow\; \tan\phi_c = \mu_s\frac{1+c}{c} \; . \tag{16.72}$$

The critical angle ϕ_c depends on the number c—it will therefore depend on the type of object rolling. For a few characteristic objects we have:

Object	I	$\tan\phi_c$
Sphere	$(2/5)mR^2$	$(7/2)\mu_s$
Cylinder	$(1/2)mR^2$	$3\mu_s$
Ring	mR^2	$2\mu_s$

A sphere will therefore *roll* down a steeper slope than a cylinder, which will start slipping at a lower angle than the sphere.

What happens if the slope is steeper than ϕ_c? In this case the object will still rotate, but it will not roll without slipping. This means that the rolling condition is no longer valid. Instead the friction force is now the dynamic friction force, which only depends on the normal force:

$$f = \mu_d N = \mu_d M g \cos\phi \ . \tag{16.73}$$

We insert this into Newton's second law for translational motion, (16.61), getting:

$$Mg\sin\phi - \mu_d Mg\cos\phi = MA_x \ , \tag{16.74}$$

which allows us to determine A_x independently of the rotational motion:

$$A_x = g\left(\sin\phi - \mu_d\cos\phi\right) \ . \tag{16.75}$$

However, the object will still rotate, since the torque around the center of mass is:

$$\tau_z = -Rf = -R\mu_d Mg\cos\phi = I\alpha \ . \tag{16.76}$$

The angular acceleration is therefore:

$$\alpha = -\frac{Rf}{I} = -\mu_d\frac{RMg\cos\phi}{I} \ . \tag{16.77}$$

16.3.3 Example: Bouncing Rod

In this example we will develop a model for a bouncing rigid rod. The model will be similar to the bouncing an rotating dumbbell model in Chap. 13, but we will now model a rigid rod, so there are no internal vibrations, and we will use our newly gained knowledge of how to model rotating rigid object for our model.

The rod is of length $L = 1$ m, mass $M = 0.5$ kg, and has a moment of inertia $I = (1/12)ML^2$ around its center of mass. We describe the rod by the position $\mathbf{R}(t)$ of its center of mass and the angular orientiation $\theta(t)$, where we assume that the rod moves in the xy-plane and rotates around an axis through the center of mass directed along the z-axis.

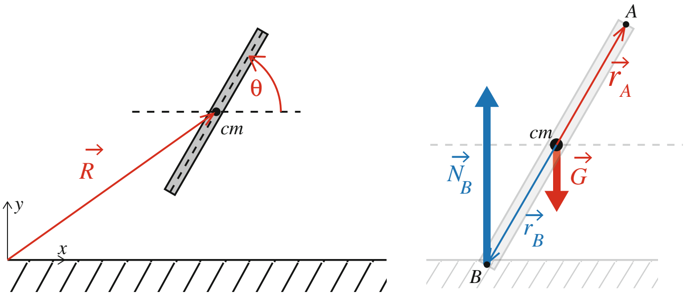


Fig. 16.16 Illustration of the rod bouncing on the floor and free-body diagram for the rod

The motion of the rod is determined by the forces acting on it. The rod is affected by gravity, $\mathbf{G} = -Mg\,\mathbf{j}$, acting at the center of mass, $\mathbf{r}_{G,cm} = \mathbf{0}$. In addition the rod will bounce on the floor. We model the force between the floor and the rod as a spring force, representing the deformation of the floor and the rod. If either end of the rod is pressed into the floor, that is, if the y-coordinate of either end is below $y = 0$, which is the position of the floor, there will be a spring force acting normal to the floor, in the positive y-direction, which depends on how far the end of the rod has been pressed down into the floor. The two ends of the rod are at positions:

$$\mathbf{r}_A = \mathbf{R} + (L/2)\hat{u} \ , \tag{16.78}$$

$$\mathbf{r}_B = \mathbf{R} - (L/2)\hat{u} \ , \tag{16.79}$$

where $\hat{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is a unit vector pointing along the rod, as illustrated in Fig. 16.16. The normal force, $\mathbf{N}_A$, due to the interaction between end A of the rod is:

$$\mathbf{N}_A = \begin{cases} -ky_A & \text{when } y_A < 0 \\ 0 & \text{when } y_A \geq 0 \ . \end{cases} \tag{16.80}$$

and similarly for end B:

$$\mathbf{N}_B = \begin{cases} -ky_B & \text{when } y_B < 0 \\ 0 & \text{when } y_B \geq 0 \ . \end{cases} \tag{16.81}$$

Here, k is the spring constant for the interaction between the rod and the floor.

The motion of the rod is determined from Newton's second law for translational and rotational motion:

$$\sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{N}_A + \mathbf{N}_B = M\mathbf{A} \ , \tag{16.82}$$

and

$$\sum_j \tau_{z,cm,j} = I\alpha \ , \tag{16.83}$$

where

$$\sum_j \boldsymbol{\tau}_{cm,j} = \mathbf{0} \times \mathbf{G} + \mathbf{r}_{A,cm} \times \mathbf{N}_A + \mathbf{r}_{B,cm} \times \mathbf{N}_B \ . \tag{16.84}$$

Here, $\mathbf{r}_{A,cm} = (L/2)\hat{u}$ and $\mathbf{r}_{B,cm} = -(L/2)\hat{u}$. Notice that both the net force and the net torque depends on both the position of the center of mass and on the angle, since the contact force between the rod and the floor depends on the position and orientiation of the rod.

Numerical: It is not simple to solve these equations analytically, but it is straight forward to implement a numerical solution. We use an Euler-Cromer scheme to integrate both the translational and the rotational motion, simultaneously:

$$\begin{aligned}
\mathbf{V}(t + \Delta t) &= \mathbf{V}(t) + \mathbf{A}(t, \mathbf{R}(t), \theta(t))\, \Delta t \\
\mathbf{R}(t + \Delta t) &= \mathbf{R}(t) + \mathbf{V}(t + \Delta t)\, \Delta t \\
\omega(t + \Delta t) &= \omega(t) + \alpha(t, \mathbf{R}(t), \theta(t))\, \Delta t \\
\theta(t + \Delta t) &= \theta(t) + \omega(t + \Delta t)\, \Delta t \ ,
\end{aligned} \tag{16.85}$$

This scheme can be implemented directly into the code, using Newton's second laws to calculate $\mathbf{A}$ and α at each time-step:

```
from pylab import *
# Physical constants
m = 0.5      # kg
g = 9.8      # m/s^2
h0 = 4.0     # m
L = 1.0      # m
time = 10.0 # s
dt = 0.001  # s
k = 1000.0  # N/m
v0 = 2.0    # m/s
I = (1.0/12.0)*m*L**2
# Variables
n = int(round(time/dt))
r = zeros((n,3),float)
v = zeros((n,3),float)
theta = zeros(n,float)
omega = zeros(n,float)
t = zeros(n,float)
# Initial conditions
r[0] = array([0,h0,0])
v[0] = array([v0,0,0])
theta[0] = 2*pi*random.rand(1)
# Calculate motion
for i in range(n-1):
    # Find force acting on each edge
    fnet = array([0,0,0])
    tnet = 0.0
```

```
u = array([cos(theta[i]),sin(theta[i]),0])
# Position of edge A
rr = r[i] + 0.5*L*u
# Collision with bottom wall
dr = rr[1]
f = -k*dr*(dr<0.0)*array([0,1,0])
fnet = fnet + f
torque = cross((rr-r[i]),f)
tnet = tnet + torque
# Position of edge B
rr = r[i] - 0.5*L*u
# Collision with bottom wall
dr = rr[1]
f = -k*dr*(dr<0.0)*array([0,1,0])
fnet = fnet + f
torque = cross((rr-r[i]),f)
tnet = tnet + torque
# Add gravity
fnet = fnet - m*g*array([0,1,0])
# Integration step - Euler-Cromer
a = fnet/m
v[i+1] = v[i] + a*dt
r[i+1] = r[i] + v[i+1]*dt
alphaz = tnet[2]/I
omega[i+1] = omega[i] + alphaz*dt
theta[i+1] = theta[i] + omega[i+1]*dt
t[i+1] = t[i] + dt
if (mod(i,20)==0):
    # Plot position of rod, with tracer
    r1 = r[i] + 0.5*L*u
    r2 = r[i] - 0.5*L*u
    xl = array([r1[0],r2[0]])
    yl = array([r1[1],r2[1]])
    ion()
    clf()
    plot(r[0:i,0],r[0:i,1],':',xl,yl,'-')
    xlabel('x [m]')
    ylabel('y [m]')
    axis([0,time*v0,0,h0])
    gca().set_aspect('equal',adjustable='box')
    draw()
```

The resulting path of the rod is shown in Fig. 16.17. Use the program to experiment and see what happens as you change parameters.

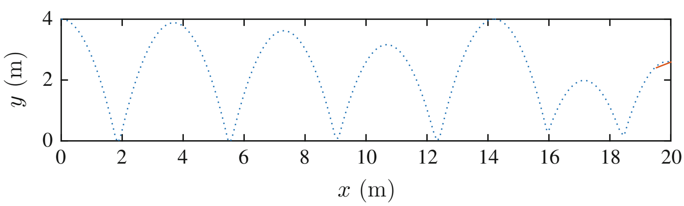


Fig. 16.17 Plot of the path of the rod

16.4 Collisions and Conservation Laws

If a meteor collides with a planet (see Fig. 16.18), the planet may gain both translational and rotational motion after the collision. We have already found that conservation principles, such as the conservation of translational momentum, allow us to determine the translational motion of the planet after the collision without knowing the details of the interactions during the collision. Can we find similar concepts and conservation laws for rotational motion—a rotational momentum and a conservation law for rotational motion—which we can use to address the rotational motion of a system during a collision? Here, we introduce the concept of rotational momentum first for a point particle, then for a system of many particles, and finally for a rotating rigid body, before we put all the pieces together and formulate a set of general principles allowing us to address collisions between several rigid, rotating object—such as during a meteor impact or during a pirouette.

Rotational Momentum for a Point Particle

We have already found that the translational momentum, **p**, is a useful concept to address collisions. This is based on the prominent place of translational momentum in Newton's second law:

$$\sum_j \mathbf{F}_j = \frac{d\mathbf{p}}{dt} . \tag{16.86}$$

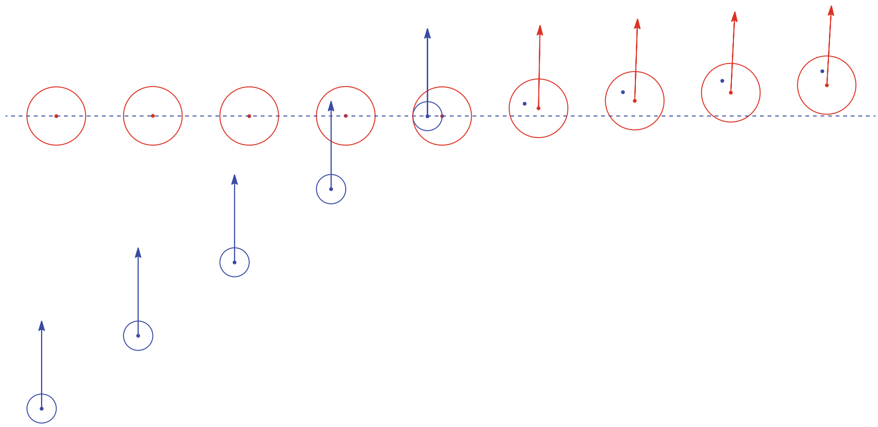


Fig. 16.18 A meteor impact on a planet may change both the translational and the rotational motion of the planet. The *arrows* show the translational momentum and the *blue dot* shows the rotational state of the planet after the collision

It is indeed this law that justifies the conservation law: When the net external force is zero, the time derivative of the translational momentum is zero and therefore conserved throughout the process. Let us see if we can justify a similar concept for rotational motion, based on Newton's second law for rotational motion:

$$\boldsymbol{\tau}^{\text{net}} = \sum_j \mathbf{r}_j \times \mathbf{F}_j \ . \tag{16.87}$$

If we study the motion of a point particle, all the torques act in the same point, $\mathbf{r}$, and the net torque is:

$$\boldsymbol{\tau}^{\text{net}} = \sum_j \mathbf{r} \times \mathbf{F}_j = \mathbf{r} \times \sum_j \mathbf{F}_j = \mathbf{r} \times \mathbf{F}^{\text{net}} \ , \tag{16.88}$$

where we can insert $\mathbf{F}^{\text{net}}$ from Newton's second law in (16.86):

$$\boldsymbol{\tau}^{\text{net}} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \ . \tag{16.89}$$

In order to get something that looks like Newton's second law in (16.86), we try to move the time derivative out in front of $\mathbf{r} \times \mathbf{p}$: What does this give?

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \underbrace{\mathbf{v} \times m\mathbf{v}}_{=\mathbf{0}} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \ . \tag{16.90}$$

Yes! Success! We can therefore rewrite (16.89) as:

$$\boldsymbol{\tau}^{\text{net}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d}{dt}\mathbf{l} \ . \tag{16.91}$$

This equation looks just like Newton's second law—we have found what we have looking for, $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is the rotational analogue for translational momentum. We call $\mathbf{l}$ the *rotational momentum* or the *angular momentum* :

$$\mathbf{l}_O = \mathbf{r} \times \mathbf{p} \qquad \text{(Rotational momentum)} \ , \tag{16.92}$$

where we use the subscript O to show that the rotational momentum is found with respect to the point O, and the vector $\mathbf{r}$ is found relative to O. We say that $\mathbf{l}_O$ is the rotational momentum around the point O or around an axis z (if the particular point along the axis is not important).

Properties of Rotational Momentum

- Rotational momentum is a **vector**. For planar motion, the rotational momentum is normal to the plane.
- The rotational momentum is calculated relative to a point—just like torque. It will be different if we choose a different point.

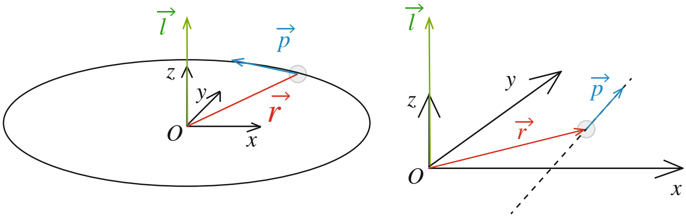


Fig. 16.19 Illustration of rotational momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ for circular motion and motion along a *straight line*

For a particle moving in a circle with radius r, in the xy-plane, as illustrated in Fig. 16.19, the velocity $\mathbf{v}$ is always normal to the position $\mathbf{r}$, and the rotational momentum around the origin at the center of the rotational motion is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v} = rmv\,\mathbf{k} \,. \tag{16.93}$$

The rotational momentum can be defined for any moving point particle, not only for a rotating point particle. For example, the rotational momentum of a particle moving along a straight line at $x = b$, $\mathbf{r} = b\,\mathbf{i} + y(t)\,\mathbf{j}$, is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v} = (b\,\mathbf{i} + y(t)\,\mathbf{j}) \times v_y\,\mathbf{j} = bv_y\,\mathbf{k} \,. \tag{16.94}$$

As illustrated in Fig. 16.19 this means that it is only the component of $\mathbf{r}$ that is normal to $\mathbf{v}$ that contributes to the rotational momentum, similar to what we previously found for torques.

Newton's Second Law for Rotational Motion of a Point Particle

We have found an alternative formulation of Newton's second law for a point particle, most useful for rotational motion, but valid for any motion:

$$\boldsymbol{\tau}^{\text{net}} = \frac{d}{dt}\mathbf{l} \,, \tag{16.95}$$

Conservation of Rotational Momentum

Based on Newton's second law in (16.95) we see that for a point particle, the rotational momentum, $\mathbf{l}$, is conserved if the net external torque is zero! While this can be used to address problems with a single particle, it first becomes really useful when we introduce similar concepts for a rigid body or a system of many particles.

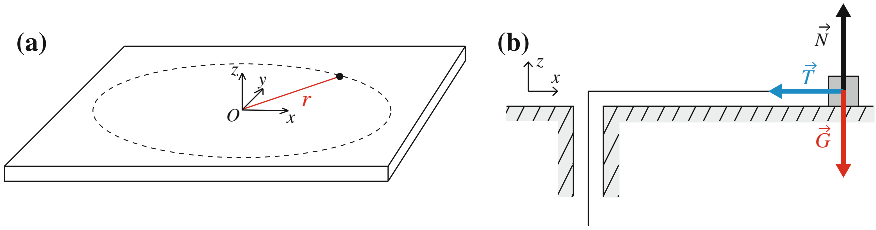


Fig. 16.20 A sketch of the motion of the block sliding on a frictionless table (**a**) and a free-body diagram for the block (**b**)

16.4.1 Example: Block on a Frictionless Table

Problem: A block of mass m is attached to a thin, massless rope passing through a hole in a frictionless table. The block starts with the angular velocity ω_0 at a distance r_0 from the hole. We pull slowly in the rope until it reaches the radius r with the angular velocity ω. Find ω.

Identify: As long as the rope is tight, the block moves in a circular path with radius r. We describe the position of the block by the radius r and its angle θ, as illustrated in Fig. 16.20.

Model: First, we find the forces acting on the block. The block is affected by gravity, $\mathbf{G}$, and the normal force, $\mathbf{N}$, from the table. Since the block is not moving in the vertical direction, the net vertical force is zero, and since two forces are acting in the same point, the net torque of the two forces (around any point) is also zero. In addition, the block is affected by the rope tension, $\mathbf{T}$.

We use Newton's second law for rotational motion to determine the motion of the block. The torque around the origin of the rope tension is:

$$\boldsymbol{\tau}_T = \mathbf{r} \times \mathbf{T} = 0 \,, \tag{16.96}$$

because the force and the position vector are parallel. This means that the net torque around the origin is zero. Newton's second law for rotational motion therefore gives that:

$$\frac{d\mathbf{l}}{dt} = \boldsymbol{\tau} = 0 \,, \tag{16.97}$$

the rotational momentum is therefore constant throughout the motion. The rotational momentum for the block is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v} \,, \tag{16.98}$$

where the velocity is normal to the radius vector at all times, therefore:

$$\mathbf{l} = rmv_{¿} . \tag{16.99}$$

We can replace the velocity by the angular velocity, using $v = R\omega$:

$$\mathbf{l} = mr^2\omega, \tag{16.100}$$

which is what we found above for circular motion.

Analyze: Since the rotational momentum is conserved, we can relate the initial and final states. Initially, the rotational momentum is:

$$\mathbf{l}_0 = mr_0^2\omega_0. \tag{16.101}$$

When the radius is r and the angular velocity ω:

$$\mathbf{l} = mr^2\omega. \tag{16.102}$$

Since $\mathbf{l} = \mathbf{l}_0$ we find ω:

$$mr_0^2\omega_0 = mr^2\omega \;\Rightarrow\; \omega = \frac{r_0^2}{r^2}\omega_0 \ . \tag{16.103}$$

This demonstrates the use of the conservation principle for rotational momentum.

Rotational Momentum for a System of Particles

The rotational momentum of a system of particles around a fixed point, O, is the sum of the rotational momentum of each particle. For a system consisting of point masses, m_i, located at points, $\mathbf{r}_i$, the total rotational momentum is:

$$\mathbf{L}_O = \sum_i \mathbf{l}_{O,i} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times m_i\mathbf{v}_i \ . \tag{16.104}$$

Similarly, we define the rotational momentum of a system of particles around their center of mass as

$$\mathbf{L}_{cm} = \sum_i \mathbf{l}_{O,i} = \sum_i \mathbf{r}_{cm,i} \times \mathbf{p}_i = \sum_i \mathbf{r}_{cm,i} \times m_i\mathbf{v}_i \ , \tag{16.105}$$

where $\mathbf{r}_{cm,i}$ is the position of mass m_i relative to the center of mass. The rotational momentum of a system around a fixed axis O can also be decomposed into the rotational momentum of the center of mass around O, assuming the center of mass moves as a point particle, and the rotational momentum of the system around its center of mass:

$$\mathbf{L}_O = \mathbf{R} \times \mathbf{P} + \mathbf{L}_{cm} \ , \tag{16.106}$$

where $\mathbf{R}$ is the position of the center of mass (relative to O), and $\mathbf{P}$ is the translational momentum of the whole system. (You can find a proof of (16.106) in Sect. A.10).

Newton's Second Law of Rotation for a Multiparticle System

Also for a multiparticle system, we find a general form for Newton's second law for rotational motion:

Newton's second law for rotational motion of a system of particles around a fixed point O:

$$\frac{d\mathbf{L}_0}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_0^{\text{ext}} , \tag{16.107}$$

where only the *external forces* are included. The internal forces cancel as long as they are central forces.

(You can find a proof in Sect. A.8). We can also derive a completely analoguous law for the rotational momentum around the center of mass of a system:

Newton's second law for rotational motion of a system of particles around its center of mass, cm:

$$\frac{d\mathbf{L}_{cm}}{dt} = \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_{cm}^{\text{ext}} , \tag{16.108}$$

where the positions $\mathbf{r}_{cm,i}$ are the positions of each mass m_i relative to the center of mass of the system.

(You can find a proof in Sect. A.11). This law is general and powerful. You will find it used frequently both theoretically, as basis for derivations, and practically, as a basis for the use of conservation laws. The law also demonstrates that as long as the torque of the external forces are constant, around a fixed point O or the center of mass, the corresponding total rotational momentum does not change. This is the law for rotational motion we have been looking for. However, in order to apply it to the collision of rigid bodies, we need to find a simplified expression for the rotational momentum of a rigid object, both around a fixed axis and around its center of mass.

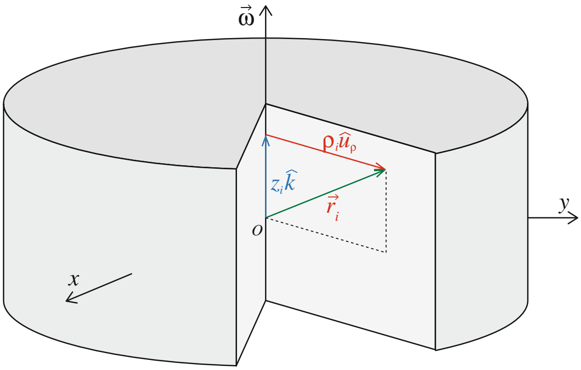


Fig. 16.21 Illustration of a rigid body rotating around the z-axis with an angular velocity $\boldsymbol{\omega}$. We describe the position $\mathbf{r}$ using cylindrical coordinates, so that $\mathbf{r} = \boldsymbol{\rho} + z\,\mathbf{k} = \rho\hat{u}_\rho + z\,\mathbf{k}$, where $\boldsymbol{\rho}$ is in the plane normal to the axis of rotation

Rotational Momentum for a Rigid Body

We know that the translational momentum of a rigid body can be written as $\mathbf{P} = M\mathbf{V}$, where $\mathbf{V}$ is the velocity of the center of mass. Can we find a similarly simple expression for the rotational momentum, $\mathbf{L}_O$ of a rigid body around a fixed axis?

The rotational momentum for a rigid body is rotating around a fixed axis z through the point O.[2]

$$\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \ , \tag{16.109}$$

For a rigid body rotating around the axis z with an angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$, each mass m_i at $\mathbf{r}_i$ has a velocity:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \ . \tag{16.110}$$

To simplify the expression, we introduce cylindrical coordinates to describe the position of mass i, where the cylindrical axis follows the z-axis, as illustrated in Fig. 16.21. The position is decomposed as:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i\,\mathbf{k} = \rho_i\hat{u}_\rho + z_i\,\mathbf{k} \ , \tag{16.111}$$

where $\hat{u}_\rho$ is a unit vector in the xy-plane pointing from the z-axis to the point $\mathbf{r}_i$, as shown in Fig. 16.21. We insert this into (16.110), getting:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \omega\,\mathbf{k} \times (\boldsymbol{\rho}_i + z_i\,\mathbf{k}) = \omega\,\mathbf{k} \times \boldsymbol{\rho}_i + \omega\underbrace{\mathbf{k} \times \mathbf{k}}_{=0} = \boldsymbol{\omega} \times \boldsymbol{\rho}_i \ . \tag{16.112}$$

We find the rotational momentum by inserting this into (16.109):

[2]Notice that O must be a point on the rotation axis.

$$\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ . \tag{16.113}$$

Hmm. How do we simplify this equation? If we only are interested in the z-component of the rotational momentum (which we typically are when we use Newton's second law for rotational motion of a rigid body), we see that it is only the $\boldsymbol{\rho}_i$ component of $\mathbf{r}_i$ that contributes to the z-component of $\mathbf{L}_O$:

$$\mathbf{L}_{O,z}\mathbf{k} = \sum_i m_i \boldsymbol{\rho}_i \times (\omega \mathbf{k} \times \boldsymbol{\rho}_i) = \sum_i m_i \rho_i^2 \omega \mathbf{k} \ , \tag{16.114}$$

because the vectors $\boldsymbol{\rho}_i$ and $\mathbf{v}_i$ are normal to each other, and the vectors $\boldsymbol{\rho}_i$ and $\boldsymbol{\omega}$ are normal to each other. Therefore:

$$L_{O,z} = \left(\sum_i m_i \rho_i^2 \right) \omega = I_z \omega \ , \tag{16.115}$$

where we recognize the moment of inertia (rotational inertia), I_z, around the z-axis.

The **rotational momentum**, $L_{O,z}$, of a rigid body rotating around a fixed axis is:

$$L_{O,z} = I_z \omega \ , \tag{16.116}$$

where I_z is the moment of inertia of the object around the z-axis, and ω is the angular velocity around the z-axis.

We can use this result also for a rigid body rotating around a fixed axis through its center of mass. We notice that if we insert this into Newton's second law for rotational motion of a multi-particle system around a fixed axis in (16.107) or (16.108), we recover Newton's second law for rotational motion:

$$\sum_j \tau_{z,O,j}^{\text{ext}} = \frac{d}{dt} L_{O,z} = \frac{d}{dt}(I_z \omega) = I_z \frac{d\omega}{dt} = I_z \alpha \ , \tag{16.117}$$

which was the result we started this chapter with. Notice the nice similarity between the translational and rotational momentum of a rigid body:

	Translational	Rotational
Inertia	M	I_z
Momentum	$\mathbf{P} = M\mathbf{V}$	$L_{O,z} = I_z \omega$
N2L	$\mathbf{F}^{\text{net}} = M\mathbf{A}$	$\tau_{O,z}^{\text{net}} = I_z \alpha$

Limitations

We have previously noted that Newton's second law for rotations of rigid bodies only are valid for rotations around a fixed axis. Similarly, the expression $L_{O,z} = I_z\omega$ is only valid for rotation around a fixed axis, and it is only valid in the z-axis. But would it not be nice if the expression was completely general:

$$\mathbf{L}_O = I\boldsymbol{\omega} \ . \tag{16.118}$$

As you will learn later, we can formulate the relation in this way, but then I is a more complicated quantity than a mere scalar. However, this expression is *not generally correct* if I is interpreted as a scalar such as I_z. (You can see a proof in Sect. A.9). We can only use the expression in (16.118) when the rigid body is rotationally symmetric around the z-axis. However, our results for the z-component, $L_{O,z} = I_z\omega$, of the rotational momentum of a rigid body are always true—as long as the body is rotating around a *fixed* axis.

Putting It All Together

Finally, we want to put all these results together to address processes where we can use conservation of rotational momentum to solve a problem without determining the details of the motion.

Redistribution of Mass

An example of a typical process where we can use conservation of rotational momentum, is the redistribution of mass within a rotating system. For example, if a skater performing a piruoutte pulls in his arms, he changes the distribution of mass, and therefore the moment of inertia around the rotation axis. Since the external forces have no significant torque around the rotation axis, the rotational momentum is conserved throughout the process:

$$L_{O,z}(t_0) = I_z(t_0)\omega(t_0) = L_{O,z}(t_1) = I_z(t_1)\omega(t_1) \ . \tag{16.119}$$

If you change the distribution of mass, you change I_z. To keep the rotational momentum constant, you must change the angular velocity correspondingly. Pulling your arms in while spinning reduces I_z. As a result the angular velocity will increase.

Collision with Rotation Around a Fixed Axis

Another example of a typical process where we can use conservation of rotational momentum is a collision where the bodies after the collision rotate as one rigid body around a fixed axis. If the external torques acting on the bodies are insignificant, the rotational momentum around the fixed axis is conserved:

$$L_{O,z}(t_0) = L_{O,z}(t_1) = I_z(t_1)\omega(t_1) \ , \tag{16.120}$$

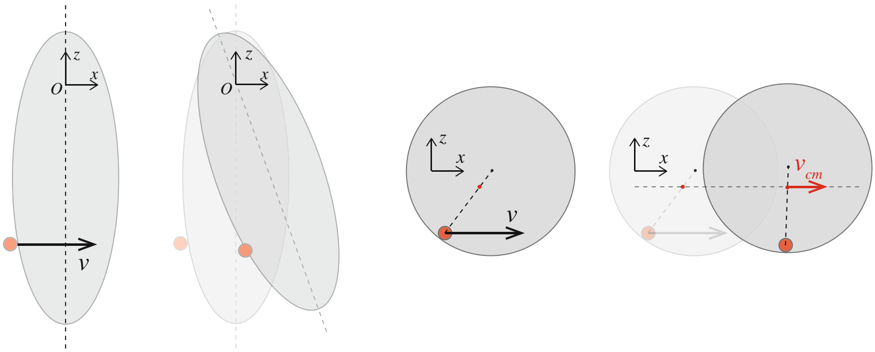


Fig. 16.22 *Left* A collision between a bullet and a rigid body hanging from the point O. *Right* A collision between a meteor and a planet

which allows us to find the angular velocity after the collision. A typical example of such a process is where a bullet is shot into a rigid body: Such as a meteor impacting on a planet or a bullet shot into a rigid pendulum, as illustrated in Fig. 16.22.

Since Newton's second law for rotational motion comes in two forms: for rotation around an axis that is fixed in space and for rotation around the center of mass, we must be careful to check what version is revelant in a given case. Notice that for a fixed axis, a typical error is to assume that the net force is zero because the net torque is zero. Unfortunately, this is generally wrong. In the case illustrated in Fig. 16.22 you cannot ignore the force from the axis on the rotating object, and the translationa momentum is therefore not conserved, while the rotational momentum is conserved. However, if the bodies are free to move, such as when a planet is hit by a meteor as illustrated in Fig. 16.22, the net external force is zero, and the translational momentum is conserved. You must therefore carefully address the net external force and the net external torque in each situation.

16.4.2 Example: Changing Your Angular Velocity

Problem: How can you control your angular velocity if you are (a) spinning around a pole (b) spinning while diving.

Solution: A rotating plate with a pole at its center can often be found on childrens playgrounds. You jump onto the plate and hold the pole, and spin the plate up with your feet. Then you can change your angular velocity by pulling yourself in towards the pole as illustrated in Fig. 16.23. How does this work?

The only external forces acting on the system consiting of you and the plate are gravity and the forces acting on the axis of rotation, but the z-component of the torques of these forces are zero. The rotational momentum around the z-axis

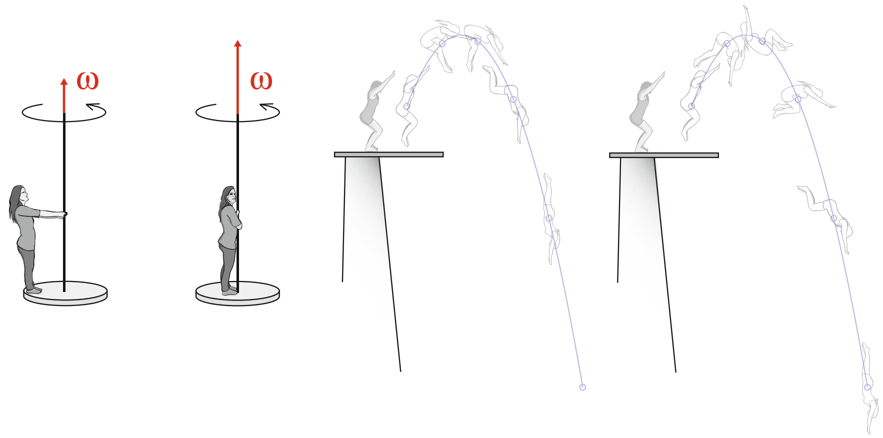


Fig. 16.23 *Left* A person on a spinning disk increases his angular velocity by pulling himself closer to the rotation axis. *Right* A diver changes her angular velocity around the center of mass by curling up, but that does not affect the path taken by the center of mass significantly

is therefore conserved. When you pull yourself towards the rotation axis, you are changing how the mass is distributed. You moment of inertia around the rotation axis is approximately $I_z = MR^2$, where R is the distance from you to the axis (if you were a single point). Conservation of rotational momentum then gives:

$$L_z(t_0) = I_z(t_0)\omega_0 = L_z(t_1) = I_z(t_1)\omega_1 \ , \tag{16.121}$$

$$MR_0^2\omega_0 = MR_1^2\omega_1 \ . \tag{16.122}$$

where ω_0 is your initial angular velocity when you are at a distance R_0 from the pole, and

$$\omega_1 = \frac{R_0^2}{R_1^2}\omega_0 \ , \tag{16.123}$$

is the angular velocity you getting. Pulling yourself towards the pole, makes $R_1 < R_0$ and therefore $\omega_1 > \omega_0$: You speed up!

Similarly, if you start your dive with a small angular velocity ω_0 around your center of mass when your body is approximately stretched out in its full length, you can increase your angular velocity, by pulling your body closer towards the center of mass, decreasing your moment of inertia around the center of mass:

$$I_{cm,z}(t_0)\omega(t_0) = I_{cm,z}(t_1)\omega(t_1) \ . \tag{16.124}$$

By stretching out again, you can decrease your angular velocity, effectively allowing you to determine how fast you spin. However, you cannot control the motion of your center of mass: The path taken by your center of mass is determined by the external forces acting on you, gravity and air resistance, and gravity is not affected by your redistribution of mass. Therefore you will typically follow the same path irrespectively of how you spin during your dive. (There may be small differences because air resistance will depend both on your spin and on how your body is curled up).

16.4.3 Example: Conservation of Rotational Momentum

Problem: A fly of mass m lands at the outer rim of a spinning DVD. The disk has mass M and radius R, and rotates with the angular velocity ω_0 before the event. Find the angular velocity of the plate after the fly has landed. (You can assume that the fly is able to grip onto and remain attached to the plate.)

Solution: The system consists of two objects: The DVD and the fly, as illustrated in Fig. 16.24. We choose the fixed rotational axis through the center of the DVD as the origin.

We plan to use conservation of rotational momentum to solve the problem. The external forces on the system consiting of the DVD and the fly, are the gravitational forces on the disk and the fly, and the force on the disk acting in the attachment point at the center of the disk. The gravitational force on the disk and the forces on the disk from the attachment point in the axis do not contribute to the net torque around the axis, since the "arm", $\mathbf{r}$, is zero for these forces. The torque due to the gravitational force on the fly acts in the xy-plane, and its z-component is therefore zero. The z-component of the net torque on the system is therefore zero, and rotational momentum is conserved:

$$L_{O,z}(t_0) = L_{O,z}(t_1) \ , \tag{16.125}$$

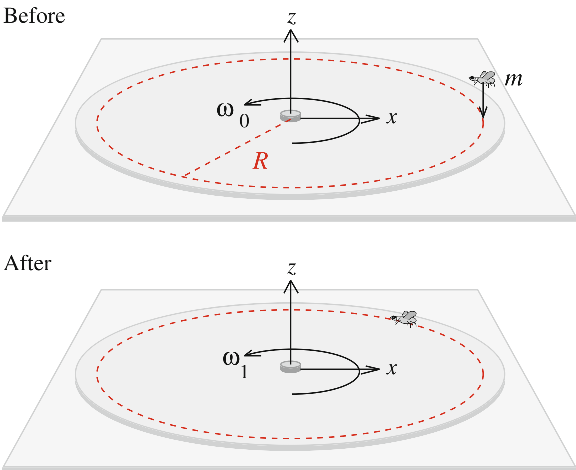


Fig. 16.24 A fly lands on a spinning DVD. The illustration shows the situation before (*top*) and after (*bottom*) the fly has landed

Immediately before the bug landed, the rotational momentum of the disk was

$$L_{O,z}(t_0) = \underbrace{I_z\omega_0}_{disk} + \underbrace{0}_{fly} , \tag{16.126}$$

where the z-component of the rotational momentum of the fly is zero since the fly is only moving vertically when it is landing.

After the landing, the fly is moving with the same velocity as the disk. The fly has therefore effectively changed the moment of inertia of the spinning disk, by adding a mass m at a distance R from the axis:

$$I_z(t_1) = I_z(t_0) + mR^2 , \tag{16.127}$$

and the rotational momentum of the disk-fly system is:

$$L_{0,z}(t_1) = (I_z + mR^2)\omega_1 , \tag{16.128}$$

Now, we can find the final angular velocity, since the rotational momentum is conserved:

$$I_z\omega_0 = \left(I_z + mR^2\right)\omega_1 , \tag{16.129}$$

$$\omega_1 = \frac{I_z}{I_z + mR^2}\omega_0 . \tag{16.130}$$

Since the disk is a cylinder, we know that the moment of inertia around the center of the disk (the center of mass) is: $I_z = MR^2/2$. The final angular velocity is therefore:

$$\omega_1 = \frac{\frac{1}{2}MR^2\omega_0}{mR^2 + \frac{1}{2}MR^2} = \frac{M}{M+2m}\omega_0 . \tag{16.131}$$

16.4.4 Example: Ballistic Pendulum

This problem is a classic in mechanics. It illustrates the conservation of angular momentum during a collision.

Problem: A thin rod of length L and mass M is hanging from a point O at one end of a the rod. A bullet of mass m is shot horizontally into the rod, and hits the rod with the horizontal velocity v at the lower end of the rod. The bullet is stuck in the rod. Find the angular velocity of the object immediately after the collision.

Approach: We divide the problem into two parts: The collision and the subsequent motion. During the collision we try to use conservation of rotational momentum to

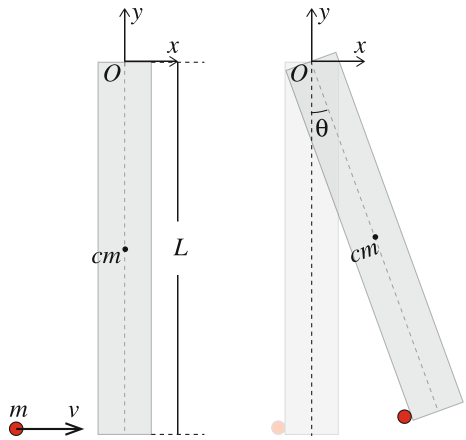


Fig. 16.25 Illustration of a collision between a bullet (*red*) and a rod (*grey*). The bullet is small compared with the rod and remains stuck on the rod after the collision. The rod rotates around an axis through the origin O

find the angular velocity after the collision. For the subsequent motion we use energy considerations to find how high the pendulum swings.

Identify: We use the rotation axis as the origin. The bullet has an initial velocity $\mathbf{v}_0 = v\,\mathbf{i}$, and the rod is initially hanging straight down, with no angular velocity. The system is shown in Fig. 16.25.

Model: The net external force is not zero throughout the collision, since the rod is affected by forces acting in the axis. What about the net torque around O? The external forces acting on the system is the force, $\mathbf{N}$, acting on the axis, $\mathbf{r}_N = \mathbf{0}$; the gravitational force, $\mathbf{G}$ of the rod; and the gravitational force $\mathbf{G}_b$ of the bullet. We assume that the rod does not move much during the collision. Consequently, the gravitational force on the rod acts approximately in the point $\mathbf{r}_G = -L/2\,\mathbf{j}$ throughout the collision, and similarly that the gravitational force on the bullet acts in $\mathbf{r}_b = -L\,\mathbf{j}$. Also, we assume that the rod is thin, so that the bullet is at $x_b = 0$ throughout the collision—therefore, we do not have to address the horizontal position of the bullet when we calculate the torques. The net external torque is therefore:

$$\boldsymbol{\tau}^{\text{net}} = \mathbf{0} \times \mathbf{N} + \mathbf{r}_G \times \mathbf{G} + \mathbf{r}_b \times \mathbf{G}_b \;, \tag{16.132}$$

where all the terms are zero! Hence the net torque is zero, and the rotational momentum around the point O is conserved throughout the collision.

We find the z-component of the total rotational momentum using the superposition principle, summing the z-component of the rotational momentum of the rod and the bullet. For the rod, $L^R_{O,z} = I_z\omega$, which is zero before the collision when the rod starts at rest. Immediately before the collision, the angular momentum of the bullet around the point O is:

$$\mathbf{L}^b_O = \mathbf{r}_b \times m\mathbf{v} = -L\,\mathbf{j} \times m v_0\,\mathbf{i} = Lmv\,\mathbf{k} \;, \tag{16.133}$$

and the z-component is:

$$L^b_{O,z} = Lmv \; . \tag{16.134}$$

The total z-component of the angular momentum of the system immediately before the collision is therefore:

$$L_{0,z} = \underbrace{Lmv}_{\text{bullet}} + \underbrace{I\omega_0}_{\text{rod}} = Lmv \; . \tag{16.135}$$

After the collision, the bullet is attached to the rod and the bullet-rod system rotates as a rigid body around the axis. The z-component of the rotational momentum of a rigid body is $L_z = I_{\text{tot}}\omega$, where I is the moment of inertia. We find the moment of inertia of the bullet-rod system using the superposition principle, summing the moment of inertia of the rod, I, and the bullet, which we consider to be a particle of mass m at a distance L from the rotation axis:

$$I_{\text{tot}} = I + mL^2 \; . \tag{16.136}$$

The rotational momentum after the collision is therefore:

$$L_{O,z} = \left(I + mL^2\right)\omega \; , \tag{16.137}$$

Solve: Conservation of angular momentum gives:

$$Lmv = \left(I + mL^2\right)\omega \;\Rightarrow\; \omega = \frac{Lmv}{I + mL^2} \; . \tag{16.138}$$

Analyze: The subsequent motion of the rod-bullet pendulum can be determined from Newton's second law for rotation around the axis O.

16.4.5 Example: Rotating Rod

Problem: A thin rod of length L and mass M is lying at rest on a frictionless surface. A bullet of the same mass, M, is shot horizontally into the rod, and hits the rod with the horizontal velocity v at a distance y from the center of the rod. The bullet is stuck in the rod. Find the translational and angular velocity of the object immediately after the collision.

Approach: Since the system is not affected by any external forces, the translational and rotational momentum is conserved, and we can use conservation laws to relate the motion before and after the collision.

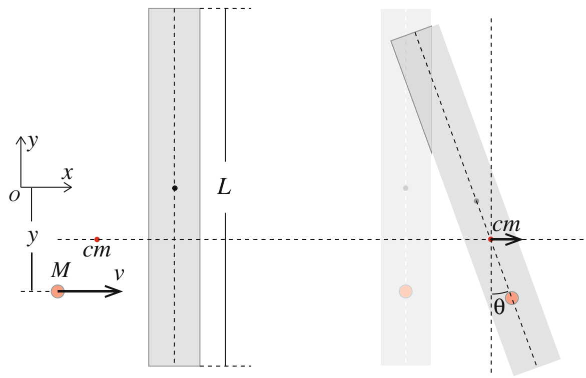


Fig. 16.26 Illustration of a collision between a bullet (*red*) and a rod (*grey*). The bullet is of the same mass as the rod, and the bullet remains stuck on the rod after the collision. The center of mass during the collision is illustrated

Identify: The system consists of the rod and the bullet. We describe the motion of the system by the motion of its center of mass and by the rotation of the system around its center of mass. The system is illustrated in Fig. 16.26.

Model: The motion of the system is determined by the external forces acting on the system. Since we can ignore frictional forces in the plane of motion, the only external forces acting are the gravitational force and the normal force from the surface. Since the rod is not moving vertically, the net external forces are zero. From Newton's second law for translational motion, this means that the (translational) momentum is conserved throughout the collision and during the motion afterwards:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{P} = M\mathbf{A} = 0 \ . \tag{16.139}$$

The center of mass of the system therefore moves with a constant velocity—before, during, and after the collision! Notice that the translational motion does not depend on where the bullet hits the rod—the velocity of the center of mass is the same independently of where the bullet hits, although the position of the center of mass of course will depend on where the bullet hits.

The velocity of the center of mass before the collision depends on the velocity of the objects: The velocity of the rod is, $\mathbf{v}_r = 0$, and the velocity of the bullet is $\mathbf{v}$, hence:

$$(M + M)\,\mathbf{V} = M\mathbf{v} + M\mathbf{0} = M\mathbf{v} \ , \tag{16.140}$$

and therefore:

$$\mathbf{V} = \frac{1}{2}\mathbf{v} = \frac{1}{2}v\,\mathbf{i} \ . \tag{16.141}$$

Since the system has no velocity component in the y-direction, the y-position of the center of mass remains constant, with the following value:

$$(M + M)\, Y = M\, 0 - M\, y \;\Rightarrow\; Y = -y/2 \;, \tag{16.142}$$

as illustrated in Fig. 16.26.

Second, we determine the motion relative to the center of mass. The rod may start rotating, depending on where the bullet hits the rod. The rotational motion around the center of mass is related to the torque of the external forces around the center of mass. But, since there are no external forces acting in the plane of motion, the z-component of the external torque is zero:

$$\boldsymbol{\tau}^{\text{ext}} = 0 \;. \tag{16.143}$$

We apply Newton's second law for rotational motion around the center of mass:

$$\boldsymbol{\tau}^{\text{ext}} = \frac{d}{dt}\mathbf{L}_{cm} = 0 \;, \tag{16.144}$$

and find that there is no change in the rotational momentum around the center of mass: The rotational momentum around the center of mass of the system is conserved. This is also the case for the z-component of the rotational momentum around the center of mass:

$$\tau_z^{\text{ext}} = \frac{d}{dt}L_{cm,z} = 0 \;. \tag{16.145}$$

Since we know the rotational momentum around the center of the mass before the collision, we can use this to find the rotational momentum around the center of mass after the collision, and from this we find the angular velocity of the object around the center of mass.

The rotational momentum around the center of mass is the sum of the rotational momentum for each of the components—*around their common center of mass*,

$$\mathbf{L}_{cm} = \mathbf{L}_{\text{rod}} + \mathbf{L}_{\text{bullet}} \;. \tag{16.146}$$

Before the collision, the rotational momentum of the rod around the center of mass is 0, since it is not rotating initially. The rotational momentum of the bullet depends on the position $\mathbf{r}_{b,cm}$ of the bullet relative the center of mass:

$$\mathbf{L}_{\text{bullet}} = \mathbf{r}_{b,cm} \times M v\, \mathbf{i} = \left(x_{b,cm}\, \mathbf{i} + y_{b,cm}\, \mathbf{j}\right) \times M v\, \mathbf{i} = -y_{b,cm}\, M v\, \mathbf{k} \;. \tag{16.147}$$

Notice that only the y-component of $\mathbf{r}_{\text{cm}}$ contributes. Since the position of the bullet is $-y$, and the position of the center of mass is $-y/2$, we find that the y-component of $\mathbf{r}_{b,cm}$ is $y_{b,cm} = -y - (-y/2)) = -y/2$, and

$$L_{cm,z}^{0} = -\left(-\frac{y}{2}\right) M v = \frac{1}{2} M v y \;. \tag{16.148}$$

After the collision, the whole object, consisting of the bullet and the rod, is rotating around the center of mass with a rotational momentum:

$$L^1_{\mathrm{cm},z} = I_z \omega \ . \tag{16.149}$$

The total moment of inertia is:

$$I_z = I_{\mathrm{rod},z} + I_{\mathrm{bullet},z} \ , \tag{16.150}$$

where both moments of inertia must be around the same axis: The axis going through the center of mass of the rod-bullet system.

For the rod, the moment of inertia around its own center of mass is given in Fig. 15.5:

$$I_{\mathrm{rod},z,cm} = \frac{1}{12} M L^2 \ . \tag{16.151}$$

The axis through the rod-bullet center of mass is a distance $s = y/2$ from the center of mass of the rod, which we use in the parallel-axis theorem to find:

$$I_{\mathrm{rod},z} = I_{\mathrm{rod},z,cm} + M \left(\frac{L}{2} \right)^2 = \frac{1}{12} M L^2 + M \frac{y^2}{4} \ . \tag{16.152}$$

For the bullet, we assume it is a point particle, located at a distance $s = y/2$ from the center of mass, hence the parallel-axis theorem gives:

$$I_{\mathrm{bullet},z} = M \left(\frac{y}{2} \right)^2 = \frac{1}{4} M y^2 \ , \tag{16.153}$$

The total moment of inertia is therefore:

$$I_z = \frac{1}{12} M L^2 + \frac{1}{4} M y^2 + \frac{1}{4} M y^2 = M \left(\frac{1}{12} L^2 + \frac{1}{2} y^2 \right) \ . \tag{16.154}$$

Finally, we find the angular velocity after the collision from:

$$\omega = \frac{L^1_{cm,z}}{I_z} = \frac{\frac{1}{2} M v y}{M \left(\frac{1}{12} L^2 + \frac{1}{2} y^2 \right)} = \frac{v}{y} \frac{6}{\left(\frac{L}{y} \right)^2 + 6} \ . \tag{16.155}$$

After the collision, the net external force is zero, and the rod-bullet system moves in a straight line with constant translational and angular velocity.

16.5 General Rotational Motion

So far we have only addressed motions where the axis of rotation does not change direction. But Newton's second law for rotations around a fixed point or around the center of mass has general applicability, it is also valid in cases where the torque is not parallel to the angular momentum. Let us address what happens in this case through two examples.

A Rotating Wheel

Let us assume that you are holding a spinning wheel—such as the wheel of a bike. You are holding onto a rod through the axis of rotation, and you want to rotate the axis by applying a pair of forces, **F** and −**F**, at the two ends of the rod. Let us find the motion of the wheel due to the force pair.

The system is illustrated in Fig. 16.27a. Initially, the wheel is rotating with the angular velocity ω around the x-axis, so that the initial angular momentum around the origin is:

$$\mathbf{L}_O = I\omega\,\mathbf{i}\,, \tag{16.156}$$

since the object is symmetric around this axis.

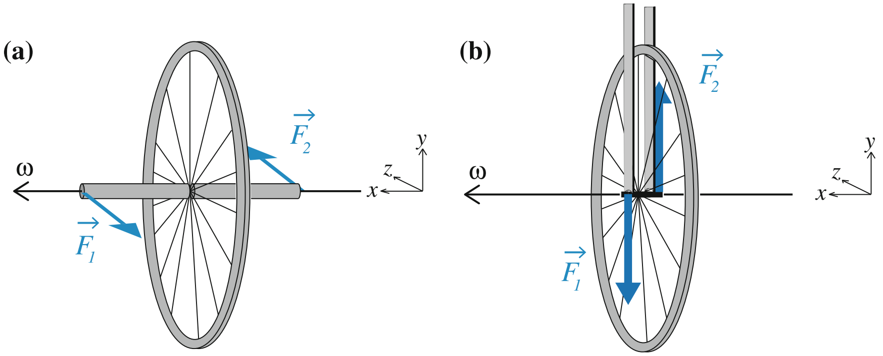


Fig. 16.27 **a** You try to change the rotation axis of a spinning wheel by applying a pair of forces **F** and −**F** at symmetric positions around the center of mass of the wheel. **b** An illustration of the front wheel of a bike rolling in the positive z-direction. The wheel is rolling, rotating around the x-axis. As you lean towards the right, you apply a pair of forces on the axis of the wheel, as illustrated. The resulting torque acts in the z-direction, causing the angular momentum to tilt towards the z-axis, rotating the wheel so that the bike turns left

We apply a pair of forces to the axis. A force $\mathbf{F}_1 = -F\,\mathbf{k}$ is acting at the position $\mathbf{r}_1 = x\,\mathbf{i}$, and a force $\mathbf{F}_2 = F\,\mathbf{k}$ is acting at the position $\mathbf{r}_2 = -x\,\mathbf{i}$. In addition, the wheel is affected by gravity, $\mathbf{W} = -mg\,\mathbf{j}$, and the axis is supported by two forces balancing the gravitational force, $\mathbf{N}_1 = \mathbf{N}_2 = mg/2\,\mathbf{j}$. The gravity and the two balancing forces have no net torque around the center of mass of the wheel.

What is the net torque around the center of mass of the system? We find:

$$\boldsymbol{\tau}_{\mathrm{cm}}^{\mathrm{net}} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = x\,\mathbf{i} \times -F\,\mathbf{k} - x\,\mathbf{i} \times F\,\mathbf{k} = -2xF\,\mathbf{j}\,. \tag{16.157}$$

Newtons' second law for rotational motion gives:

$$\frac{d\mathbf{L}_O}{dt} = \boldsymbol{\tau}_{\mathrm{cm}}^{\mathrm{net}} = -2xF\,\mathbf{j}\,. \tag{16.158}$$

The initial angular momentum is along the x-axis, in the $\mathbf{i}$ direction. After a small time interval Δt, the applied forces will have lead to a small tilting of the spin axis, but the axis is tilting in the direction of the y-axis, while the forces are applied along the z-axis. Is this strange? No, this is entirely consistent with the experience you gain while riding a bike. Consider the motion of the front wheel of your bike. Let us assume that while you are riding forwards, the wheel is spinning around the x-axis as illustrated in Fig. 16.27b. If you lean towards the left, you are transferring a pair of forces to the axis of the wheel. Leaning to the left, you pull up on the negative side (the side in the negative x-direction), and you push down on the positive side of the spinning axis. As a result, you apply a torque acting in the backward direction—along the z-axis on the figure, but backward. The rotational axis will therefore tilt in this direction, causing the wheel to turn to the left! Leaning on your bike therefore leads to the front wheel turning, as you surely have experienced while cycling.

A Spinning Top

A spinning top is a common child's toy. You spin it up by you hand (or by a string or similar device), and the top starts spinning rapidly, balancing on its bottom tip. The wheel not only rotates around its axis, in addition, the axis of rotation starts rotating slowly around the vertical axis. As the spinning wheel spins slower and slower around its own axis, its rotates faster and faster around the vertical axis. What is happnening?

Let us make a simplified physical model of the spinning wheel. We consider the wheel to be a symmetric object rotating around its axis of symmetry with an initial angular velocity ω_s, as illustrated in Fig. 16.28. The spinning wheel is standing on a tip located on the symmetry axis, and we assume that it rotates approximately without friction. In addition, the rotation axis of the spinning wheel is rotating slowly, with an angular velocity Ω, around the vertical z-axis. Can we find a relation between ω_s and Ω?

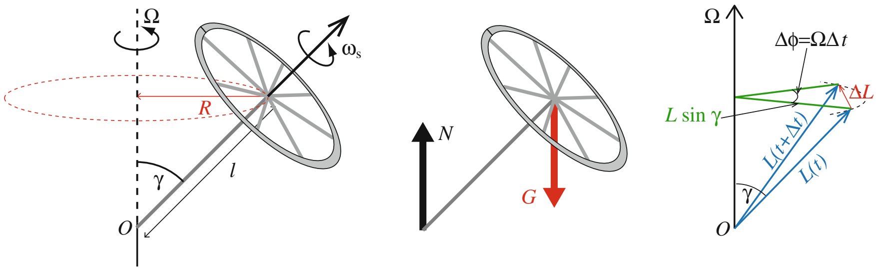


Fig. 16.28 An illustration of a spinning wheel balancing on the point O. The wheel rotates rapidly arounds its own axis with the angular velocity ω_s, in addition, the axis rotates slowly around the vertical axis with the angular velocity Ω

We start by analyzing the situation in order to determine the motion of the spinning top. Its motion is determined by the forces acting on it. The spinning top is affected by gravity, **G**, and the normal force, **N**, as illustrated the free-body diagram in Fig. 16.28. We apply Newton's second law for the motion of the center of mass:

$$\sum F_z = N - Mg = MA_z = 0 \; , \tag{16.159}$$

where we have assumed that the top does not move in the vertical direction, hence $A_z = 0$. This means that the normal force is:

$$N = Mg \; . \tag{16.160}$$

We apply Newton's second law for rotation around the point O—the contact point for the axis of the spinning top. This point is stationary—it does not move throughout the motion. We find the net torque around this point at a time when the center of mass of the spinning wheel is located along the x-axis, at the position

$$\mathbf{r} = R\,\mathbf{i} = Ml \sin\gamma\;\mathbf{i} \; , \tag{16.161}$$

where we see that $R = l \sin\gamma$ from Fig. 16.28. The net torque around O is therefore:

$$\boldsymbol{\tau}_O^{\text{net}} = \mathbf{0} \times \mathbf{N} + \mathbf{r} \times \underbrace{\mathbf{G}}_{=-Mg\,\mathbf{k}} = R\,\mathbf{i} \times (-Mg\,\mathbf{k}) = -MgR\,(-\mathbf{j}) = MgR\,\mathbf{j} \; . \tag{16.162}$$

What is the angular momentum of the spinning top? We notice that there are two contributions to the angular momentum: The spinning top is rotating around its axis of symmetry, and the center of mass is moving in a circle around the vertical axis. We use the general expression for the angular momentum for a rigid body

$$\mathbf{L}_O = \mathbf{R} \times M \underbrace{\mathbf{V}}_{=\Omega\mathbf{k}\times\mathbf{R}} + \mathbf{L}_{\text{cm}} = RM\Omega R\,\mathbf{k} + \mathbf{L}_{\text{cm}} = MR^2\Omega\,\mathbf{k} + I_{\text{cm}}\boldsymbol{\omega}_s \; , \tag{16.163}$$

where we have used that since the spinning top is rotating around its center of mass, the angular momentum around the center of mass is $\mathbf{L}_{\rm cm} = I_{\rm cm}\boldsymbol{\omega}_s$, where $\boldsymbol{\omega}_s$ points along the axis of the spinning top. In addition, we will now assume that the angular momentum of the rotation of the center of mass is much smaller than the angular momentum of the rotation of the spinning mass around its center of mass:

$$MR^2\Omega \ll I_{\rm cm}\omega_s \;\Rightarrow\; f\Omega \ll \frac{I_{\rm cm}}{MR^2}\omega_s \;, \tag{16.164}$$

where the prefactor is of the order one, since the mass is typically located a distance smaller than R from the center of mass. Consequently, we have assumed that:

$$\Omega \ll \omega_s \;, \tag{16.165}$$

that the spinning top is rotating much faster around its own axis, than the axis is rotating around the vertical axis. Our approximation is therefore:

$$\mathbf{L}_O \simeq I_{\rm cm}\boldsymbol{\omega}_s \;. \tag{16.166}$$

Newton's second law for angular motion around a fixed point states:

$$\boldsymbol{\tau}_O^{\rm net} = \frac{d\mathbf{L}_O}{dt} \;. \tag{16.167}$$

Now, since we know that the axis of the spinning top is rotating around the vertical axis with an (approximately) constant angular velocity, we know the change in angular momentum around the center of mass over a small time step Δt—it is simply found by the change in the angular momentum vector $\mathbf{L}_0$, which is (approximately) parallel to the angular momentum around the center of mass. The change in angular momentum is:

$$\Delta\mathbf{L}_O = \mathbf{L}_O(t + \Delta t) - \mathbf{L}_O(t) \;. \tag{16.168}$$

From Fig. 16.28, we see that the angular velocity vector $\boldsymbol{\omega}_s$, and therefore also the angular momentum $\mathbf{L}_O$, rotates an angle $\Delta\phi$ around the vertical axis during the time interval Δt. Since the "radius" in this circle is $L_O \sin\gamma$, we see that the change in angular momentum is approximately equal to the arc length along the circle:

$$\Delta L_O = (L_O \sin\gamma)\,\Delta\phi \;. \tag{16.169}$$

We divide by Δt, getting:

$$\frac{\Delta L_O}{\Delta t} = L_O \sin\gamma \frac{\Delta\phi}{\Delta t} = L_O \sin\gamma\,\Omega \;. \tag{16.170}$$

From Newton's second law for rotational motion, we know that

$$\frac{dL_O}{dt} = \tau_O^{\text{net}} = Mgl \sin\gamma \ , \tag{16.171}$$

hence

$$L_O \Omega = Mgl \ , \tag{16.172}$$

and since $L_O \simeq I_{\text{cm}}\omega_s$ we find:

$$\Omega = \frac{Mgl}{L_O} \simeq \frac{Mgl}{I_{\text{cm}}\omega_s} \ . \tag{16.173}$$

We have found a relation between the angular velocity ω_s of the spinning top around its center of mass, and the angular velocity Ω of the spinning axis around the vertical axis.

We notice that Ω increases as ω_s decreases, which "explains" the behavior of a spinning top running on the floor: It wobbles faster as the spinning wheel slows down.

However, when $\omega_s \to 0$, our assumption that $MR^2\Omega \ll I_{\text{cm}}\omega_s$ breaks down, and our theory is no longer valid. Our solution therefore only has limited applicability, and we should make a rule to always check such assumptions at the end, to find if we have violated them.

Summary

Torque:

- The torque of the force **F** acting at the point **r** relative to the point O is: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
- Torques of several forces can be added to find the **net torque** around a given point.
- The torque of the gravitational force on a rigid body is $\boldsymbol{\tau} = \mathbf{R} \times \mathbf{G}$, where **R** is the position of the center of mass of the body.

N2L for rotational motion around a fixed axis:

- For a rigid body rotating around a fixed axis (the z-axis) $\sum_j \tau_{z,j} = \tau_z^{\text{net}} = I_z\alpha$ where $\boldsymbol{\tau}_j = \mathbf{r}_j \times \mathbf{F}_j$ is the torque of force j acting in point $\mathbf{r}_j$, and I_z is the moment of inertia of the object around the z-axis.
- This law is only true for a rigid body rotating around a fixed axis or for a rigid body rotating around a fixed axis through its center of mass.

N2L for rotational motion around the c.m.:

- For a rigid body rotating around a fixed axis (the z-axis) through the center of mass, the acceleration of the center of mass is: $\sum_j \mathbf{F}_j = \mathbf{F}^{\text{net}} = M\mathbf{A}$ and the angular acceleration around the center of mass is: $\sum_j \tau_{z,j} = \tau_z^{\mathrm{net}} = I_z \alpha$ where $\boldsymbol{\tau}_j = \mathbf{r}_{cm,j} \times \mathbf{F}_j$ is the torque of force j acting in point $\mathbf{r}_{cm,j}$ measured relative to the center of mass, and I_z is the moment of inertia of the object around the z-axis through the center of mass.

- This law is only true for a rigid body rotating around a fixed axis or for a rigid body which is spherically symmetric around the center of mass.

Rotational momentum for a system of particles:

- The **rotational momentum** (or **angular momentum**) of a point particle with (translational) momentum $\mathbf{p}$ at the position $\mathbf{r}$ relative to the point O is: $\mathbf{l} = \mathbf{r} \times \mathbf{p}$
- Newton's second law for the motion of a point particle can be written as: $\sum_j \boldsymbol{\tau}_j = \boldsymbol{\tau}^{\mathrm{net}} = d\mathbf{l}/dt$
- The **rotational momentum** (or **angular momentum**) of a multiparticle system around the point O is: $\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i$
- Newton's second law for rotational motion of a multiparticle system around the fixed point O is: $d\mathbf{L}_O/dt = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\mathrm{ext}} = \boldsymbol{\tau}_O^{\mathrm{ext}}$
- The **rotational momentum** (or **angular momentum**) of a multiparticle system around its center of mass is: $\mathbf{L}_{cm} = \sum_i \mathbf{r}_{cm,i} \times m_i \mathbf{v}_i$
- Newton's second law for rotational motion of a multiparticle system around its center of mass is: $d\mathbf{L}_{cm}/dt = \sum_i \mathbf{r}_{cm,i} \times \mathbf{F}_i^{\mathrm{ext}} = \boldsymbol{\tau}_{cm}^{\mathrm{ext}}$

Rotational momentum of a rigid body:

- The **rotational momentum** (or **angular momentum**) of a rigid body rotating around a fixed axis is $\mathbf{L}_{O,z} = I_{O,z}\omega$
- If the net external torque around a fixed point is zero, the rotational momentum of the system around the same fixed point is conserved.
- The **rotational momentum** (or **angular momentum**) of a rigid body rotating around a fixed axis through the center of mass is $\mathbf{L}_{cm,z} = I_{cm,z}\omega$
- If the net external torque around the center of mass is zero, the rotational momentum of the system around the center of mass is conserved. This is true also when the center of mass is moving.

Collisions and conservation laws:

- If the net torque around a *fixed point O* is zero (or very small) throughout a collision, then the angular momentum around this point is conserved throughout the collision.
  - If the net torque around the *center of mass* of a system is zero (or very small) throughout a collision, then the angular momentum around the center of mass is conserved throughout the collision—independently of the motion of the center of mass.

Exercises

Discussion Questions

16.1 Opening a door. If you can push with a maximum force F, how should you push a door to open it as quickly as possible?

16.2 Opening a jar. Why does it help to use an extending shaft to open a stuck lid?

16.3 Revolving door. A friend of yours is claiming that it is easier to open a revolving door than a single door of the same size as one half of the revolving door, because the swing door has a weight on the other side that balances the movement. Is he right?

16.4 Summersaulting. Explain the principle of doing a summersault on a trampoline.

Problems

16.5 Motion of rod during a collision-like process. In this problem we will study the motion of a thin rod that falls and attaches itself to a hinge. We will look at both the motion of the center of mass of the rod and how the rotation of the rod changes in the process.

The moment of inertia of the rod about an axis through the center of mass is

$$I_{z,cm} = \frac{1}{12} M L^2 \,, \tag{16.174}$$

where M is the mass of the rod, and L its length. We hold the rod horizontally oriented with one of the ends of the rod a height h directly above a fixed point O where a hinge is located. We release the rod from rest. You can ignore air resistance (Fig. 16.29).

(**a**) What is the velocity of the center of mass v_0 of the rod when it has fallen a distance h? What is the angular velocity ω_0 about the center of mass of the rod when it has fallen this distance?

When the end of the rod hits the hinge in point 0, it attaches itself, and the whole rod starts rotating about 0. We can view this process of attachment as a collision. You can assume all movement is in the plane as shown in the figure. You can also neglect air resistance and any friction in the hinge.

(**b**) Show that the moment of inertia of the rod about an axis normal to the rod through the point O is $I_{O,z} = ML^2/3$.

(**c**) Find the angular velocity of the rod about the point O immediately after the rod becomes attached. You can assume that the rod doesn't rotate during the process of attachment and that the torque from the gravity can be disregarded.

(**d**) Find the momentum of the rod immediately after the rod becomes attached. Is the momentum conserved? Explain.

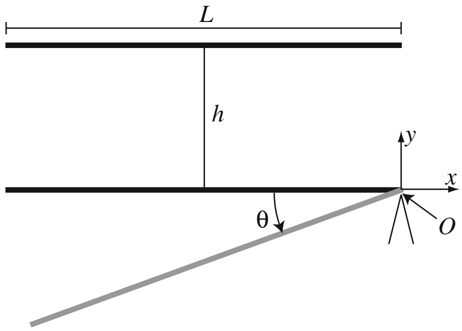


Fig. 16.29 Illustration of rod attachment

The hinge is spring-loaded and affects the rod with a torque $\tau_{O,z} = -\kappa\theta$. The potential energy related to this interaction is $U = (1/2)\,\kappa\,\theta^2$.

(**e**) Find an expression for the angular acceleration of the rod when it has rotated an angle θ about the point O.

(**f**) It is possible to find the angle of the rod as a function of time using numerical methods, finding an analytical expression is more difficult. We can instead use a different method to find the maximum angle the rod rotates. Find an equation that decides the maximum angle θ of the rod. Note that you don't have to solve this equation.

(**g**) After reaching the maximum angle, the rod will change it's direction of rotation and swing back. What is the angular velocity, ω_2, of the rod about the point O the moment the rod is horizontal again? (i.e. when the angle θ is 0.)

(**h**) What is the velocity, v_2, of the center of mass when the rod is horizontal again?

The moment the rod reaches the horizontal orientation, the hinge in point O breaks, releasing the rod so it is no longer attached. You can assume the rod is not affected by any external forces during this process and that the kinetic energy of the rod is conserved.

(**i**) Show that the velocity of the cm and the angular velocity about the cm immediately after the attachment fails is $v_3 = -(3/4)\,v_0$ and $\omega_{3,cm} = (3/2)\,(v_0/L)$

(**j**) Describe the motion of the rod after the attachment breaks.

(**k**) How high does the center of mass of the rod reach?

16.6 Collision between a rod and a block. In this problem we will study an impact between a rod and a small block. The rod is homogeneous with the mass M and length L. The rod is attached with a frictionless hinge in the point O so it can rotate as shown in the Fig. 16.30. The block is small compared to the rod. The block has the mass m and is initially at rest on a frictionless surface. The rod starts from rest at an angle θ_0 and is released. The rod hits the block when it is hanging straight down (i.e. when $\theta = 0$). The rod's moment of inertia about its center of mass is $I_{cm} = ML^2/12$.

(**a**) What is the rod's moment of inertia about the point O?

(**b**) Find the rod's kinetic energy as a function of the angle θ. You can disregard air resistance.

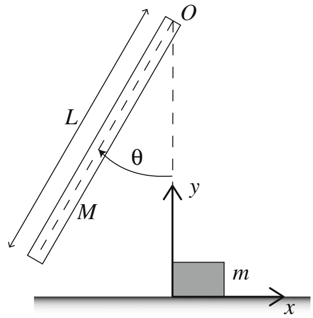


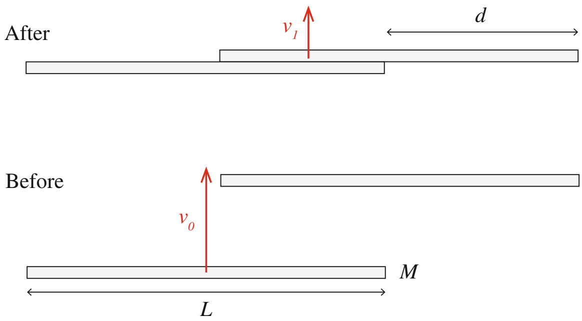
Fig. 16.30 Illustration of rod hitting a block

(**c**) Find the angular velocity of the rod, ω_0, immediately before it hits the block.
Let us first assume that the collision is perfectly elastic.
(**d**) Show that the velocity of the block immediately after the collision is $v_1 = (2\,\omega_0\,L)/(1 + (mL^2)/I_O)$.
(**e**) Show that the angular velocity of the rod immediately after the collision is $\omega_1 = \omega_0\left(1 - (2)\,/\left(1 + I_O/\left(mL^2\right)\right)\right)$.
(**f**) Discuss the motion of the block and the rod after the collisions for the cases $m \gg M$ and $m \ll M$.
(**g**) What happens in the case $m = M/3$?
Let us now assume the collision is perfectly inelastic.
(**h**) Find the angular velocity of the rod and the velocity of the block immediately after the collision.

16.7 A model of two rods colliding. We will in this problem look at a collision of two long and thin rods. This could for example be a model of how two long and linear molecules collide. The two rods are identical and remain stuck together after the collision. Each rod has a mass M and length L. For each rod the moment of inertia about its center of mass is $I_0 = ML^2/12$. The rods are gliding on a horizontal, frictionless surface as illustrated in Fig. 16.31.

The rods are parallel before the collision. One of the rods is at rest, while the other has the velocity v_0. After the collision they stick together like one rigid body, as illustrated in the figure. The starting position is characterized by the distance d as in the figure. You can neglect the width and height of the rods.
(**a**) Show that the moment of inertia around the center of mass for the body of the two rods stuck together is $I = (M/2)\left(d^2 + L^2/3\right)$
First assume $d = 0$.
(**b**) Find the velocity v_1 of the center of mass of the system after the collision.
(**c**) What is the angular velocity about the center of mass of the body of the two rods stuck together after the collision? Justify your answer.
Let us now look at the case $0 \le d \le L$.
(**d**) Find the velocity v_1 of the center of mass of the system after the collision.
(**e**) Find the angular velocity ω_1 of the entire system about its center of mass after the collision.

Fig. 16.31 Illustration of two rods colliding

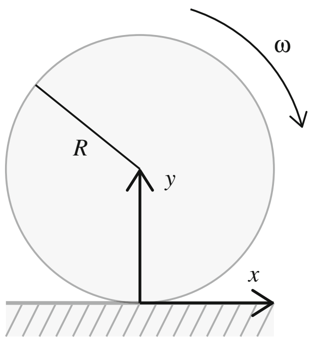


(**f**) What is the loss of energy in the collision? For what d is the loss of energy the least? Comment the result.
(**g**) Describe the motion after the collision.

16.8 Studying friction on a wheel. In this problem we will study the behavior of a spinning wheel that is lowered onto a flat, horizontal surface. The wheel has a mass m, a radius R, and a moment of inertia about its center of mass I. We let the x-axis be parallel with the surface and choose the direction of rotation to be positive in the clockwise direction, as illustrated in the figure. The coefficient of dynamic friction between the surface and the wheel is μ. The acceleration of gravity is g. The wheel is lowered onto the surface at a time $t = 0$s at the position $x(0) = x_0 = 0$. The initial velocity of the wheel is $v(0) = v_0 = 0$ and the initial angular velocity is ω_0 (Fig. 16.32).

16.9 Tarzan's swing. Tarzan jumps from a cliff and grabs a vine. He jumps horizontally from the cliff with initial velocity v_0 at the time t_0. The vine has mass M and length L. Initially, the vine is hanging straight down and is attached at its highest point, O. Tarzan jumps from a height h above the lowest point on the vine, as illustrated in Fig. 16.33. After the "collision" Tarzan remains attached to the vine,

Fig. 16.32 Illustration of a spinning wheel



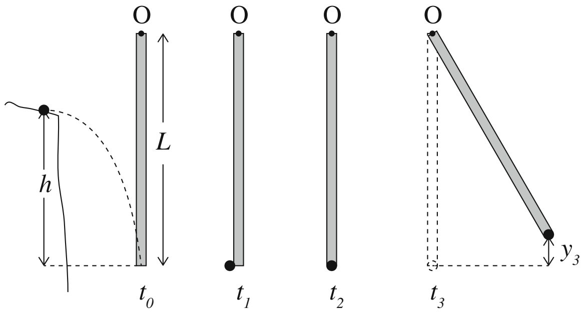


Fig. 16.33 Illustration of the motion of Tarzan and the vine: when Tarzan jumps (at t_0), immediately before he grabs the vine (t_1), immediately after he is attached to the vine (t_2), and when he reaches his highest point (t_3)

with his center of mass at the lower edge of the vine. Tarzan's mass is m. The vine behaves as a rod attached without friction to the point O. The moment of inertia of a rod around its center of mass is $I = (1/12)ML^2$. You can neglect air resistance.
(**a**) Find Tarzan's velocity immediately before the collision with the vine (at t_1).
(**b**) What is the moment of inertia of the vine about the point O?
(**c**) Show that the angular velocity of the vine (with Tarzan) immediately after the collision is $\omega_2 = m/\left((M/3) + m\right)\ (v_0/L)$.
(**d**) How far up does Tarzan swing?
(**e**) How high would Tarzan swing if he jumped from twice the height?

16.10 Rolling up a slope. In this problem we study the motion of the rotating wheel placed on a slope. The wheel has mass M and radius R. The wheel consists of three homogeneous cylinders rotating around the same axis. The middle cylinder has radius R and mass $M/2$. A cylinder with radius $R/2$ and mass $M/4$ is glued on each side of the middle cylinder, as shown in Fig. 16.34. The moment of inertia around the symmetry axis for a cylinder of mass m and radius r is $I = (1/2)mr^2$.
(**a**) Show that the moment of inertia around the z axis for the wheel is $I = (5/16)MR^2$.

The wheel starts with an angular velocity ω_0 and is placed on a slope as shown in Fig. 16.34, where the positive rotational direction is shown. The wheel starts without

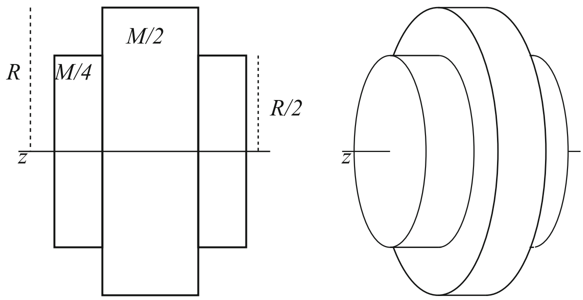


Fig. 16.34 Illustration of a wheel consisting of three cylinders that are glued together

translational velocity. The coefficient of friction between the wheel and the floor is μ. The slope makes an angle θ with the horizontal. You can neglect air resistance.
(**b**) Draw a free-body diagram for the wheel and name the forces acting on it.
(**c**) Show that the acceleration of the center of mass of the wheel in the x-direction becomes $a_x = g\,(\mu\cos\theta - \sin\theta)$.

You can assume that $\mu\cos\theta - \sin\theta > 0$. In the following, we will only study the motion of the wheel before it starts rolling without slipping. You can assume that the wheel is moving upward until it starts rolling.
(**d**) Find the velocity, $v(t)$, of the center of mass of the wheel.
(**e**) Find the angular acceleration, $\alpha(t)$, of the wheel.
(**f**) Find the angular velocity, $\omega(t)$, for the wheel.
(**g**) Find the time it times until the wheel starts rolling without slipping.
(**h**) Describe the motion after the wheel starts rolling and explain your answer.

Projects

16.11 Snow crystal In this project you will apply your knowledge of linear and angular momentum to study the aggregation of small droplets of ice to form large grains of snow.

As snow crystals form in clouds they start falling through the cloud. Due to air resistance, larger particles fall faster than smaller particles. A large particle will therefore overtake smaller particles. When a smaller particle is overtaken, it will stick to the larger particle, adding further to the size. This process forms aggregate snowflakes, which is one of the most common types of snowflakes.[3] This mechanism is often called differential sedimentation, and is a process important for pattern formation in many natural systems, and it is also a process important for many industrial processes. An example of a complex aggregate formed by a related aggregation process called Diffusion Limited Aggregation in Fig. 16.35 shows the complex geometries typically found in aggregate grains.

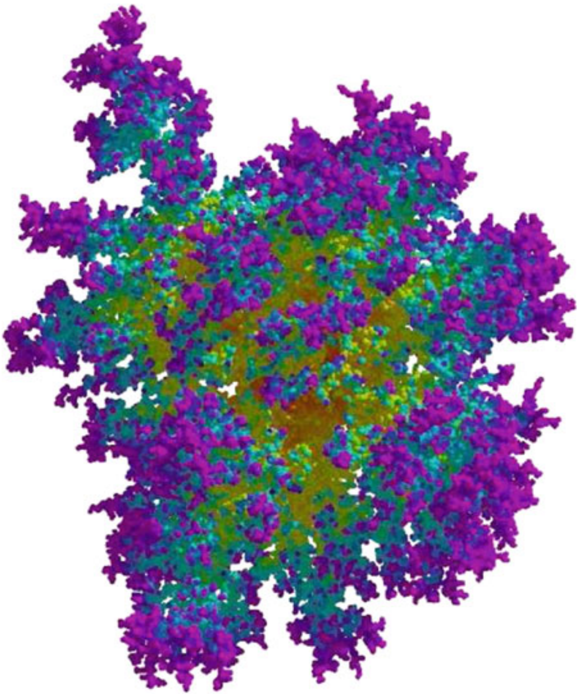
In this project, we will study the aggregation process in detail. We will study an approximately spherical grain of ice of mass M and radius R, hitting and sticking to an identical grain of ice.

First, let us address why large particles fall faster than small particles. The mass of an ice grain of radius R and mass density ρ_m is

$$M = \rho_m \frac{4\pi}{3} R^3 \ . \tag{16.175}$$

[3] You can learn more about this process, and look at how aggregate flakes look in the Ph.D. thesis of Christopher David Westbrook at http://www.met.rdg.ac.uk/sws04cdw/thesis.pdf.

Fig. 16.35 Image of a (fractal) cluster formed by diffusion limited aggregation of 10,000 particles. (Goold 2004)



We will assume that air resistance can be modelled using the approximation:

$$\mathbf{F}_v = -k_v \mathbf{v} \,, \tag{16.176}$$

where

$$k_v \simeq 20.4 R\eta \tag{16.177}$$

is a constant depending on the viscocity η of the fluid.
(**a**) Find the forces acting on an ice grain with radius R, and write down Newton's second law of motion for the grain.
(**b**) Show that the acceleration of the grain is

$$\mathbf{a} = \mathbf{g} - \frac{20.4\eta}{\rho_m \frac{4\pi}{3} R^2} \mathbf{v} \,, \tag{16.178}$$

where $\mathbf{g} = -g\mathbf{j}$ and g is the acceleration of gravity. Can you now explain why larger grains fall faster than smaller grains?

We will now study a collision between two identical ice grains. One grain is at rest relative to the reference system and the other grain has a velocity v_0 downwards. When the two grains collide, they stick together at the point of contact, and remain stuck together. We call this combination of two grains a compound grain.

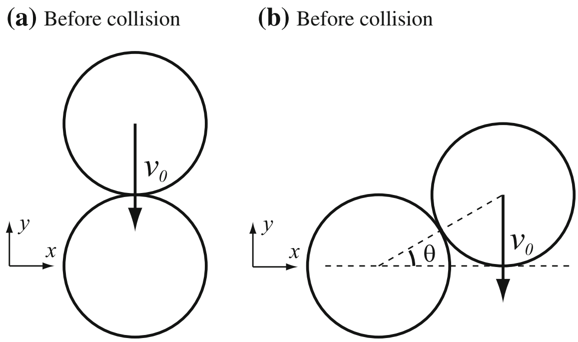


Fig. 16.36 Illustration of a collision between two identical ice grains. The *lower* grain is not moving, and the *top* grain is moving downwards with a velocity $\mathbf{v_0}$ as illustrated. In **a** the *top* grain hits at the *top* of the stationary grain, and in **b** the *top* grain hits the *lower* grain when the angle between the line connecting the centers of the grains and the *horizontal* is θ

(**c**) The moment of inertia of one ice grain around its center is $I_c = \frac{2}{5} M R^2$. Show that the moment of inertia, I, around the center of mass for a compound grain consisting of two grains sticking together is $I = (14/5)\ M\ R^2$

First, we consider a linear collision where the upper grain hits the lower grain directly in the center, as illustrated in Fig. 16.36a. We assume the collision to be instantaneous, so you can ignore the effect of air resistance and gravity during the collision.

(**d**) What is the velocity, $\mathbf{v}_1$, of the center of mass the compound grain after the collision?

(**e**) What is the angular velocity, $\boldsymbol{\omega}_1$, around the center of mass of the compound grain after the collision?

Let us now consider the more general case illustrated in Fig. 16.36b. When the two grains touch, the line between the centers of the two grains forms the angle θ with the horizontal. The upper grain still has the initial velocity v_0 downwards before the collision, and the lower grain is at rest.

(**f**) What is the velocity, $\mathbf{v}_1$, of the center of mass of the compound grain after the collision?

(**g**) What is the angular velocity, $\boldsymbol{\omega}_1$, around the center of mass of the compound grain after the collision?

(**h**) What is the loss of energy in the collision?

Let us now address the motion of the compound grain after the collision. Initially, it is rotating with the angular velocity ω_1.

(**i**) If we ignore air resistance, find $\omega(t)$ as a function of time for the subsequent motion.

In the following we will not ignore air resistance, but rather develop a simplified model for the air resistance. In order to determine the force acting on the compound object due to air resistance, we either need to perform experiments on such objects, or we can use numerical simulations of the fluid flow around the object to determine the forces.

Here, we will use a strong simplification: We assume that we may consider the compound object to consist of two separate spheres. The force on each of the spheres due to air resistance is described by (16.176), where the corresponding velocity, v, in (16.176) is the velocity of the center of the sphere, and the force acts in the center of the sphere.

The compound object has velocity $\mathbf{v}_{cm}$ and angular velocity $\boldsymbol{\omega}$.

(**j**) Argue that the velocities, $\mathbf{v}_A$ and $\mathbf{v}_B$, of each of the ice grains A and B are $\mathbf{v}_A = \mathbf{v}_{cm} + \boldsymbol{\omega} \times \mathbf{r}$ and $\mathbf{v}_B = \mathbf{v}_{cm} - \boldsymbol{\omega} \times \mathbf{r}$, where $\mathbf{r}$ describes the position of grain A relative to the center of mass of the compound grain.

(**k**) Show that the net force on the center of mass of the compound object is $\sum \mathbf{F} = 2M\mathbf{g} - 2k_v\mathbf{v}_{cm}$, where $\mathbf{g} = -g\mathbf{j}$ and g is the acceleration of gravity.

(**l**) Show that the torque around the center of mass of the compound object due to air resistance is $\boldsymbol{\tau} = -2k_v\boldsymbol{\omega}R^2$ (Hint: Use Lagrange's formula).

(**m**) Show that the angular acceleration α of the compound object around its center of mass can be written as $\alpha = d\omega/dt = -(1/t_0)\omega$, and find the characteristic time t_0.

(**n**) Describe (with words) the motion of the compound object.

(**o**) Sketch the time development of the velocity v_{cm} and the angular velocity ω of the compound object, and discuss how the behavior would change if you changed the radius, R, of the grains.

(**p**) How would our argument change if we instead studied large particles, where the air resistance force depends on the square of the velocity?

Final comment: Notice that the result above for the net force on the compound grain indicates that small and large grains have the same acceleration, which is not consistent with our initial result. This is due to our (incorrect) simplification of adding the air resistance force for each of the grains together to get the air resistance force for the compound grain. For a real ice crystal formed by aggregation, the dependence of the air resistance on the size of the compound grain is more complicated, and will also depend on the complex geometry attained by a compound grain after a few hundred collisions with smaller grains.

16.12 Collision with rotation. In this project we address a collision between two identical atoms. You will learn how to determine external and internal motion of a diatomic molecule after a collision using a combination of analytical techniques, such as conservation laws, and numerical methods to determine the motion of the molecule.

We want to address a collision between two identical atoms of mass m, and we assume that we may consider the atoms to be point particles. The atoms are not affected by any external forces.

Here, we will first analyze a simplified model for the collision—a one dimensional model—before we analyze the full collision process.

First, we address a simplified model. The system we consider consists of two atoms: atom A moves along the x-axis with the velocity v_0, and atom B is at rest in

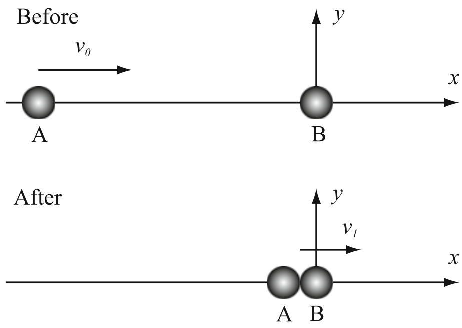


Fig. 16.37 Illustration of a simplified collision model: Atoms A and B collide on the x-axis

the origin as illustrated in Fig. 16.37. The atoms do not interact before they hit each other. After the collision they are stuck to each other.
(**a**) Find the velocity of the center of mass for the system before the collision.
(**b**) Find the velocity of the center of mass of the system after the collision.
(**c**) What is the change in the system's kinetic energy through the collision.

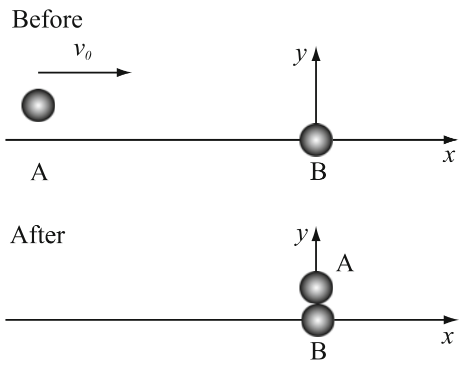
Let us make the model slightly more realstic by introducing a simplified model for the interactions between the two atoms. We will here not use a full model for the interatomic interaction, but instead assume that we can model the interatomic interaction using a spring force model. When atom A reaches a distance b from atom B, the two atoms become attached by a massless spring with spring constant k and equilibrium length b. The atoms remain attached with this spring throughout the collision and the subsequent motion.
(**d**) What is the velcoity of the center of mass immediately after the atoms are attached with the spring, that is, when atom A is at the distance b from atom B? What is the change in kinetic energy for the system before and immediately after the collision?

Let us now address the motion of the atoms after the attachment. The positions of the atoms are x_A and x_B.
(**e**) Show that the force on atom A is: $F = k((x_A - x_B) - b)$, and find a corresponding expression for the force on atom B.
(**f**) Find expressions for the acceleration for atom A and B, and formulate the differential equations you need to solve to find the motion of the atoms, including the initial conditions.
(**g**) Write a program to determine the positions and velocities of atom A and atom B as a function of time. Assume $m = 0.1$, $k = 20$, $b = 0.2$, $v_0 = 1.0$, and $\Delta t = 0.001$.
(**h**) Plot the position as a function of time for the center of mass of the system and for each of the atoms.
(**i**) What is the maximum distance between the two atoms?

The collision we have addressed so far is a special case—the case of a central colilsion. Let us now address a non-central collision.

First, we address a simplified model for a non-central collision, as illustrated in Fig. 16.38. Atom A moves in the x-direction along the line $y = b$ with velocity v_0, and atom B is at rest at the origin. The atoms do not interacti until they hit each

Fig. 16.38 Illustration of a simplified collision model: Atoms A and B collide in a non-central collision



other, which occurs when atom A reaches $x = 0$. After the collision, the atoms form a diatomic molecule, and the atoms remain attached at a fixed distance b from each other. (We are not studying a model without the spring force, but with a non-central collision. We will add the spring force again further on to get a complete, but still simplified model).

(**j**) What is the velocity of the center of mass and the angular velocity around the center of mass immediately after the collision?

Let us now introduce a more advanced model for this collision: When atom A is in the position $x = 0$, $y = b$, and atom B is in the position $x = 0$ and $y = 0$, the two atoms become attached with a massless spring with spring constant k and equilibrium length b. The atoms remain attached throughouth the subsequent motion.

(**k**) Rewrite your program to model the motion of the atoms in this case.

(**l**) Plot the motion of the atoms and the center of mass after the collision.

(**m**) Discuss the motion of the angular velocity for the rotation about the center of mass for the motion after the collision.

16.13 Modelling a Bouncing Ball. In this project we will study a ball that bounces on a flat surface. We will only look at a single collision between the ball and the surface, but we will use different models for the interactions during the collision.

Both the ball and the surface are deformed during the collision, but you can assume that this deformation is small, this means that the forces from the surface on the ball will only act in a single point on the ball throughout the collision, and that the distance from this point to the center of mass of the ball does not change. The ball slips against the surface during the collision and the coefficient of dynamic friction between the ball and the surface is the constant μ. You can neglect air resistance. The ball has a mass m and a radius R, the acceleration of gravity is g, the moment of inertia of the ball about its center of mass is I.

You throw the ball from a height h with only a horizontal velocity. The velocity immediately before the collision with the surface is $\mathbf{v}(t_0) = v_{0x}\,\mathbf{i} + v_{0y}\,\mathbf{j}$, where v_{0x} is positive and v_{0y} is negative.

(**a**) Draw a free-body diagram for the ball while it is in contact with the surface. Identify the forces.

Let us first assume that the normal force from the surface on the ball is constant, N_0.
(**b**) Find the vertical component of the velocity, $v_y(t)$, and the vertical position, $y(t)$, of the ball while it is in contact with the surface.
(**c**) How long is the ball in contact with the surface?
(**d**) Find the horizontal component of the velocity of the ball as a function of time, $v_x(t)$, while it is in contact with the surface. What is the horizontal component of the velocity of the ball, v_{1x}, immediately after the collision?
(**e**) Find the angular velocity as a function of time, $\omega(t)$, as well as the angular velocity, ω_1, of the ball immediately after the collision. Describe the motion of the ball after the collision.
(**f**) Is the energy of the ball conserved during the collision? Does the ball bounce back to the height h after the collision? Justify your answers.

Now assume that the force from the surface on the ball is $N = k(R - y)^{3/2}$ when the ball is in contact with the surface, i.e. when $y < R$.
(**g**) Find expressions for the accelerations a_x and a_y of the ball while it is in contact with the surface.
(**h**) Find an expression for the angular acceleration a_z of the ball while it is in contact with the ground.
(**i**) Write a program that finds the motion of the ball's centre of mass as a function of time.
(**j**) Use your program to plot the motion and the velocities of the ball as a function of time from $t = 0$s to $t = 1$s when the ball has a radius of $R = 0.15$ m and is released from a height $h = 1$ m with an initial velocity $v_{0x} = 3$ m/s. The spring constant is $k = 10{,}000$ N/m, the dynamic friction is $\mu = 0.3$, the mass of the ball $m = 1$ kg, the acceleration of gravity $g = 9.8$ m/s^2, and the moment of inertia $I = (2/3)$ kgm^2. Use a timestep of $dt = 0.001$ s.

Appendix A
Proofs

This appendix contains proofs that were not included in the main exposition of the material in the book.

A.1 Derivation of Formula for Two-Particle, Linear Elastic Collision

Let us study a collision between two objects—drawn as carts in Fig. A.1—where cart A has mass m_A and initial velocity $v_{A,0}$, and cart be has mass m_B and initial velocity $v_{B,0}$. In this case, we assume that cart B starts at rest, $v_{B,0} = 0$. However, we can address any problem like this, since we can always place our coordinate system so that it follows the motion of cart B before the collision. We assume that there are no external forces acting on either cart during the collision—only internal forces acting between the cart.

Since there are no external forces acting, the total translational momentum, P_x, is conserved in the x-direction. The translational momentum is:

$$P = p_A + p_B = m_A v_A + m_B v_B \ . \tag{A.1}$$

The momentum is conserved throughout the collision, and it is therefore the same at the time t_0 before the collision and the time t_1 after the collision:

$$P_0 = m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} = P_1 \ , \tag{A.2}$$

where we simplify by introducing $v_{B,0} = 0$m/s, allowing us to rewrite (A.2) to:

$$m_A \left(v_{A,0} - v_{A,1} \right) = m_B v_{B,1} \ . \tag{A.3}$$

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4

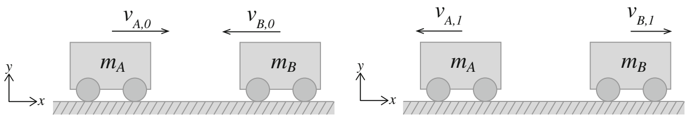

Fig. A.1 Illustration of a collision between two carts rolling along a friction-free, *horizontal* surface

This equation represents the conservation of momentum, and is true as long as the net external force is zero. We consider $v_{A,0}$ a know quantity. Then we are left with two unknowns $v_{A,1}$ and $v_{B,1}$. But we only have one equation. We do not know how the momentum is distributed between the two carts.

Now, if we assume that the collision is *elastic*, that is, that the mechanical energy is conserved throughout the collision, we can introduce an addition equation. This assumption means that we assume that the internal forces acting between the two carts are conservative, which is an additional assumption compared to what we did above. In this case, we can an additional equation for the conservation of mechanical energy:

$$\frac{1}{2}m_A v_{A,0}^2 + \frac{1}{2}m_B v_{B,0}^2 = \frac{1}{2}m_A v_{A,1}^2 + \frac{1}{2}m_B v_{B,1}^2 \ . \tag{A.4}$$

We now solve these two equations ((A.2) and (A.4)) to find $v_{A,1}$ and $v_{B,1}$. We start by rewriting (A.4) to:

$$m_A \left(v_{A,0}^2 - v_{A,1}^2\right) = m_B v_{B,1}^2 \ . \tag{A.5}$$

which can also be written as:

$$m_A \left(v_{A,0} - v_{A,1}\right)\left(v_{A,0} + v_{A,1}\right) = m_B v_{B,1}^2 \ . \tag{A.6}$$

We divide (A.6) by (A.3), getting:

$$v_{A,0} + v_{A,1} = v_{B,1} \ , \tag{A.7}$$

and we insert this for $v_{B,1}$ in (A.3):

$$m_A \left(v_{A,0} - v_{A,1}\right) = m_B \left(v_{A,0} + v_{A,1}\right) \ , \tag{A.8}$$

We solve for $v_{A,1}$:

$$m_A v_{A,0} - m_A v_{A,1} = m_B v_{A,0} + m_B v_{A,1} \ \Rightarrow \ \frac{m_A - m_B}{m_A + m_B} v_{A,0} = v_{A,1} \ , \tag{A.9}$$

We insert the result for $v_{A,1}$ into (A.3), and find:

$$m_A\left(v_{A,0}-v_{A,1}\right)=m_B v_{B,1} \;\Rightarrow\; m_A\left(1-\frac{m_A-m_B}{m_A+m_B}\right)v_{A,0}=m_B v_{B,1}\ , \tag{A.10}$$

$$v_{B,1}=\frac{2m_A}{m_A+m_B}v_{A,0}\ . \tag{A.11}$$

This proves the solutions provided in (12.67).

A.2 Derivation of Formula for Two-Particle, Linear Collisions

Let us address a general collision, elastic, inelastic, and perfectly inelastic, characterized by a coefficient of restitution r.

The coefficient of restitution is defined as the ratio of the relative velocities after the collision to the relative velocities before the collision:

$$r=-\frac{v_{A,1}-v_{B,1}}{v_{A,0}-v_{B,0}}\ . \tag{A.12}$$

The case $r=1$ corresponds to an elastic collision, $r=0$ corresponds to a perfectly inelastic collision, and the case $0<r<1$ corresponds to an inelastic collision.

From (A.12), the velocity of object B after the collision is:

$$v_{B,1}=v_{A,1}+r\left(v_{A,0}-v_{B,0}\right)\ , \tag{A.13}$$

which we combine with conservation of momentum:

$$m_A v_{A,0}+m_B v_{B,0}=m_A v_{A,1}+m_B v_{B,1}\ , \tag{A.14}$$

to get:

$$m_A v_{A,0}+m_B v_{B,0}=m_A v_{A,1}+m_B v_{A,1}+r m_B\left(v_{A,0}-v_{B,0}\right)\ . \tag{A.15}$$

We solve for $v_{A,1}$, finding:

$$v_{A,1}=\frac{\left(m_A-r m_B\right)v_{A,0}+\left(1+r\right)m_B v_{B,0}}{m_A+m_B}\ . \tag{A.16}$$

We find the velocity of object B in exactly the same way, or simply by exchanging the indexes for A and B:

$$v_{B,1} = \frac{(m_B - rm_A)\, v_{B,0} + (1 + r)\, m_A v_{A,0}}{m_A + m_B} \ . \tag{A.17}$$

These are the completely general solutions to the collision problem, including the special cases of an elastic collision ($r = 1$) and a perfectly inelastic collision ($r = 0$).

A.3 Kinetic Energy of a Multi-particle System

The total kinetic energy of a multi-particle system is:

$$K = \sum_{i=1}^{N} \frac{1}{2} m \left(\mathbf{v}_i\right)^2 = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{d\mathbf{r}_i}{dt}\right)^2 \ . \tag{A.18}$$

We divide the motion into the motion of the center of mass of the system and the motion relative to the center of mass:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} \ , \ \mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i} \ . \tag{A.19}$$

We insert this into the total kinetic energy of the system:

$$\begin{aligned}
K &= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\mathbf{v}_i\right)^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{V} + \mathbf{v}_{\mathrm{cm},i}\right)^2 \\
&= \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{V}\right)^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(2\mathbf{V} \cdot \mathbf{v}_{\mathrm{cm},i}\right) + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 \\
&= \frac{1}{2} M \left(\mathbf{V}\right)^2 \ + \ \mathbf{V} \cdot \underbrace{\sum_{i=1}^{N} m_i \mathbf{v}_{\mathrm{cm},i}}_{=\mathbf{P}_{\mathrm{cm}}=0} \ + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 \\
&= \frac{1}{2} M \left(\mathbf{V}\right)^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \left(\mathbf{v}_{\mathrm{cm},i}\right)^2 \ .
\end{aligned} \tag{A.20}$$

This proves that the kinetic energy can be subdivided into the kinetic energy of the translational motion of the center of mass and the kinetic energy of the motion relative to the center of mass.

A.4 Proof of the Superposition Principle

The proof of the superposition principle follows directly from the definition of the moment of inertia: The moment of inertia of a system consisting of both system A and system B around the axis O is:

$$I_O = \sum_j m_i \rho_i^2 \; , \tag{A.21}$$

where the sum is over all particles in object A and all particles in object B. We can split this sum into two parts: One sum of all the particles in object A and one sum over all the particles in object B:

$$I_O = \sum_{j=1}^{N_A} m_i \rho_i^2 + \sum_{j=N_A+1}^{N_A+N_B} m_i \rho_i^2 = I_{O,A} + I_{O,B} \; , \tag{A.22}$$

which is a proof of the superposition principle.

A.5 Proof of the Parallel-Axis Theorem:

The moment of inertia of an object around the axis A is:

$$I_A = \sum_i m_i \rho_{A,i}^2 \; , \tag{A.23}$$

where $\rho_{A,i}$ is the distance from a point i to the axis. Since the vector $\mathbf{s}$ points from axis A to axis C through the center of mass, we can write the vector from axis A to point i as (see Fig. A.2):

$$\boldsymbol{\rho}_{A,i} = \mathbf{s} + \boldsymbol{\rho}_{C,i} \; , \tag{A.24}$$

where $\boldsymbol{\rho}_{C,i}$ is a vector from axis C to point i. We insert this into the sum in (A.23):

$$\begin{aligned}
I_A &= \sum_i m_i \left(\boldsymbol{\rho}_{A,i}\right)^2 = \sum_i m_i \left(\mathbf{s} + \boldsymbol{\rho}_{C,i}\right)^2 = \sum_i m_i \left(s^2 + 2\boldsymbol{\rho}_{C,i} \cdot \mathbf{s} + \rho_{C,i}^2\right) \\
&= \sum_i m_i s^2 + \sum_i m_i 2\boldsymbol{\rho}_{C,i} \cdot \mathbf{s} + \sum_i m_i \rho_{C,i}^2 = Ms^2 + 2\mathbf{s} \cdot \underbrace{\sum_i m_i \boldsymbol{\rho}_{C,i}}_{=0} + \underbrace{\sum_i m_i \rho_{C,i}^2}_{=I_C} \\
&= I_C + Ms^2 \; .
\end{aligned} \tag{A.25}$$

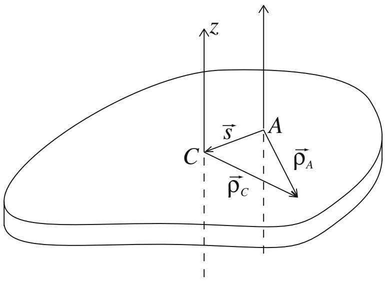

Fig. A.2 Illustration of the parallel-axis theorem. The moment of inertia around an axis C through the center of mass can be used to find the moment of inertia around an axis A—if the axis A is parallel to the axis C through the center of mass. The vector $\mathbf{s}$ is perpendicular to both axes, and points from the origin of axis C to the origin of axis A

Here we have used that the position of the center of mass in the center of mass system is zero, hence $\sum_i m_i \boldsymbol{\rho}_{C,i} = 0$.

A.6 Rotational Momentum and Newton's Second Law for a Point Particle

The angular momentum of the point particle with mass m, velocity $\mathbf{v}$, and position, $\mathbf{r}$ is defined as:

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} \; . \tag{A.26}$$

We find Newton's second law by taking the time derivative of this equation:

$$\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \underbrace{\frac{d\mathbf{r}}{dt} \times \mathbf{p}}_{=0} + \mathbf{r} \times \underbrace{\frac{d\mathbf{p}}{dt}}_{=\sum \mathbf{F}} \; , \tag{A.27}$$

where we have used that $d\mathbf{r}/dt = \mathbf{v}$ is parallel to $\mathbf{p}$ and that the cross-product $\mathbf{v} \times \mathbf{p}$ is zero. This gives Newton's second law for a point particle on an alternative form:

$$\frac{d\mathbf{l}}{dt} = \sum_j \mathbf{r} \times \mathbf{F}_j \; , \tag{A.28}$$

where all the forces are acting in the point $\mathbf{r}$, since the point particles does not have any physical extent.

A.7 Rotational Momentum of a Multiparticle System

From the momentum $\mathbf{p}_i$ of a particle i, we introduced the total momentum,

$$\mathbf{P} = \sum_i \mathbf{p}_i \;, \tag{A.29}$$

of a multiparticle system. This allowed us to formulate a generalized version of Newton's second law for translational motion. Similarly, we can introduce the rotational momentum of a system of particles, simple as the sum of the rotational momentum of each particle. Particle i is in position $\mathbf{r}_i$ relative to the origin, hence the rotational momentum of particle i is:

$$\mathbf{l}_i = \mathbf{r}_i \times m_i \mathbf{v}_i \;, \tag{A.30}$$

and the total rotational momentum of a system of particles around a point O, the origin, is defined as:

$$\mathbf{L}_O = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \;. \tag{A.31}$$

A.8 Newton's Second Law for Rotation Around a Fixed Axis

Since we have alread found that for a single point particle, Newton's second law can be written as

$$\frac{d\mathbf{l}_i}{dt} = \boldsymbol{\tau}_i \;, \tag{A.32}$$

we can try to find a similar relation for a multiparticle system. We start from the definition of the total angular momentum $\mathbf{L}_O$:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \;, \tag{A.33}$$

and take the time derivative on both sides:

$$\frac{d\mathbf{L}_O}{dt} = \frac{d}{dt} \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_{i=1}^{N} \underbrace{\frac{d\mathbf{r}_i}{dt} \times m_i \mathbf{v}_i}_{=0} + \sum_{i=1}^{N} \mathbf{r}_i \times \underbrace{\frac{d m_i \mathbf{v}_i}{dt}}_{=\mathbf{F}_i^{\text{net}}} \;. \tag{A.34}$$

Here, the net force on part i is the sum of the external forces acting on part i and the internal forces. An internal force must originate in one of the other parts of the system. We can therefore write all the internal forces as: $\mathbf{F}_{j,i}$, meaning the force from part j on part i. The net force on part i is therefore:

$$\mathbf{F}_i^{\text{net}} = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} \ , \tag{A.35}$$

which we insert into the equations, getting:

$$\frac{d\mathbf{L}_O}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{j,i} \right) = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} \ . \tag{A.36}$$

Let us look at the last term, which is the sum of the torques on part i from all parts j in the system. From Newton's third law, we know that $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$. Therefore, we rewrite the equation so that we explicitely include action/reaction terms. We do this through a "trick" you will often meet in physics: We realize that the sum over all the internal torques:

$$\sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} \ , \tag{A.37}$$

is a sum over all pairs, i, j, so that $i \neq j$. We could also write this as:

$$\sum_{i=1}^{N} \sum_{j \neq i} \mathbf{r}_i \times \mathbf{F}_{j,i} = \sum_{i,j:i \neq j} \mathbf{r}_i \times \mathbf{F}_{j,i} \ , \tag{A.38}$$

where the last sum is over all possible values of i and j as long as they are not equal. However, in this sum, there are pairs of torques that are related by action/reaction forces. There is a torque on particle i due to the force $\mathbf{F}_{j,i}$ from particle j on particle i, but there is also a torque on particle j due to the force $\mathbf{F}_{i,j}$ from particle i on particle j. If we want to include both of these terms explicitely in the sum, the sum must only be over half of the i and j values so that we do not include any term twice. We ensure this by summing over all pairs i and j so that $i < j$:

$$\sum_{i,j:i \neq j} \mathbf{r}_i \times \mathbf{F}_{j,i} = \sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \mathbf{F}_{i,j} \right) \ . \tag{A.39}$$

We now use that $\mathbf{F}_{j,i} = -\mathbf{F}_{i,j}$:

$$\sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \mathbf{F}_{i,j}\right) = \sum_{i,j:i<j} \left(\mathbf{r}_i \times \mathbf{F}_{j,i} + \mathbf{r}_j \times \left(-\mathbf{F}_{j,i}\right)\right)$$
$$= \sum_{i,j:i<j} \left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_{j,i} \; . \tag{A.40}$$

Hmmm. What can we do about this sum? We realize that if the force on particle i from particle j acts along the line between particle i and j:

$$\mathbf{F}_{j,i} = C_{j,i} \left(\mathbf{r}_i - \mathbf{r}_j\right) \; , \tag{A.41}$$

then the last term in (A.40) is zero. We call such forces central forces. In a rigid body, we assume that all forces are central. However, in many other types of systems the forces are also central. For example, gravitational forces, typical two-particle interatomic forces (given by a two-particle potential energy), and electro-static forces are central forces: This means that many forces from the atomic to the galactic scale are indeed central forces. It is therefore a reasonable assumption to assume that the forces are central forces, and that the sum of internal torques is zero.

The rate of change of the total angular momentum of the system is therefore equal to the net external torque on the system:

$$\frac{d\mathbf{L}_O}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_{\text{ext}} \; , \tag{A.42}$$

which is what we call the net external torque around the point O.

A.9 Rotational Momentum of a Ridig Body

We have now found a general formulation for Newton's second law for rotational motion of a multiparticle system. However, we are often interested in rigid bodies. Can we simplify the relation by first finding the rotational momentum $\mathbf{L}_O$, of a rigid body rotating around a fixed axis, and then use this to find a simplified expression for Newton's second law for rotation of rigid bodies?

First, we find the rotational momentum of a rigid body. Figure A.3 shows a rigid body rotating around the z-axis with an angular velocity $\boldsymbol{\omega} = \omega\,\mathbf{k}$. The rigid body consists of a set of mass point, m_i, located at positions $\mathbf{r}_i$. The rotational momentum of this system is then:

$$\mathbf{L}_O = \sum_i \mathbf{l}_i = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \; . \tag{A.43}$$

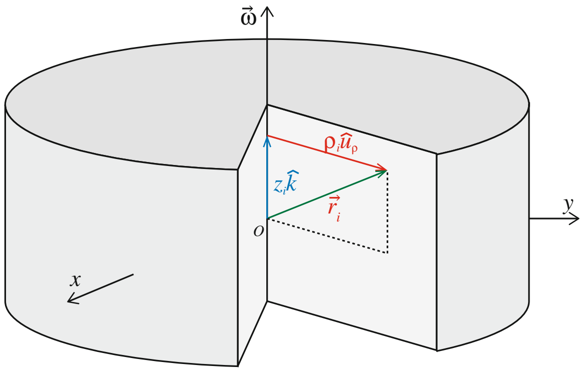

Fig. A.3 Illustration of a *rigid* body rotating around the z-axis. The cylindrical coordinate system is illustrated

Since the rigid body is rotating with angular velocity $\boldsymbol{\omega}$, the velocity of point i at $\mathbf{r}_i$ is

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \ , \tag{A.44}$$

where we can decompose the position $\mathbf{r}_i$ using a cylidrical coordinate system with a radius vector, $\boldsymbol{\rho} = \rho \hat{u}_\rho$, from the z-axis and out to the point $\mathbf{r}_i$ and a coordinate z_i along the z-axis:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i \, \mathbf{k} \ . \tag{A.45}$$

We insert this into the expression for $\mathbf{l}_i$, getting:

$$\mathbf{l}_i = \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i \times (\omega \mathbf{k} \times (\boldsymbol{\rho}_i + z_i \, \mathbf{k})) \ , \tag{A.46}$$

where we notice that the $z_i\mathbf{k}$ term is parallel to $\omega\mathbf{k}$, and therefore this part of the cross product is zero, giving:

$$\mathbf{l}_i = \mathbf{r}_i \times \left(\boldsymbol{\omega} \times \boldsymbol{\rho}_i + \underbrace{\boldsymbol{\omega} \times z_i \, \mathbf{k}}_{=0} \right) = \mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ . \tag{A.47}$$

We use Lagrange's formula for the cross product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \, (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \, (\mathbf{a} \cdot \mathbf{b}) \ , \tag{A.48}$$

getting:

$$\begin{aligned} \mathbf{l}_i &= m_i \, (\boldsymbol{\omega} \, (\mathbf{r} \cdot \boldsymbol{\rho}) - \boldsymbol{\rho} \, (\mathbf{r} \cdot \boldsymbol{\omega})) = m_i \, (\boldsymbol{\omega} \, ((\boldsymbol{\rho} + z \, \mathbf{k}) \cdot \boldsymbol{\rho}) - \boldsymbol{\rho} \, ((\boldsymbol{\rho} + z \, \mathbf{k}) \cdot \boldsymbol{\omega})) \\ &= m_i \left(\rho_i^2 \boldsymbol{\omega} - \boldsymbol{\omega} z_i \boldsymbol{\rho}_i \right) \ . \end{aligned} \tag{A.49}$$

The total rotational momentum around the point O is therefore:

$$\begin{aligned} \mathbf{L}_O &= \sum_i \mathbf{l}_i = \sum_i m_i \left(\rho_i^2 \boldsymbol{\omega} - \boldsymbol{\omega} z \boldsymbol{\rho}_i \right) \\ &= \underbrace{\sum_i m_i \rho_i^2}_{=I_{O,z}} \boldsymbol{\omega} - \sum_i \boldsymbol{\omega} m_i z_i \boldsymbol{\rho}_i = I_{O,z} \boldsymbol{\omega} - \boldsymbol{\omega} \sum_i m_i z_i \boldsymbol{\rho}_i \ . \end{aligned} \tag{A.50}$$

Notice the second part of this equation. This term will be zero if the object is rotationally symmetric around the rotation axis. Otherwise, we will need to include this term.

However, if we are only interested in the z-component of the total rotational momentum of a rigid body, then we get a simplified result:

$$L_{O,z} = \mathbf{L}_O \cdot \mathbf{k} = \left(I_{O,z} \boldsymbol{\omega} - \boldsymbol{\omega} \sum_i m_i z_i \boldsymbol{\rho}_i \right) \cdot \mathbf{k} = I_{O,z} \underbrace{\boldsymbol{\omega} \cdot \mathbf{k}}_{=\omega} - \boldsymbol{\omega} \sum_i m_i z_i \underbrace{\boldsymbol{\rho}_i \cdot \mathbf{k}}_{=0} = I_{O,z} \boldsymbol{\omega} \ . \tag{A.51}$$

Although we must be careful with this expression, because it is tempting to generalize it to a vector equations, which we found above is only correct if the object is rotationally symmetric around the rotation axis.

A.10 Subdivision of Rotational Momentum

The total rotational momentum $\mathbf{L}_O$ around a fixed point O can be decomposed into the rotational momentum of the center of mass moving as a point particle relative to O and the rotational momentum relative to the center of mass:

$$\mathbf{L}_O = \mathbf{R} \times \mathbf{R} + \mathbf{L}_{cm} \ , \tag{A.52}$$

where $\mathbf{R}$ is the position and $\mathbf{P} = M\mathbf{V}$ is the momentum of the center of mass. We can show this by starting from the definition of the rotational momentum around a fixed point O:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{l}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \ . \tag{A.53}$$

We decompose the position of mass m_i into:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{cm,i} \tag{A.54}$$

where $\mathbf{R}$ is the position of the center of mass, and $\mathbf{r}_{cm,i}$ is the position of mass i relative to the center of mass. Inserted into (A.53), we get:

$$\begin{aligned}
\mathbf{L}_O &= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} \right) \\
&= \sum_{i=1}^{N} m_i \left[\left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \times \frac{d}{dt} \left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \right] \\
&= \sum_{i=1}^{N} m_i \left[(\mathbf{R} \times \mathbf{V}) + \left(\mathbf{R} \times \mathbf{v}_{cm,i} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{V} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \right] \\
&= M\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \frac{d}{dt} \underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i}}_{=0} + \underbrace{\left(\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \right)}_{=0} \times \mathbf{V} + \sum_{i=1}^{N} \left(m_i \mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i}
\end{aligned} \tag{A.55}$$

A.11 Newton's Second Law for Rotation Around the Center of Mass

We can always describe the motion of a particle i in a multiparticle system as:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i} \ , \tag{A.56}$$

where $\mathbf{R}$ is the position of the center of mass, and $\mathbf{r}_{\mathrm{cm},i}$ is the position of the particle relative to the center of mass, as illustrated in Fig. A.4.

Similarly, we may split the total rotational momentum of a system into the rotational momentum of the center of mass, and the rotational momentum relative to the center of mass:

$$\mathbf{L}_O = \sum_{i=1}^{N} \mathbf{l}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i \ , \tag{A.57}$$

where we now introduce $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\mathrm{cm},i}$ and $\mathbf{v}_i = \mathbf{V} + \mathbf{v}_{\mathrm{cm},i}$:

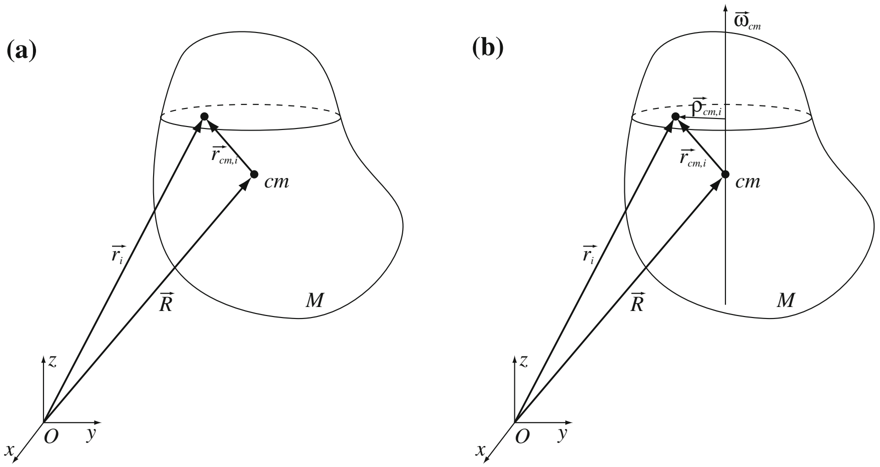

Fig. A.4 Illustration of an object rotating around the center of mass. **a** Position of a point relative to the center of mass. **b** Position of a point relative to the axis of rotation

$$
\begin{aligned}
\mathbf{L}_O &= \sum_{i=1}^{N} m_i \left(\mathbf{r}_i \times \frac{d\mathbf{r}_i}{dt} \right) \\
&= \sum_{i=1}^{N} m_i \left[\left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \times \frac{d}{dt} \left(\mathbf{R} + \mathbf{r}_{cm,i} \right) \right] \\
&= \sum_{i=1}^{N} m_i \left[(\mathbf{R} \times \mathbf{V}) + \left(\mathbf{R} \times \mathbf{v}_{cm,i} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{V} \right) + \left(\mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \right] \\
&= M\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \frac{d}{dt} \underbrace{\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i}}_{=0} + \underbrace{\left(\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \right)}_{=0} \times \mathbf{V} + \sum_{i=1}^{N} \left(m_i \mathbf{r}_{cm,i} \times \mathbf{v}_{cm,i} \right) \\
&= \mathbf{R} \times \mathbf{P} + \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i}
\end{aligned}
\tag{A.58}
$$

The first term is the rotational momentum of the center of mass motion around the axis O, the second term is the rotational momentum of the object relative to the center of mass:

$$
\mathbf{L}_{\mathrm{cm}} = \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{p}_{cm,i} \ . \tag{A.59}
$$

Let us now use Newton's second law for rotations of multiparticle systems to find out what determines the change in rotational momentum around the center of mass.

From Newton's second law for rotations, we know that the rotational momentum and torque around the *fixed point* O are related by:

$$\frac{d\mathbf{L}_O}{dt} = \boldsymbol{\tau}_O = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} \ , \tag{A.60}$$

First, we insert the result for the rotational momentum to determine what is on the left hand side:

$$\begin{aligned}\frac{d}{dt}\mathbf{L}_O &= \frac{d}{dt}\left(\mathbf{R} \times \mathbf{P} + \mathbf{L}_{\text{cm}}\right) = \frac{d\mathbf{R}}{dt} \times \mathbf{P} + \mathbf{R} \times \frac{d\mathbf{P}}{dt} + \frac{\mathbf{L}_{\text{cm}}}{dt} \\ &= \underbrace{\mathbf{V} \times M\mathbf{V}}_{=0} + \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} \ .\end{aligned} \tag{A.61}$$

Second, we rewrite the right-hand side of (A.60) by introducing the center of mass coordinates:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{\text{cm},i} \ , \tag{A.62}$$

We can therefore rewrite

$$\begin{aligned}\boldsymbol{\tau}_O &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \sum_{i=1}^{N} \left(\mathbf{R} + \mathbf{r}_{\text{cm},i}\right) \times \mathbf{F}_i^{\text{ext}} = \sum_{i=1}^{N} \mathbf{R} \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \\ &= \mathbf{R} \times \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \ ,\end{aligned} \tag{A.63}$$

where we have introduce the net external force as:

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} \ . \tag{A.64}$$

We insert this result and the result from (A.61) into (A.60), getting:

$$\begin{aligned}\frac{d\mathbf{L}_O}{dt} &= \boldsymbol{\tau}_O \mathbf{R} \times \mathbf{F}^{\text{ext}} + \frac{\mathbf{L}_{\text{cm}}}{dt} = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \frac{\mathbf{L}_{\text{cm}}}{dt} \\ &= \sum_{i=1}^{N} \mathbf{r}_{\text{cm},i} \times \mathbf{F}_i^{\text{ext}} \frac{\mathbf{L}_{\text{cm}}}{dt} = \boldsymbol{\tau}_{\text{cm}} \ ,\end{aligned} \tag{A.65}$$

where we have introduced:

$$\boldsymbol{\tau}_{\mathrm{cm}} = \sum_{i=1}^{N} \mathbf{r}_{\mathrm{cm},i} \times \mathbf{F}_i^{\mathrm{ext}} \,, \tag{A.66}$$

as the torque around the center of mass.

We have therefore proven Newton's second law for rotational motion around the center of mass:

$$\frac{d\mathbf{L}_{\mathrm{cm}}}{dt} = \boldsymbol{\tau}_{\mathrm{cm}} \,. \tag{A.67}$$

We notice that in this derivation we have not assumed anything about the motion of the center of mass—it can be accelerated or executing any type of motion—the law is still valid!

Appendix B
Solutions

Chapter 2

B.1 Seconds.

(**a**) `s = 3600*h, print s` (**b**) 5400 s, 43200 s, 86499 s

B.5 Plotting the normal distribution.

(**a**)

```
def normal(x,mu,sigma):
    P = 1.0/sqrt(2*pi*sigma**2)*exp(-(x-mu)**2/2*sigma**2)
    return P
```

(**b**)

```
x = linspace(-5,5,1000)
P = norpdf(x,0,1)
plot(x,P), show()
```

(**c**)

```
plot(x,P), hold('on')
P = normpdf(x,0,2); plot(x,P,'-r')
P = normpdf(x,0,0.5); plot(x,P,'-g')
hold('off'), show()
```

(**d**)

```
x = linspace(-5,5,1000)
subplot(3,1,1)
P = normpdf(x,0,1)
plot(x,P,'-b')
subplot(3,1,2)
P = normpdf(x,1,1);
plot(x,P,'-g')
subplot(3,1,3)
P = normpdf(x,2,);
plot(x,P,'-g'), show()
```

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A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4

B.6 Plotting $1/x^n$.

(a)

```
def fvalue(x,n):
    f = 1.0/x**n
    return f
```

(b)

```
x = linspace(-1,1,1000)
f1 = fvalue(x,1)
plot(x,f1), hold('on')
f2 = fvalue(x,2)
plot(x,f2)
f3 = fvalue(x,3)
plot(x,f3), hold('off'), show()
```

B.7 Plotting $\sin(x)/x^n$.

(a)

```
def gvalue(x,n):
    g = 1.0/x**n
    return g
```

(b)

```
x = linspace(-5,5,1000)
g1 = gvalue(x,1)
plot(x,g1), hold('on')
g2 = gvalue(x,2)
plot(x,g2)
g3 = gvalue(x,3)
plot(x,g3), hold('off'), show()
```

B.8 Logistic map.

(a)

```
def logistic(x,r):
    g = r*x*(1-x)
    return g
```

(b)

```
r = 1.0
n = 100
x = zeros(n,float)
for i in range(n-1):
    x[i+1] = logistic(x,r)
i = r_[0:n-1], plot(i,x), show()
```

B.9 Euler's method.

(a)

```
def acceleration(v,x,k,C):
    a = -k*x - C*v
    return a
```

(b)

```
k = 10, C = 5, n = 100, deltat = 0.01, n = 100
x = zeros(n,float), v = zeros(n,float)
a = zeros(n,float), t = zeros(n,float)
x[0] = x0, v[0] = v0
for i in range(n-1):
```

```
    a[i] = acceleration(v[i],x[i],k,C)
    v[i+1] = v[i] + a[i]*deltat
    x[i+1] = x[i] + v[i]*deltat
    t[i+1] = t[i] + deltat
subplot(3,1,1)
plot(t,a), xlabel('t'), ylabel('a')
subplot(3,1,2)
plot(t,v), xlabel('t'), ylabel('v')
subplot(3,1,3)
plot(t,x), xlabel('t'), ylabel('x')
show()
```

(**c**) You only need to change the function `acceleration`.

```
def acceleration(v,x,k,C):
    a = k*sin(x)-C*v
    return a
```

B.10 Throwing two dice.

(**a**) In vectorized notation:

```
def dice(n):
    x1 = randint(1,6,n)
    x2 = randint(1,6,n)
    z = x1+x2
    return z
```

Using loops:

```
def dice(n):
z = zeros(n,float)
for i in range(n):
   x1 = randint(6)
   x2 = randint(6)
   z[i] = x1 + x2
return
```

B.11 Reading data.

(**a**)

```
t,x,y=loadtxt('trajectory.d',usecols=[0,1,2],unpack=True)
```

(**b**)

```
subplot(2,1,1)
plot(t,x), xlabel('t (s)'), ylabel('x (m)')
subplot(2,1,2)
plot(t,y), xlabel('t (s)'), ylabel('y (m)')
```

(**c**)

```
plot(x,y), xlabel('x (m)'), ylabel('y (m)')
```

B.12 Numerical integration of a data-set.

(**a**)

```
t,v=loadtxt('velocityy.d',usecols=[0,1],unpack=True)
```

(**b**)

```
plot(t,v), xlabel('t (s)'), ylabel('v (m/s)')
```

(**c**)

```
n = len(t)
y = zeros(n,float)
y0 = 0.0
y[0] = y0
for i in range(n-1):
    y[i+1] = y[i] + v[i]*(t[i+1]-t[i])
```

(**d**)

```
subplot(2,1,1)
plot(t,y), xlabel('t (s)'), ylabel('y (m)')
subplot(2,1,2)
plot(t,v), xlabel('t (s)'), ylabel('v (m/s)')
```

Chapter 3

B.1 Kilometers per hour.

40 m/s

B.2 Miles per hour.

(**a**) 43 mph (**b**) 89 km/h

B.3 Acceleration of gravity.

(**a**) $g = 32.2\,\text{ft/s}^2$ (**b**) $g = 1.3\,10^5\,\text{km/h}^2$

B.4 Bacterial volume.

(**a**) $4\pi\ (\mu m)^3$ (**b**) $4\pi\ 10^-18\,\text{m}^3$ (**c**) 4π femtoliter

B.5 Ruler length.

2.2 m

B.6 Sphere mass and volume.

(**a**) $7.2\,\text{mm}^3$ (**b**) $5.6\,10^{-5}\,\text{kg} = 56\,\text{milligram}$

B.7 Laserlength.

11.2 m

B.8 Salmon speed.

(**a**) 3.05 m/s (**b**) 3.05 m/s (**c**) Works if you are limited by your accuracy in time.

Chapter 4

B.12 Capturing the motion of a falling ball.

(**d**) $v_{\text{max}} \simeq -5.75\,\text{m/s}$

B.17 The fastest indian.

(**a**) 898 m (**b**) 11.1 s

B.18 Meeting trains.

(**a**) 12 minutes (**b**) 10 km

B.19 Catching up.
(**a**) 900 m (**b**) 0 m (**d**) 2160 s (**e**) 36 m (**l**) 2.32 km

B.20 Electron in electric field.
(**a**) $\sqrt{14000\,\text{m}^2/\,\text{s}^2} = 118\,\text{m/s}$

B.21 Archery.
(**a**) $1.8\,\text{km/s}^2$

B.22 Collision.
(**a**) $a = -50\,\text{m/s}^2$

B.23 Braking distance.
(**a**) $x = v_0^2/(10\,\text{m/s}^2)$ (**b**) $x_{\text{old tires}} = (3/2)\ x_{\text{new tires}}$ (**c**) $x_{\text{stop, new tires}} = 26.2\,\text{m}$, $x_{\text{stop, old tires}} = 35.9\,\text{m}$

B.28 A swimming bacterium.
(**a**) $v = v_0 + a_0 T/(2\pi)\,[1 - \cos{(2\pi t/T)}]$ (**b**) $x = v_0 t + a_0 (T/2\pi)\,[t - (T/2\pi)\sin{(2\pi t/T)}]$ (**c**) $v_{av} = v_0 + a_0(T/2\pi)$

Chapter 5

B.18 Pulling a train.
(**a**) $2\,\text{m/s}^2$ (**b**) $1.66\,\text{m/s}^2$

B.19 Firing a bullet.
$v = 141\,\text{m/s}$

B.20 Jumping into snow.
$6mg$

B.22 Vertical throw.
(**b**) $a = -g$ (**c**) $t = \left(v_0 + \sqrt{2gh_0 + v_0^2}\right)/g$ (**d**) $v = -\sqrt{v_0^2 + 2gh_0}$ (**e**) $v = -\sqrt{v_0^2 + 2gh_0}$

B.23 Reaction time.
(**b**) $x(t) = -(1/2)gt^2$ (**c**) $t = \sqrt{2h/g}$ (**d**) $x_{\text{car}} = v_{\text{car}}\sqrt{2h/g}$

B.24 Terminal velocity of heavy and large objects.
(**b**) $a = -g + Dv^2/m$ (**c**) Largest mass has largest acceleration (**d**) $a = -g + (6C_0 v^2)/(\pi \rho d)$ where ρ is the mass density (**e**) The object with the largest diameter has the largest magnitude of the acceleration.

B.25 Space shuttle with air resistance.
(**b**) $a = F/m - g$ (**c**) 153.8 m/s, 1538 m

B.26 Experiments in Pisa.
(**a**) Gravity and air resistance. (**b**) Air resistance is the same for both spheres (**c**) $a = g - f(v)/m$ (**d**) The solid sphere reaches the ground first.

B.27 Stretching an aluminum wire.
$k = 98\,\text{kN/m}$

B.28 Two masses and a spring.
$k = 98100\,\text{N/m}$

Chapter 6

B.12 Alpha particle.
(**a**) 2235 m/s (**b**) $\mathbf{r} = \mathbf{v}t = 1000\,\text{m/s}\,t\,\mathbf{i} + 2000\,\text{m/s}\,t\,\mathbf{j}$ (**c**) 2235 m

B.13 Airplane collision.
(**a**) $x(t) = 0$, $y(t) = 472.2\,\text{m/s}\,t$ (**b**) $x(t) = -1.0e4\,\text{m} + 29.2\,\text{m/s}\,t$, $y(t) = 8.0e4\,\text{m} + 251.4\,\text{m/s}\,t$, (**d**) No (**e**) Yes

B.14 Motion of spaceship.
(**b**)

$$\mathbf{v}(t) = \begin{cases} 1000\,\text{m/s}\,\mathbf{i} + 10\,\text{m/s}^2\,t\,\mathbf{j}\ , \text{when } t < 10\,\text{s} \\ 1000\,\text{m/s}\,\mathbf{i} + 100\,\text{m/s}\,\mathbf{j}\ \ , \text{when } t \geq 10\,\text{s} \end{cases} \tag{B.1}$$

(**c**)

$$\mathbf{r}(t) = \begin{cases} 1000\,\text{m/s}\,t\,\mathbf{i} + 5\,\text{m/s}^2\,t^2\,\mathbf{j} & \text{when} t < 10\,\text{s} \\ 1000\,\text{m/s}\,t\,\mathbf{i} + 500\,\text{m}\,\mathbf{j} + 100\,\text{m/s}\,t\,\mathbf{j} & \text{when} t \geq 10\,\text{s} \end{cases} \tag{B.2}$$

B.15 Controlling the electron beam.
(**a**) $v_x(t) = 100\,\text{m/s}$, $v_y(t) = -20\,\text{m/s}^2\,t - 5\,\text{m/s}^3\,t^2$. (**b**) $x(t) = 100\,\text{m/s}\,t$, $y(t) = -10\,\text{m/s}^2\,t^2 - (5/3)\,\text{m/s}^3\,\text{t}^3$. (**c**) $t = 1/50\,\text{s}$ (**d**) $y = -4.01 \times 10^{-3}\,\text{m}$ (**e**) $\alpha = -0.23°$.

B.17 Running inside a bus.
(**a**) 40 km/h (**b**) 60 km/h

B.18 Jumping onto a running train.
(**a**) $-10\,\text{m/s}$ (**b**) $5\,\text{m/s}^2$ (**c**) $v = -10\,\text{m/s} + 5\,\text{m/s}^2\,t$, when $t < 2\,\text{s}$, $v = 0\,\text{m/s}$, when $t > 2\,\text{s}$ (**d**) $v = 5\,\text{m/s}^2\,t$, when $t < 2\,\text{s}$, $v = 10\,\text{m/s}$ when $t > 2\,\text{s}$

B.19 A plane in crosswinds.
(**a**) 78.5 degrees over west (**b**) $v = 293.9\,\text{km/h}$

Chapter 7

B.11 Chandelier.
(**b**) $T = 490.5 * \sqrt{(}h^2 + 8)/h$ (**c**) $h = 0.6424\,\text{m}$

B.12 Three-pointer.
(**b**) $x(t) = 4.7\,textm/s\,t$, $y(t) = y_0 + 8.1\,\text{m/s}\,t - 4.9\,\text{m/s}^2\,t^2$ (**c**) 2.2 m (**d**) $-6.5\,\text{m/s}$

B.13 Hitting an apple.
(**b**) $x(t) = 50\,\text{m/s} \cdot t$ (**c**) 1.23 m (**d**) 3.675 m (**e**) 4.5 m

B.14 Hitting the target.
$v = 3.50\,\text{m/s}$

B.15 Long jump world record.
9.20 m

B.18 Weather balloon.
(**b**) $a = (B/m) - g$ (**c**) $v(t) = v(0) + (B/m - g)\,t$, $z(t) = z(0)(1/2)\,(B/m - g)\,t^2$. (**f**) $v_z^2 = (B/m - g)/(D/m)$ (**g**) $\mathbf{F}_D = -D\,|\mathbf{v} - \mathbf{w}|\,(\mathbf{v} - \mathbf{w})$. (**i**) $\mathbf{a} = (B/m)\mathbf{k} - g\mathbf{k} - (D/m)\,|\mathbf{v} - w\,\mathbf{i}|\,(\mathbf{v} - w\,\mathbf{i})$. (**m**) $v_z = ((B/m) - g)/(D/m)$ (**o**) It is the same.

Chapter 8

B.5 Skier pulled up a slope.
(**a**) $v(t) = at$ (**b**) $s(t) = \frac{1}{2}at^2$ (**c**) $\mathbf{r}(t) = s(t)\,(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})$ (**d**) $|\mathbf{v}(t)| = |at|$

B.6 Skiing down a slope.
(**a**) $v(t) = a\,t$ (**b**) $s(t) = (1/2)\,a\,t^2$ (**c**) $\mathbf{r}(t) = h\,\mathbf{j} + (g/2)\sin\alpha t^2\,(\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j})$ (**d**) $t = \sqrt{h/g}\,(1/\sin\alpha)$

B.7 Bead on a line.
(**a**) $v(t) = a\,t$ (**b**) $s(t) = (1/2)\,a\,t^2$ (**c**) $h(t) = -s(t)\ \cos\alpha$

B.8 Acceleration of 200 m sprinter.
(**a**) $R = 100\,\text{m}/\pi$ (**b**) $a = v^2/R$ toward the center of the circle

B.9 Velocity of point on helicopter rotor blade.
(**a**) $v \simeq 105\,\text{m/s}$ (**b**) $a = 2.2\,\text{km/s}$ towards the center of the blade

B.10 Turning a high-speed train.
(**a**) $a = v^2/R$ where R is the radius of the circle (**b**) $R = v^2/a \simeq 3.15\,\text{km}$ (**c**) $t = \pi R/(2v) \simeq 89\,\text{s}$

B.11 Acceleration on the equator.
(**a**) $v \simeq 464\,\text{m/s}$ (**b**) $a = v^2/R \simeq 0.03\,\text{m/s}^2 = 0.0034\,g$

B.12 Artificial gravity in space travel.
(**a**) $n \simeq 4.2$ (**b**) $\Delta a = (2\pi/T)^2 \cdot 2\,\text{m} = 0.4\,\text{m/s}^2$

B.13 Probe in tornado.
(**a**) $\bar{\mathbf{a}} = -3.3\,\text{m/s}^2\,\mathbf{i} - 18.1\,\text{m/s}^2\,\mathbf{j}$ (**b**) $R \simeq 40$, $\mathbf{r}_{circle} \simeq 5\,\text{m}\,\mathbf{i} + 10\,\text{m}\,\mathbf{j}$

B.14 Bead on ring.
(**a**) $v = R\cos\theta\ (2\pi n)/(60\text{s})$ (**b**) $a = R\cos\theta(2\pi n/(60\text{s}))^2$

B.16 Car in a wire.
(**a**) $v = a_t t$ (**b**) $a_r = v^2/R = a_t^2 t^2/R$ (**c**) $v = \sqrt{100 a_t R}$

Chapter 9

B.6 Rope with finite mass.

(**a**) $S = mg/(2\sin\alpha)$ (**b**) $S = mg/(2\sin\alpha)$ (**c**) No

B.7 Fireman on pole.

(**b**) $F_\mu = mg$ (**c**) $N = mg/\mu_d$

B.8 Pulling a box.

(**b**) $N = mg - T\sin(\alpha)$ (**c**) $a = (T/m)(\cos(\alpha) + \mu\sin(\alpha)) - \mu g$, d) $\alpha = \pi/4$ (**d**) $\alpha = \pi/4$

B.9 Hanging rope.

(**b**) $T = x\,(M/L)\,g$ (**c**) $N = (L - x)\,/L\,Mg$ (**d**) $x = \mu/\,(\mu + 1)\;L$

B.10 Pulling out a book.

(**a**) $F > \mu_2(m_1 + m_2)g$ (**b**) $F > (\mu_1(m_1 + m_2) + \mu_2 m2)\,g$

B.11 Forces on a 200 m runner.

(**a**) $f = mv^2/R$ (**b**) $\mu = v^2/(gR)$

B.12 Rope through a hole.

$v = \sqrt{MgR/m}$

B.13 Bead on a wire.

$\alpha = \sin^{-1}\left(T^2 g\right)/\left(R(2\pi)^2\right)$

B.14 Man in a wheel.

$v = \sqrt{gR/\mu_s}$

B.15 Motorcycle in a loop.

$v \geq \sqrt{gR}$

B.16 Stick-slip friction.

(**b**) $x_b(t) = b + ut$ (**e**) $N = mg$ (**f**) $a = 0$ (**g**) $\Delta L = (\mu_d mg)/k$ (**h**) $x(t) = x_b(t) - b - (\mu_d mg)/k$ (**j**) $\Delta L = (\mu_s mg)/k$ (**k**) $f = k\,u\,t$

B.17 Feather in tornado.

(**b**) $a = -g + (D/m)v^2$ (**d**) $D/mg = (t/h)^2 = (4.8\,\mathrm{s}/2.4\,\mathrm{m}) = 4.0\,\mathrm{s}^2\,\mathrm{m}^{-2}$ (**e**) $a = d^2z/dt^2 = -g - D|v_z|v_z$, $y(0) = h$, and $v(0) = 0$ (**f**)

```
# Program for a falling feather
from pylab import *
h = 2.4
Dmg = 4.0
g = 9.8
time = 10.0
dt = 0.001
n = int(round(time/dt))
t = zeros(n,float)
x = zeros(n,float)
v = zeros(n,float)
a = zeros(n,float)
```

```
x[0] = h
v[0] = 0.0
i = 1
while (i<n) and (x[i]>=0.0):
    a[i] = -g - g*Dmg*v[i]*abs(v[i])
    v[i+1] = v[i] + a[i]*dt
    x[i+1] = x[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
    i = i + 1
i = i - 1, x(i), t(i)
subplot(2,1,1)
plot(t[0:i],x[0:i])
xlabel('t [s]'), ylabel('x [m]')
subplot(2,1,2)
plot(t[0:i],v[0:i])
xlabel('t [s]'), ylabel('v [m/s]')
```

(h) $\mathbf{a} = -g{\text{-}}g(D/mg)\,|\mathbf{v} - \mathbf{w}|\,(\mathbf{v} - \mathbf{w})$ **(i)** $w_T = v_T = v_0$, $\mathbf{a} \simeq v_0^2/r_0$ **(j)** No. **(k)**

```
from pylab import *
vT = 0.18 # Terminal velocity
Dmg = 4.0
R = 20.0 # Size in meters
U = 100.0 # Velocity in m/s
g = 9.8
time = 15.0
dt = 0.001
n = int(round(time/dt))
t = zeros((n,1),float)
r = zeros((n,3),float)
v = zeros((n,3),float)
a = zeros((n,3),float)
t[0] = 0.0;
r[0,:] = array([-1.0*R,0.0,2.4])
v[0,:] = array([0.0,0.0,0.0])
i = 1
while ((r(i,3)>=0.0)and(i<n-1)):
    rr = norm(r[i,:])
    u = U*array([-r[i,1)],r[i,0],0.0])*exp(-rr/R)/R
    vrel = v[i,:] - u
    aa = -g*array([0,0,1]) - g*Dmg*norm(vrel)*vrel
    a[i = aa
    v[i+1] = v[i] + aa*dt
    r[i+1] = r[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
    i = i + 1
imax = i
figure(3)
i = range(imax)
ii = range(0,imag,100)
plot(r[i,0],r[i,1],'-',r[i,0],r[ii,1],'o')
axis('equal'), xlabel('x [m]'), ylabel('y [m]')
```

B.18 Modelling Atomic Interactions.

(a) $x = 0$ is an unstable equilibrium point, while $x = \pm d$ are stable equilibrium points. **(c)** $v_0 = \pm\sqrt{18 U_0/m}$ **(d)** $|v| \geq \sqrt{2U_0/m}$ **(f)** $a = F/m$; $v(t + \Delta t) = v(t) + a\,\Delta t$; $x(t + \Delta t) = x(t) + v(t + \Delta t)\,\Delta t$ **(g)**

```
from pylab import *
m = 1 #pkg
d = 0.1 #nm
U_0 = 1 #nJ
dt = 0.01 #ns
T = 10 #ns
```

```
n = int(round(T/dt))
t = zeros(n,float);
v = zeros(n,float);
x = zeros(n,float);
for i in range(n-1):
        F = - ((4*U_0)/(d**4))*(x[i]^3-x[i]*d**2)
        a = F/m
        t[i+1] = t[i] + dt
        v[i+1] = v[i] + a*dt
        x[i+1] = x[i] + v[i+1]*dt
```

(**j**) $\mathbf{a} = -(4U_0)/\left(md^4\right)\left(r^3 - rd^2\right)(\mathbf{r}/r)$ (**k**)

```
for i in range(n-1):
    rr = norm(r[i])
    F = -(4*U_0)/(d**4)*(rr**3-rr*d**2)*(r[i]/rr)
    a = F/m
    t[i+1] = t[i] + dt
    v[i+1] = v[i] + a*dt
    r[i+1] = r[i] + v[i+1]*dt
```

(**m**) For the atom to move in a circular orbit with a constant radius, the initial conditions have to be $r > d$ and $v = \sqrt{((4U_0)/(md^4)\,(r^3 - rd^2)r}$, with the velocity directed orthogonal to the position.

Chapter 10

B.7 Dragging a cart.

(**a**) $W_F = F\,s = 90\,\text{N} \times 10\,\text{m} = 900\,\text{J}$ (**b**) $W_f = f\,s = -30\,\text{N} \times 10\,\text{m} = -300\,\text{J}$ (**c**) $v = \sqrt{2(W_F + W_f)/m} = 11\,\text{m/s}$

B.8 Toboggan slide.

(**a**) $W_f = (1/2)mv^2 - mgh = -275\,\text{J}$.

B.9 Crate on conveyor belt.

(**b**) $s = \left(v_0^2\right)/(2\mu_d g)$ (**d**) $W = \int_0^x -f\,dx = -fx = -\mu_d mgx$ (**e**) $x = -\left(v_c^2\right)/(2\mu_d g)$

B.10 Volleyball smash.

(**a**) $x \simeq 0.043\,\text{m}$

B.11 A bouncing ball.

(**a**) $v_1 = -\sqrt{2gh}$ (**c**) $W_G = -mgy$, $W_k = -k\,(2/5)\,(-y)^{5/2}$ (**d**) $mg\,(-y) - (2k/5)\,(-y)^{5/2} = -mgh$ (**e**) $v_3 = -v_1 = \sqrt{2gh}$

B.12 Power of the heart.

(**a**) $W = 70.53\,\text{kJ}$ (**b**) $P = 0.816\,\text{W}$

B.13 Power station.

(**a**) $P = 9800\,\text{W}$

B.14 Accelerating car.

(**a**) $t = (1/2)\,(mv^2)/P = 2.48\,\text{s}$

B.15 An accelerating motorbike.
(**a**) $v = \sqrt{2\,P\,t/m}$ (**b**) $a(t) = dv/dt = \sqrt{P/2mt}$ (**c**) $x(t) = (2/3)\sqrt{2Pt^3/m}$

B.16 Driving efficiently.
(**c**) $a = \left((P_0/v) - Dv^2\right)/m$ (**g**) $W_E = mv^2/2$ (**h**) $W_D = Dv^2 L$

Chapter 11

B.8 The loop.
(**a**) $v_B = \sqrt{2gh}$ (**b**) $v_C = \sqrt{2g(h - 2R)}$ (**c**) $v_C \geq \sqrt{gR}$ (**d**) $h \geq (5/2)R$ (**e**) $s = h/\mu$

B.9 Sliding on a cylinder.
(**a**) $v = \sqrt{2gR\,(1 - \cos(\theta))}$ (**b**) $\cos\theta = 2/3$

B.10 Vertical pendulum.
(**a**) $v = \sqrt{v_0^2 - 4gL}$ (**b**) $v_0 \geq \sqrt{5\,g\,L}$

B.11 Two-point pendulum.
(**a**) $v_A = \sqrt{2gL}$ (**b**) $v_B = \sqrt{2g(2h - L)}$ (**c**) $h > (2/3)\,L$

B.12 Lennard-Jones Potential.
(**a**) $F = U_0\left(12\left(a^{12}/r^{13}\right) - 6\left(b^6/r^7\right)\right)$ (**c**) $r_{1,2} = \pm 2^{(1/6)}\,(a^2/b)$

B.13 A bouncing ball—part 1.
(**a**) R (**b**) $v = \sqrt{2g(h - R)}$ (**c**) $\delta y = \sqrt{(2mg/k)\;(h - R)}$

B.14 A bouncing ball—part 2.
(**a**) $\mathbf{v} = (v_0, -\sqrt{2g(h - R)})$ (**b**) $\mathbf{v} = (v_0, 0)$ (**c**) $\delta y = \sqrt{(2mg/k)\;(h - R)}$
(**d**) $\mathbf{v} = (v_0, +\sqrt{2g(h - R)})$

B.15 Shooting Ions.
(**b**) $x_1 = C/\left((1/2)mv_0^2 + C/b\right)$ (**c**) $v_\infty^2 = v_0^2 + 2\,C/(m\,b)$ (**e**) $\mathbf{a} = (C/m)\mathbf{r}/r^3$, $\mathbf{r}(0) = b\,\mathbf{i} + d\,\mathbf{j}$, $\mathbf{v}(0) = v_0\,\mathbf{i}$. (**g**)

```
m = 1.0    # mass in dimensionless units
b = 1.0    # length in dimensionless units
d = 0.2    # length in dimensionless units
C = 1.0
v0 = 2.5
time = 4.0/v0 # time in dimensionless units
dt = 0.001 # dt
n = int(round((time/dt))
r = zeros((n,2),float)
v = zeros((n,2),float)
t = zeros(n,float)
# Initial conditions
r[0] = array([b,d])
v[0] = array([-v0,0.0])
# Solve eqns. of motion
for i in range(n-1):
     rnorm3 = norm(r[i])**3
     F = C/rnorm3*r[i]
     a = F/m
     v[i+1] = v[i] + a*dt
     r[i+1] = r[i] + v[i+1]*dt
     t[i+1] = t[i] + dt
plot(r(:,1),r(:,2),'-')
xlabel('x/b'), ylabel('y/b'), axis('equal')
```

B.4 A bike and a car.
(**a**) $v = 600\,\text{km/h}$

Chapter 12

B.5 Kicking a ball.
(**a**) $\Delta p = 8.6\,\text{kg m/s}$ (**b**) $J = 8.6\,\text{kg m/s}$ (**c**) $F_{\text{avg}} = 86\,\text{N}$ (**d**) $F_{\text{avg}} = 172\,\text{N}$

B.6 Stopping a car.
(**a**) $F = 100\,\text{kN}$ (**b**) $F = 3.3\,\text{kN}$

B.7 Ball reflected from wall.
(**a**) $\Delta p = 2mv_0 \sin\theta$ (**b**) $J = 2mv_0 \sin\theta$ (**c**) $F = 2mv_0 \sin\theta/\Delta t$ (**d**) $\theta = 90^\circ$

B.8 Snowball on ice.
(**a**) $\mathbf{p} = 34.6\,\text{kg m/s}\,\mathbf{i} + 20\,\text{kg m/s}\,\mathbf{j}$ (**b**) $\mathbf{v}_{\text{you}} = -0.43\,\text{m/s}\,\mathbf{i}$, $\mathbf{v}_{\text{son}} = 0\,\text{m/s}\,\mathbf{i}$ (**c**) $\mathbf{v}_{\text{you}} = -0.43\,\text{m/s}\,\mathbf{i}$, $\mathbf{v}_{\text{son}} = 1.73\,\text{m/s}\,\mathbf{i}$

B.9 Toppling a book.
(**a**) You should choose the ball that bounces back

B.10 Bullet and a block.
(**a**) $v_0 = 20.8\,\text{m/s}$ (**b**) $\Delta E_k = -20.6\,\text{J}$

B.11 Stopping a ball.
(**a**) Yes

B.12 Pendulum and block.
(**a**) $v = -\sqrt{2gL}/3$, $V = 2\sqrt{2gL}/3$ (**b**) $h = L/9$

B.14 Newton's cradle.
(**a**) $v_0 = \sqrt{2gh_0}$ (**b**) $v_1^A = 0$ and $v_1^B = v_0$. (**c**) $h_1 = h_0$. (**d**) $h_1 = h_0/4$. (**e**) $v_0 = v_1^A + (1+r)\,v_0/2$, and $v_1^A = (1-r)\,v_0/2$ (**f**) The result of the first collision is to give ball B velocity v_0 and ball A velocity 0. The result of the second collision is to give ball C velocity v_0 and ball B velocity 0. (**g**) There are two equations with three unknowns.

B.15 Catching an atom.
(**b**) $F(x) = -k\,(x-b)$ when $b-d < x < b+d$, $F(x) = 0$ when $x > b+d$ and the atom cannot move to $x < b-d$. (**c**) $v_{A,1} = -\sqrt{v_{A,0}^2 + (2U_0/m)}$ (**d**) $v_2 = \frac{1}{2}v_{A,1}$. (**e**) $v_0 \geq \sqrt{U_0/m}$. (**g**)

```
k = 100.0
m = 1.0
b = 1.0
d = 0.5
r0 = array([1.0,0.0])
v0 = array([0.0,2.8])
time = 5.0
dt = 0.001
n = round(time/dt)
```

```
t = zeros(n,float)
r = zeros((n,2),float)
v = zeros((n,2),float)
a = zeros((n,2),float)
v[0] = v0
r[0] = r0
for i in range(n-1):
    rr = norm(r[i])
    if (rr>b+d):
        F = array([0.0,0.0]);
    elseif (rr>b-d):
        F = -k*(rr-b)*r[i]/rr
    else # Collision - reverse velocity in radial direction
        ur = r[i]/rr
        vprojur = dot(v[i],ur)
        v[i] = v[i] - vprojur*ur + abs(vprojur)*ur
    a[i] = F/m
    v[i+1] = v[i] + a[i]*dt
    r[i+1] = r[i] + v[i]*dt
    t[i+1] = t[i] + dt
plot(r[:,0],r[:,1])
xlabel('x/b'), ylabel('y/b')
```

(**l**) Not possible.

Chapter 13

B.5 Two-particle system.

(**a**) $x = 14/3\,\text{m}$

B.6 Center of mass of Earth-Moon system.

(**a**) 0.763 Earth-radii from the centre of the Earth

B.7 Carbon-monoxide.

(**a**) 48.37 pm from the Oxygen molecule

B.8 Three-particle system.

(**a**) $\mathbf{r} = 2\,\text{m}\,\mathbf{i} + 3\,\text{m}\,\mathbf{j}$ (**b**) By placing the particle at the center of mass of the system

B.9 Tetrahedron.

(**a**) $\mathbf{R} = (0, 0, 0)$ (**b**) $\mathbf{R} = (0, 0.4, 0.4)$

B.10 Cubic hole.

(**a**) $\mathbf{R} = -(L - d/2)\,(d/L)^3/\left(1 - (d/L)^3\right)\,\mathbf{i}$, where the origin is at the centre of the large cube and the small cube is cut out on the positive side of the x-axis

B.11 Triangle.

(**a**) $R_{CM} = (0, (2/3)a)$, where the origin is at the bottom centre.

B.12 Triangle.

(**a**) $R_{CM} = (0, (b/\sqrt{3}))$, where the origin is at the bottom centre

B.13 A piece of pie.

(**a**) $X = (2/3)\ (R\sin\theta)\,/\theta$, $Y = (2/3)\ (R(1-\cos\theta))\,/\theta$

B.14 Person in a boat.
(**a**) 2.4 m in the opposite direction of John

B.15 Car on a train.
(**a**) 5 m in the opposite direction

Chapter 14

B.4 Flywheel position.
(**a**) $\omega = (c_1/t_1) + 2c_2\left(t/t_2^2\right)$ (**b**) $\alpha = \left(2c_2/t_2^2\right)$

B.5 Unbalanced wheel.
(**a**) $\omega = 2.5\cos\left(t/(2\,\mathrm{s})\right)$ rad/s (**b**) $\alpha = -1.25\sin\left(t/(2\,\mathrm{s})\right)$ rad/s^2

B.6 Earth and Sun.
(**a**) 1.99×10^{-7} rad/s (**b**) 7.27×10^{-5} rad/s

B.7 Engine.
(**a**) 6.98 rad/s^2 (**b**) 375

B.8 Spinning down.
(**a**) $\omega(t) = 10\,\mathrm{rad/s^2}\ t$ (**b**) $\theta(t) = 5\,\mathrm{rad/s^2}t$ (**c**) $\omega(t) = 30\,\mathrm{rad/s} - 0.1\,\mathrm{rad/s^2}\,t$ (**d**) $\theta(t) = 45\,\mathrm{rad} - 0.05\,\mathrm{rad/s^2}\,t^2$ (**e**) 300 s (**f**) 600 s

B.9 A slippery wheel.
(**a**) $\omega = \omega_0\exp(-k_\omega t)$ (**b**) 23.0 s

B.10 Running the curve.
(**a**) $\omega = 0.20$ rad/s (**b**) $\alpha = 0$ (**c**) $a = 2\,\mathrm{m/s^2}$

B.11 Rotating Earth.
(**a**) $\omega_0 = 7.27\,10^{-5}$ rad/s (**b**) ω_0 (**c**) $v = \omega_0 R = 463.8$ m/s (**d**) ω_0 (**e**) $v = \omega_0(R\cos\alpha)$ (**f**) $\alpha = 0$ (**g**) $a = \frac{v^2}{R} = \omega_0^2 R = 0.034\,\mathrm{m/s^2}$ (**h**) $a = \omega_0^2\rho = \omega_0^2(R\cos\alpha) = 0.017\,\mathrm{m/s^2}$ directed in towards the rotational axis

B.12 Rolling wheel.
(**b**) 0 m/s (**c**) $2v$ (**d**) 0 m/s^2 along the surface and v^2/R normal to the surface toward the center of the wheel (**e**) 0 m/s^2 along the surface and v^2/R normal to the surface, toward the center of the wheel

Chapter 15

B.4 Three-particle system.
(**a**) $\mathbf{R} = (0, -a/3)$ (**b**) $I_{cm} = 6ma^2$ (**c**) $I_{0,z} = (6 + 1/9)ma^2$ (**d**) $I_{0,x} = 3ma^2$ (**e**) $I_{0,Y} = 2ma^2$

B.5 Compound system.
(**a**) $(1/12)mL^2 + (4/5)MR^2 + 2M(L/2)^2$ (**b**) $(4/5)MR^2$ (**c**) $(4/5)MR^2 + (1/12)mL^2$
$+\, m(L/2)^2 + ML^2$

B.6 Water molecule.
(**a**) $I_{cm} = 1.92\,\mathrm{u}\,a^2$ (**b**) $I_O = 2\,\mathrm{u}\,a^2$

B.7 Compound system.
(**a**) $(4/5)MR^2 + 4MR^2$ (**b**) $\boldsymbol{\omega} = \sqrt{(5/6)\,(g/R)\,sin(\theta)}$

B.8 Atwood's fall machine.
(**a**) $v = \sqrt{(gh(m_1 - m_2))\,/\,(M + m_1 + m_2)}$
(**b**) $\boldsymbol{\omega} = (1/R)\sqrt{(g\,h(m_1 - m_2))\,/\,(M + m_1 + m_2)}$

B.9 Triangular pendulum.
(**a**) $I_O = 2mL^2$ (**c**) $\boldsymbol{\omega} = \left(\left(\sqrt{3}/2\right)(g/L)\right)^{1/2}$ (**d**) It continues with the same angular velocity around a center of mass that follows a parabolic path.

B.10 Spinning toy car.
(**c**) $\boldsymbol{\omega} = \omega_0 - \mu\,(g/Rc)\,t$ (**d**) $t = (\boldsymbol{\omega}_0 R)/(\mu g)\,1/\,(1/(2+c) + (1/c))$

B.11 Micro-electromechanical system.
(**a**) $X = L/2, Y = L/2.$ (**c**) $I_y = ML^2/3$
(**h**) $\boldsymbol{\omega} = \sqrt{(15\,g\,sin\theta)/(11\,L) - (3\,\kappa\,\theta^2)/(22\,M\,L^2)}$ (**j**) $\theta = 10\,(M\,L\,g/\kappa)$

Chapter 16

B.5 Motion of rod during a collision-like process.
(**a**) $v_0 = -\sqrt{2gh}$, $\boldsymbol{\omega}_0 = 0$ (**c**) $\boldsymbol{\omega}_1 = -(3/2)\,(v_0/L)$ (**d**) $p_1 = (3/4)p_0$ (**e**) $\alpha = (3/2)(g/L)\cos(\theta) - (3\kappa)/(ML^2)\,\theta$ (**f**) $I_{O,z}\boldsymbol{\omega}_1^2 = \kappa\theta^2 - MgL\sin(\theta)$ (**g**) $\boldsymbol{\omega}_2 = -\boldsymbol{\omega}_1$ (**h**) $v_2 = (3/4)v_0$ (**k**) $y_4 = (9/16)\,h$

B.6 Collision between a rod and a block.
(**a**) $I_O = (1/3)\,M\,L^2$ (**b**) $E_{k,1} = (MgL)/2\,(\cos(\theta) - \cos(\theta_0))$
(**c**) $\boldsymbol{\omega}_0 = \sqrt{(3g/L)(1 - \cos(\theta_0))}$ (**g**) The rod stops completely, and the block gains the "velocity" of the rod. (**h**) $v_1 = (\boldsymbol{\omega}_0\,L)\,/\,(1 + 3\,(m/M))$

B.7 A model of two rods colliding.
(**b**) $v_1 = v_0/2$ (**c**) 0 (**d**) $v_1 = v_0/2$ (**e**) $\boldsymbol{\omega}_1 = -dv_0/\left(d^2 + (L^2/3)\right)\,\mathbf{k}$ (**f**) $K_0 - K_1 = (Mv_0^2/4)\,\left(1 - d^2/\left(d^2 + (L^2/3)\right)\right)$

B.9 Tarzan's swing.
(**a**) $v_{x1} = v_0\;,\;\; v_{y1} = \sqrt{2gh}$ (**b**) $I_{O,z} = ML^2/3$ (**d**) $y_3 = \left((1/2)I_{O,z}\boldsymbol{\omega}_2^2\right)/\,((m + (M/2))\,g)$ (**e**) The same height.

B.10 Rolling up a slope.
(**c**) $a_x = g\,(\mu\cos\theta - \sin\theta)$ (**d**) $v(t) = g\,(\mu\cos\theta - \sin\theta)\,t$ (**e**) $\alpha = fR/I$
(**f**) $\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + (f\,R/I)\,t$ (**g**) $t = R\boldsymbol{\omega}_0/\,[(f/I) + g\,(\mu\cos\theta - \sin\theta)]$

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